Left-invariant metrics and submanifold geometry

TAMARU, Hiroshi

Hiroshima University

Differential Geometry, Lie Theory and Low-Dimensional Topology (La Trobe University, Melbourne), 20/Dec./2016

Abstract (1/2)

Background

- Left-invariant (Riemannian) metrics on Lie group:
- $\exists$ many "nice" such metrics, e.g., Einstein, Ricci soliton, ...

Our Framework

- A left-invariant metric $\langle , \rangle$ defines a submanifold $[\langle , \rangle]$, in some noncompact Riemannian symmetric space $\widetilde{\mathfrak{M}}$.
- Expectation: a "nice" metric corresponds to a "nice" submfd.

Abstract

- The above expectation is true for several cases.

Abstract (2/2)

Contents

- Introduction (to our framework)
- Case 1: Easy cases
- Case 2: Low-dim. solvable Lie groups
- Case 3: Some general cases
- Summary

Intro (1/6)

Our Framework (recall)

- $\langle , \rangle$: a left-inv. metric on $G \longrightarrow [\langle , \rangle]$: a submfd in $\widetilde{\mathfrak{M}}$.

Basic Fact

$\exists$ 1-1 correspondence between

- a left-inv. (Riemannian) metric on G,
- a (positive definite) inner product $\langle , \rangle$ on $\mathfrak{g} := \mathrm{Lie}(G)$.

Def. (the ambient space)

The **space of left-inv. metrics** on G is defined by

- $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\}$.

Intro (2/6)

Recall

- $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\}$.

Prop. (well-known)

If $\dim G = n$, then

- $\widetilde{\mathfrak{M}} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$

  where $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \widetilde{\mathfrak{M}}$ by $g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$;
- Hence $\widetilde{\mathfrak{M}}$ is a noncompact Riemannian symmetric space.

Note

- Finding a nice left-inv. metric on G

  $\leftrightarrow$ Finding a nice point on $\widetilde{\mathfrak{M}}$...?

...but every point on $\widetilde{\mathfrak{M}}$ looks the same.

Intro (3/6)

Def.

Let $\langle , \rangle_1, \langle , \rangle_2 \in \widetilde{\mathfrak{M}}$. We say $\langle , \rangle_1 \sim \langle , \rangle_2$ (**isometric up to scalar**)

$:\Leftrightarrow\ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}),\ \exists c > 0 : c\varphi.\langle , \rangle_1 = \langle , \rangle_2$.

Note

$\langle , \rangle_1 \sim \langle , \rangle_2$

$\Rightarrow$ all Riemannian geometric properties of them are the same.

Def. (the submfd)

We define the **corresponding submfd** of $\langle , \rangle$ by

- $[\langle , \rangle] :=$ "the isometry and scaling class of $\langle , \rangle$" ($\subset \widetilde{\mathfrak{M}}$).

Intro (4/6)

Recall

- $\langle , \rangle_1 \sim \langle , \rangle_2 :\Leftrightarrow \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \exists c > 0 : c\varphi.\langle , \rangle_1 = \langle , \rangle_2$.
- $[\langle , \rangle] :=$ "the isometry and scaling class of $\langle , \rangle$" $(\subset \widetilde{\mathfrak{M}})$.

Prop. (cf. Kodama-Takahara-T. 2011)

- $[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$,

where $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}_n(\mathbb{R})$ acts naturally on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

We got:

- $\widetilde{\mathfrak{M}} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$: a noncpt Riem. symmetric space.
- $\widetilde{\mathfrak{M}} \supset [\langle , \rangle]$: a homogeneous submanifold.

Intro (5/6)

Note

- Finding a nice left-inv. metric on G

  $\leftrightarrow$ Finding a nice submfd $[\langle , \rangle]$ in $\widetilde{\mathfrak{M}}$...

Note that $[\langle , \rangle_1]$ and $[\langle , \rangle_2]$ are different in general.

Note (why this framework would be interesting)

- This connects two different areas:

  — geometry of left-inv. metrics vs submfd geometry.
- Both have been studied actively in these years.

Intro (6/6)

Note (Both have been studied actively)

Geometry of left-inv. metrics:

- Among others, Nikolayevsky, Nikonorov, T., ...

Homog. submfds (isometric actions) in noncpt symmetric spaces:

- Berndt, DíazRamos, DomínguezVázquez, Kollross, T., ...

We hope that

- characterize nice left-inv. metrics in terms of submfds...
- obtain nice submfds (isom. actions) from left-inv. metrics...

Case 1: Easy cases (1/5)

Recall (our expectation)

- A "nice" metric $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Ex. ($\mathrm{Sec} \equiv 0$)

For $\mathfrak{g} = \mathbb{R}^n$ (abelian),

- $\forall \langle , \rangle$ is flat.
- $\forall \langle , \rangle$, one has $[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle = \widetilde{\mathfrak{M}}$.

  (i.e., $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is transitive)

Case 1: Easy cases (2/5)

Ex. (Nice metric; including the case $\mathrm{Sec} \equiv c > 0$)

Let $\mathfrak{g}$ be compact simple, and $\langle , \rangle_{\mathrm{K}}$ the Killing metric. Then

- $\langle , \rangle_{\mathrm{K}}$ is Einstein, $\mathrm{Sec} \geq 0$.
- $[\langle , \rangle_{\mathrm{K}}] = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle_{\mathrm{K}} = \mathbb{R}^{\times}.\langle , \rangle_{\mathrm{K}} \cong \mathbb{R}$: geodesic.

  (since it is bi-inv.; other orbits have larger dimensions)

Note

$[\langle , \rangle]$ contains information of

- how large the symmetry of $\langle , \rangle$ is ...

  (large symmetry $\leftrightarrow$ $\mathrm{Aut}(\mathfrak{g})_{\langle , \rangle}$ is large $\leftrightarrow$ $[\langle , \rangle]$ is small)
- also the "position" of $\langle , \rangle$.

Case 1: Easy cases (3/5)

Prop. (Nice action; Lauret 2003, Kodama-Takahara-T. 2011)

The action $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is transitive

$\Leftrightarrow \ \mathfrak{g} = \mathbb{R}^n, \ \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}, \ \mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Ex. ($\mathrm{Sec} \equiv c < 0$; Milnor 1976, Lauret 2003, KTT 2011)

For $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n} = \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$,

- $\forall \langle , \rangle$ is const. negative sectional curvature;
- $\forall \langle , \rangle$, one has $[\langle , \rangle] = \widetilde{\mathfrak{M}}$.

Case 1: Easy cases (4/5)

Recall

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is transitive $\Leftrightarrow$ $\mathfrak{g} = \mathbb{R}^n,\ \mathfrak{g}_{\mathbb{R}\mathrm{H}^n},\ \mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Note

- $\mathfrak{h}^3$: 3-dim. Heisenberg;
- $\forall \langle , \rangle$ on $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ is a non-Einstein (algebraic) Ricci soliton.

Def.

$(\mathfrak{g}, \langle , \rangle)$ is an **algebraic Ricci soliton** (**ARS**)

$:\Leftrightarrow\ \exists c \in \mathbb{R},\ \exists D \in \mathrm{Der}(\mathfrak{g}) : \mathrm{Ric}_{\langle , \rangle} = c \cdot \mathrm{id} + D.$

Case 1: Easy cases (5/5)

Recall

- $(\mathfrak{g}, \langle , \rangle)$: **ARS** $:\Leftrightarrow \exists c \in \mathbb{R}, \exists D \in \mathrm{Der}(\mathfrak{g}) : \mathrm{Ric}_{\langle , \rangle} = c \cdot \mathrm{id} + D$.

Fact (Lauret 2001, 2011, Jablonski 2014)

ARS is "almost" equivalent to Ricci soliton;

- $(\mathfrak{g}, \langle , \rangle)$: ARS
  $\Rightarrow$ the corresponding simply-conn. $(G, \langle , \rangle)$ is Ricci soliton.
- $(G, \langle , \rangle)$ is left-inv. Ricci soliton
  $\Rightarrow$ it is "isometric" to ARS (as Riem. mfd).

Note

A (complete) Riemannian manifold is said to be **Ricci soliton**

$:\Leftrightarrow \exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(M) : \mathrm{Ric}_g = c \cdot \mathrm{id} + \mathcal{L}_X g$.

Case 2: Low-dim. solvable Lie groups (1/5)

Recall (our expectation)

- A "nice" metric $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Thm. (Hashinaga-T.)

Let $\mathfrak{g}$ be a 3-dim. solvable Lie algebra. Then

- $\langle , \rangle$ is ARS $\Leftrightarrow$ $[\langle , \rangle]$ is minimal.

Idea of Proof

Study them case-by-case... In fact,

- One knows the classification of 3-dim. solvable Lie algebras.
- We can see $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohomogeneity at most one.

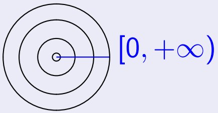
Case 2: Low-dim. solvable Lie groups (2/5)

More on 3-dim. case

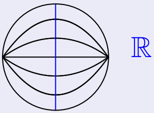
Let $\mathfrak{g}$ be a 3-dim. solvable Lie algebra with $[\langle , \rangle] \neq \widetilde{\mathfrak{M}}$.

- $\exists$ 3 families of such $\mathfrak{g}$.
- Consider $H := (\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$.
- Then, for each $\mathfrak{g}$, $H \curvearrowright \widetilde{\mathfrak{M}}$ satisfies one of the following:

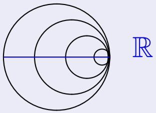
Picture



type (K)



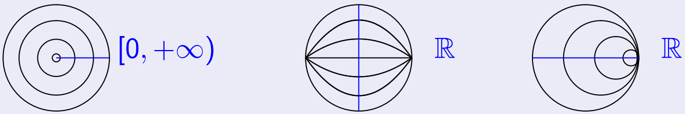
type (A)



type (N)

Case 2: Low-dim. solvable Lie groups (3/5)

Picture (recall)



type (K) type (A) type (N)

Note

- (K) $\exists 1$ singular orbit;
- (A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;
- (N) $\nexists$ singular orbit, all orbits are congruent to each other.

Case 2: Low-dim. solvable Lie groups (4/5)

Thm. (recall, 3-dim. solvable case)

- $\langle , \rangle$ is ARS $\Leftrightarrow$ $[\langle , \rangle]$ is minimal.

Note (for higher dim. case; good news)

We know that

- $\exists$ several $\mathfrak{g}$ satisfying the above "$\Leftrightarrow$".

Note (for 4-dim. case; bad news)

Hashinaga (2014) proved that

- $\exists \ \mathfrak{g}$: the above "$\Leftarrow$" does not hold.
- $\exists \ \mathfrak{g}$: the above "$\Rightarrow$" does not hold.

Case 2: Low-dim. solvable Lie groups (5/5)

Recall

Our expectation is:

- a "nice" metric $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Note

Our studies imply that

- the minimality of $[\langle , \rangle]$ is not enough, in general.

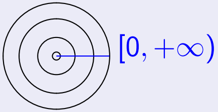
Case 3: Some general cases (1/5)

Recall

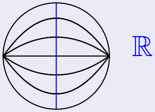
$\mathfrak{g}$: 3-dim. solvable, $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ of type (K)

$\Rightarrow$ $[\langle , \rangle]$ is a singular orbit iff $\langle , \rangle$ is ARS.

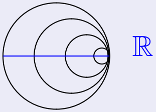
Picture (recall)



type (K)



type (A)



type (N)

Case 3: Some general cases (2/5)

Claim

- For actions of type (K), we can generalize it.

Thm. (Taketomi)

Let $\mathfrak{g}$ be any Lie algebra, and assume that

- $0 \neq \forall \xi \in T^{\perp}_{\langle,\rangle}([\langle,\rangle])$, $\exists \varphi \in \mathrm{Aut}(\mathfrak{g}) : (d\varphi)_{\langle,\rangle}\xi \neq \xi$.

Then $\langle,\rangle$ is ARS.

Cor.

Let $\mathfrak{g}$ be any Lie algebra, and assume that

- $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is a cohomogeneity one action of type (K),
- $[\langle,\rangle]$ is a singular orbit.

Then $\langle,\rangle$ is ARS.

Case 3: Some general cases (3/5)

Recall (Assumption by Taketomi)

- $\forall$ nonzero normal vector of $[\langle , \rangle]$ can be moved by $\mathrm{Aut}(\mathfrak{g})$.

Idea of Proof $+\alpha$

- The above assumption yields that $[\langle , \rangle]$ is minimal.
  (the mean curv. vector is a normal vector, fixed by $\mathrm{Aut}(\mathfrak{g})$)
- The above assumption yields that $\mathrm{ric}_p^{\perp} = 0$ ($\forall p \in [\langle , \rangle]$).
  (Hence ric is tangential to $[\langle , \rangle]$ at any point)

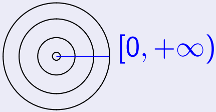
Case 3: Some general cases (4/5)

Recall

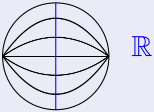
$\mathfrak{g}$: 3-dim. solvable, $(\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ of type (N)

$\Rightarrow$ $\nexists\langle , \rangle$ which is ARS.

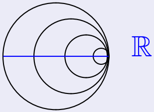
Picture (recall)



type (K)



type (A)



type (N)

Case 3: Some general cases (5/5)

Conjecture (Taketomi-T., to appear)

All orbits of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are congruent to each other

$\Rightarrow$ $\nexists \langle , \rangle$ which is ARS.

Note

The assumption of the conjecture means that

- all orbits are looks the same (i.e., no distinguished orbit)...

Prop. (Taketomi-T., to appear)

$\forall n \geq 3$, $\exists \mathfrak{g}$: Lie algebra of dim. n :

- all orbits of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are congruent to each other, and
- $\nexists \langle , \rangle$ on $\mathfrak{g}$ which is ARS.

Summary (1/4)

Our Framework/Expectation

- $\langle , \rangle$ on $\mathfrak{g}$ defines a submfd $[\langle , \rangle] \subset \widetilde{\mathfrak{M}}$.
- Does a "nice" $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$...?

Our Results

For 3-dim. solvable case,

- there is a very nice correspondence.

For 4-dim. solvable case,

- not so nice as the 3-dim. case...

For general cases,

- Taketomi obtained a sufficient condition for $\langle , \rangle$ to be ARS;
- We conjecture an obstruction for the existence of ARS.

Summary (2/4)

Related Topics

- (Taketomi 2015)
  Constructed examples of $\mathfrak{g}$ s.t. $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: hyperpolar.
- (Kubo-Onda-Taketomi-T. 2016)
  A study on left-inv. pseudo-Riem. metrics.

Problems (1/3)

Can we characterize ARS in terms of $[\langle , \rangle]$...?

- Certainly, the minimality is not enough.
- Taketomi's sufficient condition cannot be a necessary cond.
- So, what else?
- What happens for some other Lie algebras?
  — need to select a nice class of Lie algebras.

Summary (3/4)

Problems (2/3)

Find special classes: can we classify

- $\mathfrak{g}$ such that $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is special

  (e.g., cohomogeneity one, hyperpolar, polar, ...)
- $(\mathfrak{g}, \langle , \rangle)$ such that $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is special

  (e.g., totally geodesic, minimal, austere, ...)

Problems (3/3)

For the existence of a left-inv. "nice" metric (on a given $\mathfrak{g}$),

- a necessary and sufficient condition seems to be very hard;
- so, can we get an obstruction?

  — our conjecture is one possibility, but the condition is not easy to check.

Summary (4/4)

Ref. (just for our papers)

- Hashinaga, T.: Hiroshima Math. J. (2014)
- Hashinaga, T., Tamaru, H.: arXiv:1501.05513
- Hashinaga, T., Tamaru, H., Terada, K.: J. Math. Soc. Japan (2016)
- Kodama, H., Takahara, A., Tamaru, H.: Manuscripta Math. (2011)
- Kubo, A., Onda, K., Taketomi, Y., Tamaru, H.: Hiroshima Math. J. (2016)
- Taketomi, Y.: Topology Appl. (2015)
- Taketomi, Y.: submitted
- Taketomi, Y., Tamaru, H.: Transf. Groups, to appear

Thank you very much!