Symmetric spaces, submanifold geometry, and solvmanifolds

Hiroshi TAMARU (田丸 博士)

Hiroshima University

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Abstract

Contents

- Introduction of our geometry group
- Introduction of researches of my seminar
  - symmetric spaces;
  - submanifold geometry;
  - solvmanifolds.

Our Group (1/3)

Members

Geometry and Topology:

- Makoto SAKUMA (P)
- Hiroshi TAMARU (P)
- Hideo DOI (AP)
- Yuya KODA (AP)
- Takayuki OKUDA (A)

Geometric and Algebraic Analysis:

- Yoshio AGAOKA (P)
- Kazuhiro SHIBUYA (AP)

Our Group (2/3)

Members

Geometry and Topology:

- Makoto SAKUMA (P): retired at March/2020
- Hiroshi TAMARU (P)
- Hideo DOI (AP): retired at March/2020
- Yuya KODA (AP)
- Takayuki OKUDA (A)

Geometric and Algebraic Analysis:

- Yoshio AGAOKA (P): retired at March/2020
- Kazuhiro SHIBUYA (AP)

Our Group (3/3)

Aspects of our group

- becomes a quit young group;
- many students in every year, usually.

Students of our group

- 1 Doctor Thesis in the last year (8 in the last 4 years);
- 7 Master Theses in the last year;
- 8 Undergraduate Theses in the last year.

Introduction (1/3)

Topics

- Symmetric Spaces;
- Submanifold Geometry;
- Solvmanifolds.

A Picture

$$
\begin{tikzcd}
 & \text{Solvmfds} \arrow[dd, leftrightarrow] \\
\text{Symm Sps} \arrow[ur] \arrow[dr] & \\
 & \text{Submfd Geom}
\end{tikzcd}
$$

Introduction (2/3)

Def.

A Riemannian manifold M is **symmetric** if

- $\exists s_x$: a "symmetry" at each $x \in M$.

Note

- A symmetry s_x is an involutive ($s_x^2 = \mathrm{id}$) isometry.
- Spheres, Projective spaces, Hyperbolic spaces, Grassmannians, … are examples of symmetric spaces.
- Every symmetric space satisfies $\nabla R \equiv 0$.
- $\exists$ many structure theories.

Introduction (3/3)

Contents

- Topic 1: Application of Symmetric Spaces to Solvmfds
- Topic 2: Application of Symmetric Spaces to Submfd Geom
- Topic 3: Interplay between Solvmfds and Submfd Geom

A Picture

$$\begin{tikzcd}
& \text{Solvmfds} \arrow[dd, leftrightarrow] \\
\text{Symm Sps} \arrow[ur] \arrow[dr] & \\
& \text{Submfd Geom}
\end{tikzcd}$$

Topic 1: Solvmanifolds (1/4)

Def.

$(S, \langle , \rangle)$ is a **solvmanifold** if

- S is a solvable Lie group;
- $\langle , \rangle$ is a left-invariant Riemannian metric on S.

Note

- Solvmfds provide "nice" examples; Einstein, Ricci soliton, ...
- Conjecture: noncompact homogeneous Einstein $\Rightarrow$ solvmfd.

General Problem

- Construct $(S, \langle , \rangle)$ which are Einstein or Ricci soliton.

Topic 1: Solvmanifolds (2/4)

Thm. (T.: Math. Ann. (2011))

Let

- $M = G/K$: any symmetric space of noncompact type,
- $G \supset Q_\Phi$: any "parabolic" subgroup,
- $Q_\Phi \supset S_\Phi$: its "solvable part".

Then we have

- $S_\Phi \cong (S_\Phi).o$ ($\subset M$) is always an Einstein solvmfd.

Note

- This theorem provides a lot of (new) examples.

Topic 1: Solvmanifolds (3/4)

Fact

- $G > Q_\Phi$ is **parabolic** if it is "sufficiently large";
- $\exists$ nice decomposition $Q_\Phi = M_\Phi S_\Phi$, called **Langlands decomposition**, where S_Φ is solvable;
- They are controlled by the "root system".

Ex.

- $\mathfrak{sl}(3,\mathbb{R}) \supset \mathfrak{q}_\Phi = \left\{ \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array} \right) \mid \mathrm{tr} = 0 \right\}$ is parabolic;
- $\mathfrak{q}_\Phi = \mathfrak{sl}(2,\mathbb{R}) \oplus \left\{ \left(\begin{array}{cc|c} a & 0 & * \\ 0 & a & * \\ \hline 0 & 0 & -2a \end{array} \right) \right\}$ is our decomposition.

Topic 1: Solvmanifolds (4/4)

Our Projects

- Take some orbit $H.p \subset M = G/K$ (where $H < G$), and study whether it is Einstein or Ricci soliton.

Recent Results

- (Hamada-Hoshikawa-T.: J. Geom. (2012))
  Curvature properties of some hypersurfaces in $\mathbb{C}\mathrm{H}^n$.
- (Hashinaga-Kubo-T.: Tohoku Math. J. (2016))
  Ricci soliton hypersurfaces in $\mathbb{C}\mathrm{H}^n$.
- (Cho-Hashinaga-Kubo-Taketomi-T.: J. Geom. Phy.)
  Ricci soliton contact hypersurfaces in $G_2^*(\mathbb{R}^n)$.

Topic 2: Submanifold Geometry (1/4)

General Problem

- For $M = G/K$: symmetric spaces, construct/classify "nice" isometric actions $H \curvearrowright M$.

Def.

$H \curvearrowright M$ is of **cohomogeneity one** if

- regular orbits have codimension one.

Note

- $\exists$ many studies for **compact** symmetric spaces $M = G/K$.
- We are interested in **noncompact** case.

Topic 2: Submanifold Geometry (2/4)

Thm. (Berndt-T.: JDG (2003), Crelle (2013))

Let

- M : irreducible symm. sp. of **noncompact** type;
- $H \curvearrowright M$: **cohomogeneity one** with H being connected.

Then it satisfies one of the following:

(K) $\exists 1$ singular orbit;

(A) $\nexists$ singular orbit, and $\exists 1$ minimal orbit;

(N) $\nexists$ singular orbit, and all orbits are congruent.

Note

- Actions of type (A), (N) are classified completely.

Topic 2: Submanifold Geometry (3/4)

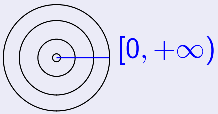
Recall

(K) $\exists 1$ singular orbit;

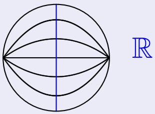
(A) $\nexists$ singular orbit, and $\exists 1$ minimal orbit;

(N) $\nexists$ singular orbit, and all orbits are congruent.

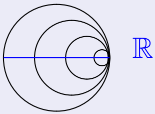
Picture



type (K)



type (A)



type (N)

Topic 2: Submanifold Geometry (4/4)

Recent Results

- (Berndt-DiazRamos-T.: JDG (2010))
  Hyperpolar actions without singular orbits.
- (Kubo-T.: Geom. Dedicata (2013))
  Study the "congruency" of orbits.
- (Fujimaru-Kubo-T.: Springer Proc. Math. Stat. (2014))
  Totally geodesic surfaces in "some" symmetric spaces.
- (Gondo-T.: ongoing)
  $H \curvearrowright M$: cohomogeneity one with H being disconnected.

Topic 3: Solvmanifolds vs Submanifold Geometry (1/4)

General Problem

For a given (solvable) Lie group G,

- study whether G admits "nice" left-inv. metrics or not.

Note

The above problem is very difficult in general, because

- $\{\langle , \rangle : \text{inner product on } \mathfrak{g}\} \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, if $\dim \mathfrak{g} = n$.

Topic 3: Solvmanifolds vs Submanifold Geometry (2/4)

Note

Consider $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) < GL(n, \mathbb{R})$.

- If $\langle , \rangle' \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$,
  then $\langle , \rangle$ and $\langle , \rangle'$ have the same curvature properties.

Such studies are begun in

- (Kodama-Takahara-T.: Manuscripta Math. (2011))
  Study $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$.
- (Hashinaga-T.-Terada: J. Math. Soc. Japan (2016))
  A generalization of "Milnor frames".

Topic 3: Solvmanifolds vs Submanifold Geometry (3/4)

Our Expectation

- The submfd $[\langle , \rangle]$ should reflect the geometry of $\langle , \rangle$...

Thm. (Hashinaga-T.: Internat. J. Math. (2017))

Let $\mathfrak{g}$ be a 3-dim. solvable Lie algebra. Then

- $\langle , \rangle$ is (algebraic) Ricci soliton $\Leftrightarrow$ $[\langle , \rangle]$ is minimal.

Topic 3: Solvmanifolds vs Submanifold Geometry (4/4)

Our Expectation

- A "nice" $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Recent Results

- (Hashinaga: Hiroshima Math. J. (2014))
  Study on $[\langle , \rangle]$ for 4-dim. solvable cases.
- (Kubo-Onda-Taketomi-T.: Hiroshima Math. J. (2016))
  Study on the "pseudo-Riemannian" version.
- (Taketomi-T.: Transf. Groups (2017))
  Study the "nonexistence" in view of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-actions.
- (Taketomi: Hiroshima Math. J. (in press))
  Give a sufficient condition for $\langle , \rangle$ to be Ricci soliton.

Summary (1/2)

A Picture for our studies

$$
\begin{tikzcd}
 & \text{Solvmfds} \arrow[dd, leftrightarrow] \\
\text{Symm Sps} \arrow[ur] \arrow[dr] & \\
 & \text{Submfd Geom}
\end{tikzcd}
$$

Comments

- Symmetric spaces could be “classical” subjects, but still have many interesting applications.

Summary (2/2)

A Picture for our studies (with additional info)

$$
\begin{tikzcd}
 & \text{Solvmfds} \arrow[d, leftrightarrow] \\
\text{Symm Sps} \arrow[ur] \arrow[r] \arrow[d, "\text{``discretization''}"'] & \text{Submfd Geom} \\
\text{``Quandles''} &
\end{tikzcd}
$$

Thank you very much!