Realizations of some contact metric manifolds as Ricci soliton real hypersurfaces

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The 11th OCAMI-RIRCM Joint Differential Geometry Workshop on Submanifolds and Lie Theory
(Osaka City University)
2016/March/21

Preface (1/2)

Question (contact geometry)

For a simply-connected contact (κ, μ)-space,

- nongradient Ricci soliton $\Rightarrow (\kappa, \mu) = (0, 4)$.
- Does the converse hold?

Main Result (submanifold geometry)

$(0, 4)$-space can be realized as a homogeneous hypersurfaces in $G_2^*(\mathbb{R}^n)$ (noncompact Grassmann).

Corollary (left-invariant Ricci solitons)

$(0, 4)$-space is a Ricci soliton.

Preface (2/2)

Note

This talk is based on a joint work with

- Jong Taek Cho (Chonnam National University)
- Takahiro Hashinaga
  (National Inst. Tech., Kitakyushu College)
- Akira Kubo (Hiroshima Shudo University)
- Yuichiro Taketomi (Hiroshima University)

Contents

§1: Contact (κ, μ)-spaces

- We recall some facts on contact (κ, μ)-spaces.

(§2: Homogeneous hypersurfaces in $G_2^*(\mathbb{R}^n)$)

(§3: Left-invariant Ricci solitons)

Contact (κ, μ)-spaces (2/8)

Def.

$(M^{2n+1}; \eta, \xi, \varphi, g)$ is **contact metric mfd**

$:\Leftrightarrow$ (contact form) η : 1-form;

(characteristic vector field) $\xi \in \mathfrak{X}(M)$, $\eta(\xi) = 1$;

φ : $(1,1)$-tensor, $\varphi^2(X) = -X + \eta(X)\xi$;

g : Riem. metric, $g(\varphi(\cdot), \varphi(\cdot)) = g - \eta \otimes \eta$;

(fundamental 2-form) $\Phi := g(\cdot, \varphi(\cdot)) = d\eta$.

Contact (κ, μ)-spaces (3/8)

Def. (Blair-Koufogiorgos-Papantoniou, 1995)

$(M; \eta, \xi, \varphi, g)$ is a **(κ, μ)-space** $(\kappa, \mu \in \mathbb{R})$

$:\Leftrightarrow \forall X, Y \in \mathfrak{X}(M),$

$$R(X, Y)\xi = (\kappa \, \mathrm{id} + (\mu/2)\mathcal{L}_\xi \varphi)(\eta(Y)X - \eta(X)Y).$$

$\exists$ interesting examples of (κ, μ)-spaces.

Ex. (1/3)

Sasakian $\Rightarrow$ (κ, μ)-space. (with $\kappa = 1$, $\mathcal{L}_\xi \varphi = 0$)

Contact (κ, μ)-spaces (4/8)

$\exists$ interesting examples of (κ, μ)-spaces.

Ex. (2/3)

$M := T_1(M_n(c))$ is a $(c(2-c), -2c)$-space, where

- $M_n(c)$: the space of const. curvature c,
- $T_1(\cdot)$: the unit sphere bundle,

Note: $c = 1 \Rightarrow$ Sasakian.

Ex. (3/3)

G : 3-dim. nonabelian unimodular Lie group

$\Rightarrow \exists\ (\eta, \xi, \varphi, g)$ which is left-invariant (κ, μ)-space.

Contact (κ, μ)-spaces (5/8)

Non-Sasakian (κ, μ)-spaces have been classified.

Prop. (Blair-Koufogiorgos-Papantoniou, 1995)

Let $(M; \eta, \xi, \varphi, g)$ be a (κ, μ)-space. Then

- $\kappa \leq 1$.
- $\kappa = 1 \Leftrightarrow$ Sasakian.

Prop. (Boeckx, 2000)

$\forall(\kappa, \mu)$ with $\kappa < 1$,

- $\exists(M; \eta, \xi, \varphi, g)$ which is a (κ, μ)-space;
- it is unique if conn., simply-conn., and complete.

Contact (κ, μ)-spaces (6/8)

$\exists$ study on Ricci soliton (κ, μ)-spaces.

(Ricci soliton $:\Leftrightarrow \mathrm{Ric}_g = c \cdot \mathrm{id} + \mathcal{L}_X g$)

Thm. (Ghosh-Sharma, 2014)

(κ, μ)-space is

- gradient Ricci soliton $\Leftrightarrow$ $(0, 0)$-space;
- non-gradient Ricci soliton $\Rightarrow$ $(0, 4)$-space.

Rem.

$(0, 4)$-space with dimension n is

- $n = 3 \Rightarrow \mathrm{Sol}$, non-gradient Ricci soliton;
- $n \geq 5 \Rightarrow$?

Contact (κ, μ)-spaces (7/8)

$(0,4)$-space (with $\dim \geq 5$) is of our interest.

Prop. (Boeckx, 2000)

The Lie group $G_{0,2}$ of $\mathfrak{g}_{0,2}$ becomes a $(0,4)$-space, where

- $\mathfrak{g}_{0,2} := \mathrm{span}_{\mathbb{R}}\{\xi, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$,

$$
\begin{aligned}
[\xi, Y_i] &= 2X_i && (i \geq 1) \\
[Y_2, Y_i] &= 2Y_i && (i \neq 2) \\
[X_2, Y_2] &= 2\xi && \\
[X_2, Y_i] &= 2X_i && (i \neq 2) \\
[X_i, Y_i] &= -2X_2 + 2\xi && (i \neq 2)
\end{aligned}
$$

Contact (κ, μ)-spaces (8/8)

Summary of this section.

A question

$(0, 4)$-space with dim ≥ 5 is a Ricci soliton?

A question (modified)

What is the Lie group $G_{0,2}$ (or the Lie algebra $\mathfrak{g}_{0,2}$)?

Test

- $\dim \mathfrak{g}_{0,2} = 2n + 1$, $\quad \mathfrak{n} := [\mathfrak{g}_{0,2}, \mathfrak{g}_{0,2}]$: $2n$-dim.
- $\dim[\mathfrak{n}, \mathfrak{n}] = n$, $\quad \dim[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 1$.
- Conclusion: $\mathfrak{g}_{0,2}$ is solvable, $\mathfrak{n}$ is of 3-step nilpotent.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (1/9)

Contents

(§1: Contact (κ, μ)-spaces)

§2: Homogeneous hypersurfaces in $G_2^*(\mathbb{R}^n)$

- We recall some general results (by Berndt-T.); and show that $G_{0,2} \subset G_2^*(\mathbb{R}^{n+3})$ as homog. hypersurface.

(§3: Left-invariant Ricci solitons)

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (2/9)

We review some results on homog. hypersurfaces in symmetric spaces of noncompact type.

- Ref. Berndt-T.: JDG 2003, Crelle 2013.

Def.

$H \curvearrowright (M, g)$ is of **cohomogeneity one**
$:\Leftrightarrow$ regular orbits have codimension one.

Def.

- regular orbits := orbits of maximal dim.
- singular orbits := orbits of smaller dim.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (3/9)

Prop.

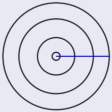
$\mathrm{SL}(2,\mathbb{R}) = KAN$: Iwasawa dec.

Then, $K, A, N \curvearrowright \mathbb{R}\mathrm{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ satisfies

(K): $\exists 1$ singular orbit.

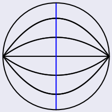
(A): $\nexists$ singular orbit, $\exists 1$ minimal orbit.

(N): $\nexists$ singular orbit, all orbits are congruent.



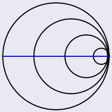
$[0, +\infty)$

type (K)



$\mathbb{R}$

type (A)



$\mathbb{R}$

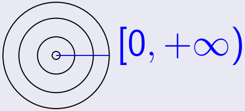
type (N)

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (4/9)

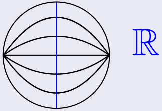
Thm. (Berndt-T. 2003, 2013)

M : irreducible symmetric space of noncpt type.
$H \curvearrowright M$: cohomogeneity one (with H connected).
Then, it is of type (K), (A), or (N):

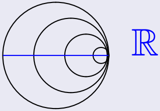
- (K): $\exists 1$ singular orbit.
- (A): $\nexists$ singular orbit, $\exists 1$ minimal orbit.
- (N): $\nexists$ singular orbit, all orbits are congruent.



type (K)



type (A)



type (N)

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (5/9)

Classification of actions of type (N).

Set Up

- $M = G/K$: irr. symmetric space of noncpt type.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa dec.

Thm, (Berndt-T., 2003)

Take $0 \neq X \in \mathfrak{a}$, and put $\mathfrak{s}_X := (\mathfrak{a} \ominus \mathbb{R}X) \oplus \mathfrak{n}$. Then

- $S_X \curvearrowright M$: cohomogeneity one of type (N).
- $\forall$ cohomogeneity one action of type (N) can be obtained in this way, up to orbit equivalence.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (6/9)

Examples of actions of type (N). (rank one)

Ex.

Assume that $\mathrm{rk}(M) = 1$. Then

- $\dim \mathfrak{a} = 1$.
- Thus $0 \neq X \in \mathfrak{a}$ satisfies $\mathfrak{s}_X = \mathfrak{n}$.
- $N \curvearrowright M$ induces a horosphere foliation.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (7/9)

Iwasawa dec. of $\mathrm{SO}^0_{2,n+1}$.

Ex.

$M = G_2^*(\mathbb{R}^{n+3}) = \mathrm{SO}^0_{2,n+1}/\mathrm{SO}_2\mathrm{SO}_{n+1}$. Then

- The root system Σ is of type B_2 $(= C_2)$.
- $\alpha_1 := \varepsilon_1 - \varepsilon_2, \ \ \alpha_2 := \varepsilon_2$.
- $\Sigma = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$.
- $\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3$, where
$$\mathfrak{n}^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}, \ \ \mathfrak{n}^2 = \mathfrak{g}_{\alpha_1+\alpha_2}, \ \ \mathfrak{n}^3 = \mathfrak{g}_{\alpha_1+2\alpha_2}.$$
- Note: $\dim \mathfrak{n}^1 = n, \ \ \dim \mathfrak{n}^2 = n-1, \ \ \dim \mathfrak{n}^3 = 1$.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (8/9)

It relates to contact $(0,4)$-space $G_{0,2}$.

Thm. (Cho-Hashinaga-Kubo-Taketomi-T.)

Consider $M = G_2^*(\mathbb{R}^{n+3})$. Then

- $\exists X \in \mathfrak{a} : (S_X).o \cong G_{0,2}$ as contact metric mfds.

Recall

M : Kähler mfd., $M \supset M'$: a real hypersurface. Then

- $\exists$ almost contact metric str. (η, ξ, φ, g) on M'.

Homog. hypersurfaces in $G_2^*(\mathbb{R}^n)$ (9/9)

Some comments.

Idea of Proof

We establish the "solvable model" of $G_2^*(\mathbb{R}^{n+3})$.

- i.e., o.n.b. of $\mathfrak{s}$ with explicit bracket relations.

Note (Berndt-Suh (PAMS 2015))

They classified CMC contact hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$.

- Our hypersurface is contained in their list.
- ("a horosphere whose center at infinity is the equivalence class of an $\mathcal{A}$-principal geodesic")

Left-invariant Ricci solitons (1/5)

Contents

(§1: Contact (κ, μ)-spaces)
(§2: Homogeneous hypersurfaces in $G_2^*(\mathbb{R}^n)$)
§3: Left-invariant Ricci solitons

- A general theory shows that $G_{0,2}$ is a Ricci soliton.

Left-invariant Ricci solitons (2/5)

A general theory shows that $G_{0,2}$ is a Ricci soliton.

- Ref. Lauret: Math. Ann. 2001, Crelle 2011.

Def.

A metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is **algebraic Ricci soliton**
$:\Leftrightarrow$ $\exists c \in \mathbb{R}, \exists D \in \mathrm{Der}(\mathfrak{g}) : \mathrm{Ric} = c \cdot \mathrm{id} + D$.

Note

$$\mathrm{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D[\cdot, \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)]\}$$
$$= \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})).$$

Left-invariant Ricci solitons (3/5)

Prop. (Lauret, 2001, 2011)

$(\mathfrak{g}, \langle , \rangle)$ is an algebraic Ricci soliton

$\Rightarrow$ the corresponding $(G, \langle , \rangle)$ is a Ricci soliton.

(G : simply-connected)

Comment

$\exists$ many $(G, \langle , \rangle)$: Ricci soliton, not Einstein.

- Interesting case: $\mathfrak{g}$ is solvable.
- Note: $\mathfrak{g}$ is solvable $\Leftrightarrow$ $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Left-invariant Ricci solitons (4/5)

A construction of ARS (:= algebraic Ricci soliton).

Notation

- $(\mathfrak{s}, \langle , \rangle)$: a solvable metric Lie algebra.
- $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$, which is nilpotent.
- $\mathfrak{a} := \mathfrak{n}^{\perp}$.

Prop. (Lauret 2011, CHKTT)

Assume $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$ is Einstein with $c < 0$

$\Rightarrow$ $\forall \mathfrak{a}' \subset \mathfrak{a}$ (subspace), $(\mathfrak{a}' \oplus \mathfrak{n}, \langle , \rangle)$ is an ARS.

Left-invariant Ricci solitons (5/5)

Prop. (Recall)

Assume $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$ is Einstein with $c < 0$

$\Rightarrow \forall \mathfrak{a}' \subset \mathfrak{a}$ (subspace), $(\mathfrak{a}' \oplus \mathfrak{n}, \langle , \rangle)$ is an ARS.

Cor.

The contact $(0, 4)$-space $G_{0,2}$ is a Ricci soliton.

$\because$ $G_2^*(\mathbb{R}^{n+3}) \cong AN$: Einstein, solvable, $c < 0$.

$\exists X \in \mathfrak{a}$: $\mathfrak{g}_{0,2} \cong (\mathfrak{a} \ominus \mathbb{R}X) \oplus \mathfrak{n}$.

Hence the above prop. proves the assertion. $\square$

Summary

Question (contact geometry):

- What are the contact $(0, 4)$-spaces?

Answer (submanifold geometry):

- They are homogeneous hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$.

Corollary (left-invariant metrics):

- They are left-invariant Ricci solitons.

Further Plans

- submfds / groups actions on symmetric spaces;
- geometry of left-inv. metrics;
- and their combinations.

Thank you very much for your attention!