On totally geodesic surfaces in symmetric spaces and applications

Hiroshi Tamaru

Hiroshima University

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Preface (1/2)

Preface

- TG := totally geodesic.
- $\exists$ many studies on TG-submfds in symmetric spaces.

Thanks

Our studies are very influenced by

- Naitoh: a survey talk in Yuzawa 2009.

Preface (2/2)

Contents

Main Result:

§1: TG-surfaces in symmetric spaces

Its applications:

§2: TG-complex curves in Hermitian symm. sp.

§3: TG-submfds in symmetric spaces of type AI

Note

This talk is based on joint works with

- Kentaro Kimura, Takayuki Okuda (Hiroshima U.)
- Akira Kubo (Hiroshima Shudo U.)
- Katsuya Mashimo (Hosei U.)

Introduction (1/3)

Def.

$(\overline{M}, g) \supset M$ is **TG** (**totally geodesic**)

$:\Leftrightarrow$ [Second Fundamental Form] $\equiv 0$

$\Leftrightarrow$ "γ : geodesic in $M \Rightarrow \gamma$: geodesic in $\overline{M}$".

Note

We always assume that M is connected, complete.

Fundamental Problem

For a given (irreducible) symmetric space $(\overline{M}, g)$,
classify TG-submfds (up to isometric congruence).

Introduction (2/3)

Note

Classifications of TG-submfds are known only for

- $\mathrm{rk}(\overline{M}) = 1$ by Wolf (1963),
- $\mathrm{rk}(\overline{M}) = 2$ by Klein (2008–10).

Other cases would remain open.

Note

It is hence natural to study particular TG-submfds:

- cplx (in Hermitian) by Satake (1965), Ihara (1967),
- reflective by Leung (1973–79),
- symmetric TG-submfds by Naitoh (1984–86),
- cf. Chen-Nagano, Ikawa-Tasaki, Berndt-Olmos, ...

Introduction (3/3)

Our starting point

Mashimo (cf. Hashimoto et. al) studies **TG-surfaces**:

- in $\overline{M} = G/K$: symmetric space of cpt type,
- in terms of representations $\mathrm{su}(2) \to \mathfrak{g}$.

What we thought

We study TG-surfaces in $\overline{M}$ of **noncpt type**,

- the problem is essentially the same as the cpt case.

An advantage is

- one can use Iwasawa dec., solvable groups, ...

TG-surfaces in symmetric spaces (1/7)

$\overline{M} = G/K$: irreducible symmetric sp. of noncpt type.

Problem

Classify **TG-surfaces** in $\overline{M}$.

Problem (almost equivalent)

Classify **nonflat TG-surfaces** in $\overline{M}$.

Problem (almost equivalent)

Classify **nonabelian 2-dim. LTS** in $\mathfrak{p}$.

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition.
- $\mathfrak{p} \supset V$: **Lie triple system** $:\Leftrightarrow [[V, V], V] \subset V$.

TG-surfaces in symmetric spaces (2/7)

Thm. (primitive version)

There is a correspondence between

- nonabelian 2-dim. LTS in $\mathfrak{p}$,
- $X \in \mathfrak{n} \setminus \{0\}$ satisfying

  (C1) $[\theta X, X] \in \mathfrak{a}^+$;

  (C2) $\exists c > 0 : [[\theta X, X], X] = cX$.

Notation

- $\theta : \mathfrak{g} \to \mathfrak{g}$: the Cartan involution.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: the Iwasawa decomposition.
- $\mathfrak{a}^+ :=$ [positive closed Weyl chamber].

TG-surfaces in symmetric spaces (3/7)

Recall

There is a correspondence between

- nonabelian 2-dim. LTS in $\mathfrak{p}$,
- $X \in \mathfrak{n} \setminus \{0\}$ satisfying

  (C1) $[\theta X, X] \in \mathfrak{a}^+$;

  (C2) $\exists c > 0 : [[\theta X, X], X] = cX$.

Proof

$(\leftarrow)$ For such X, LTS is $\Sigma_X := \mathrm{Span}\{[\theta X, X], (1 - \theta)X\}$.

$(\rightarrow)$ It follows from the congruency of $\mathfrak{a}$, $\mathfrak{a}^+$, ... $\square$

TG-surfaces in symmetric spaces (4/7)

Simplest Case

$\overline{M} = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n) \Rightarrow$ everything is linear algebra:

- $\theta X = -{}^t X$, $\mathfrak{p} = \mathrm{Sym}^0(n, \mathbb{R})$.
- $\mathfrak{n} = \{$upper triangular$\}$,
- $\mathfrak{a} = \{$diagonal $\mid \mathrm{tr} = 0\}$,
- $\mathfrak{a}^+ = \{\mathrm{diag}(a_1, \ldots, a_n) \in \mathfrak{a} \mid a_1 \geq \cdots \geq a_n\}$.

Prop. (Fujimaru-Kubo-T.)

In $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, up to isometric congruence,

- $n = 3 \Rightarrow \exists$ exactly 2 nonflat TG-surfaces;
- $n = 4 \Rightarrow \exists$ exactly 4 nonflat TG-surfaces.

TG-surfaces in symmetric spaces (5/7)

Recall

$\exists$ correspondence between

- nonabelian 2-dim. LTS in $\mathfrak{p}$,
- $X \in \mathfrak{n} \setminus \{0\}$ satisfying (C1), (C2).

Note (general theory behind)

Mostow (1955):

- nonabelian 2-dim. LTS $\leftrightarrow$ subalgebras $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{g}$.

Jacobson-Morozov theorem:

- such subalgebras $\leftrightarrow$ nilpotent orbits in $\mathfrak{g}$.

TG-surfaces in symmetric spaces (6/7)

Thm. (sophisticated version)

$G := \mathrm{Isom}(\overline{M})$

$\Rightarrow \exists$ one-to-one correspondence between

- $\{$nonflat TG-surfaces in $\overline{M}\}/G$;
- $\{$nilpotent orbits $\mathrm{Ad}_G(X)$ of $\mathfrak{g}\}/\{\pm 1\}$.

Remark

For nilpotent orbits,

- $\{\mathrm{Ad}_{G^0}(X)\}$ is well studied.
- $\{\mathrm{Ad}_G(X)\}$ is understandable, for some $\overline{M}$.

TG-surfaces in symmetric spaces (7/7)

Cor.

Let $\overline{M} := \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.
Then $\exists$ one-to-one correspondence between

$$\{\text{nonflat TG-surface in } \overline{M}\}/\mathrm{Isom}(\overline{M})$$
$$\{\text{partition of } n\} \setminus \{[1^n]\}.$$

Ex.

- $n = 3$: $\#\{[3], [2, 1]\} = 2$.
- $n = 4$: $\#\{[4], [3, 1], [2, 2], [2, 1, 1]\} = 4$.
- $n = 5$:
  $\#\{[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1]\} = 6$.

TG-complex curves (1/6)

Topic of this section

- $\overline{M}$: irr. Hermitian symmetric space of noncpt type.
- $\overline{M} \supset M$: TG-complex curve.

  (i.e., $\dim_{\mathbb{C}} M = 1$, $\dim_{\mathbb{R}} M = 2$, J-invariant.)

Thm. (Kubo-Okuda-T.)

Let $\overline{M}$ be as above. Then

- $\# \left(\{\text{TG-cplx curves in } \overline{M}\} / \mathrm{Isom}(\overline{M}) \right) = \mathrm{rk}(\overline{M})$.

TG-complex curves (2/6)

Recall

$$\#\left(\{\text{TG-cplx curves in } \overline{M}\}/\mathrm{Isom}(\overline{M})\right) = \mathrm{rk}(\overline{M}).$$

Ex. $(\overline{M} := \mathbb{C}\mathrm{H}^n)$

(1) $\mathrm{rk}(\mathbb{C}\mathrm{H}^n) = 1$,

(2) $\exists 1$ TG-complex curve (up to $\mathrm{Isom}(\overline{M})$)
(TG-cplx submfds are $\mathbb{C}\mathrm{H}^n \supset \mathbb{C}\mathrm{H}^{n-1} \supset \cdots \supset \mathbb{C}\mathrm{H}^1$).

TG-complex curves (3/6)

Recall

$$\#\left(\{\text{TG-cplx curves in } \overline{M}\}/\mathrm{Isom}(\overline{M})\right) = \mathrm{rk}(\overline{M}).$$

Ex. ($\overline{M} := G_2^*(\mathbb{R}^n)$, $n \geq 4$)

(1) $\mathrm{rk}(G_2^*(\mathbb{R}^n)) = 2$,

(2) $\exists$ two TG-complex curves (up to $\mathrm{Isom}(\overline{M})$):

- $\overline{M} \supset G_2^*(\mathbb{R}^4) \cong \mathbb{C}\mathrm{H}^1 \times \mathbb{C}\mathrm{H}^1$: TG-complex submfd.
- TG-complex curves are:

  $\mathbb{C}\mathrm{H}^1 \times \{\mathrm{pt}\}$, and "diagonal $\mathbb{C}\mathrm{H}^1$".

TG-complex curves (4/6)

Step 1 (cf. Satake (1966), Hermann (1962))

$\overline{M}$: Hermitian, $r := \mathrm{rk}(\overline{M})$

$\Rightarrow$ $\exists M$ (TG-complex submfd) : $M \cong (\mathbb{C}\mathrm{H}^1)^r$.

$(\because)$ By taking the strongly orthogonal roots. □

Step 2 (construction)

One can construct r TG-maps $\varphi : \mathbb{C}\mathrm{H}^1 \to (\mathbb{C}\mathrm{H}^1)^r$,

- The image lives in $k \in \{1, 2, \ldots, r\}$ components.
- Note: Their sectional curvatures are different.

TG-complex curves (5/6)

Step 3 (they exhaust all)

$\overline{M} \supset M$: TG-complex curve. Then

- $\exists X \in \mathfrak{n}$: $T_o M = \mathrm{Span}\{[\theta X, X], (1-\theta)X\}$.

Since $T_o M$ is J-invariant, we have

- X is in a good position (i.e., $(1-\theta)X \in J(\mathfrak{a})$).

By looking at the root spaces, we conclude

- $M \in$ {previous examples}.

Note that, in particular, $M \subset (\mathbb{C}\mathrm{H}^1)^r$.

TG-complex curves (6/6)

Recall

$$\#\left(\{\text{TG-cplx curves in } \overline{M}\}/\mathrm{Isom}(\overline{M})\right) = \mathrm{rk}(\overline{M}).$$

Comment

$\overline{M} \supset M$: TG-complex curve. Then

- $M \subset (\mathbb{C}\mathrm{H}^1)^r$ is actually known by Satake (1966).
- We determined the isometry classes.

Key Tool (recall)

$\exists$ correspondence between

- nonabelian 2-dim. LTS in $\mathfrak{p}$,
- $X \in \mathfrak{n} \setminus \{0\}$ satisfying (C1), (C2).

TG-submfds in AI (1/7)

In this section,

- we propose a procedure to classify TG-submfds,
- and apply it to $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with $n = 3, 4$.

Procedure

(Step 1) Classify all nonflat TG-surfaces Σ in $\overline{M}$.

(This is a topic of the previous sections.)

(Step 2) For each Σ, classify nonflat TG-submfds ($\supset \Sigma$).

Key Fact

$\forall$ nonflat TG-submfd contains nonflat TG-surface.

TG-submfds in AI (2/7)

Thm. (Klein, cf. Kimura)

$\forall$ max. TG-submfd in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is congruent to

- $[\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)] \times \mathbb{R}^+$, or
- $\mathrm{SO}^0(1,2)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2))$.

Note

$$\begin{aligned}\mathbb{R}\mathrm{H}^2 &\cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \\ &\cong \mathrm{SO}^0(1,2)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2)).\end{aligned}$$

TG-submfds in AI (3/7)

Step 1 of Proof (Fujimaru-Kubo-T.)

$\exists$ exactly 2 nonflat TG-surfaces in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$:

- $\mathfrak{L}^1 := \mathrm{Span}\left\{ \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$,
- $\mathfrak{L}^2 := \mathrm{Span}\left\{ \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$.

TG-submfds in AI (4/7)

Step 2 of Proof

Consider $\mathfrak{L}^1$ (ToDo: Classify LTS $\mathfrak{L}$ ($\supsetneq \mathfrak{L}^1$)):

- $[\mathfrak{L}^1, \mathfrak{L}^1] \ni \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =: X.$
- $\mathfrak{L}$ must be ad_X-invariant.
- ad_X-weight space dec.: $\mathfrak{p} \ominus \mathfrak{L}^1 = V^1(0) \oplus V^2(\pm i)$.
- Candidates: $\mathfrak{L} = \mathfrak{L}^1 \oplus V^1(0)$ or $\mathfrak{L}^1 \oplus V^2(\pm i)$.
- The former is LTS, but the latter is not.

TG-submfds in AI (5/7)

Step 2 of Proof (Continued)

Consider $\mathfrak{L}^2$ (ToDo: Classify LTS $\mathfrak{L}$ ($\supsetneq \mathfrak{L}^2$)):

- By similar calculations, $\nexists$ such $\mathfrak{L}$.

Thm. (recall)

$\forall$ max. TG-submfd in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is congruent to

- $[\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)] \times \mathbb{R}^+$, or
- $\mathrm{SO}^0(1, 2)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2))$.

Comment

Both TG-submfds are reflective.

TG-submfds in AI (6/7)

Thm. (Kimura)

$\forall$ max. TG-submfd in $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$ is congruent to

- $[\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)] \times \mathbb{R}^{+}$,
- $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(2)$,
- $[\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)] \times [\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)] \times \mathbb{R}^{+}$,
- $\mathrm{SO}^{0}(2,2)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(2))$,
- $\mathrm{SO}^{0}(1,3)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(3))$.

Note

This would be a new result. $(\mathrm{rk}\,(\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)) = 3)$

TG-submfds in AI (7/7)

Proof

(Step 1) Recall: $\exists$ exactly 4 nonflat TG-surfaces.
(Step 2) Classify TG-submfds containing one of them.

Cor.

$\overline{M} = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with $n = 3, 4$,
$\overline{M} \supset M$: max. TG-submfd
$\Rightarrow$ M is reflective.

Question

- Why?
- What happens for $n \geq 5$?

Summary and Problems

Our Studies

- TG-surfaces in symmetric spaces.
- TG-complex curves in Hermitian symmetric spaces.
- TG-submfds in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with $n = 3, 4$.

Further Problems

- $\# \left(\{\text{nonflat TG-surfaces in } \overline{M}\}/\mathrm{Isom}(\overline{M}) \right) = \, ?$
- Classify TG-submfds in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with $n \geq 5$.
- Classify TG-submfds for other $\overline{M}$.
- Which $\overline{M}$ satisfies "maximal TG $\Rightarrow$ reflective" ?

Thank you!

Congratulations on your retirement,
and
Wishing you a future filled with happiness!!