Left-invariant metrics and submanifold geometry

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Abstract (1/2)

Background

- Left-invariant (Riemannian) metrics on Lie group:
- $\exists$ many "nice" such metrics, e.g., Einstein, Ricci soliton, ...

Our Framework

- A left-invariant metric $\langle , \rangle$ defines a submanifold $[\langle , \rangle]$, in some noncompact Riemannian symmetric space $\widetilde{\mathfrak{M}}$.
- Expectation: a "nice" metric corresponds to a "nice" submfd.

Abstract

- The above expectation is true for several cases.

Abstract (2/2)

Contents

Intro (1/5)

Our Framework (recall)

- $\langle , \rangle$: a left-inv. metric on $G \longrightarrow [\langle , \rangle]$: a submfd in $\widetilde{\mathfrak{M}}$.

Basic Fact

$\exists$ 1-1 correspondence between

- a left-inv. (Riemannian) metric on G,
- a (positive definite) inner product $\langle , \rangle$ on $\mathfrak{g} := \mathrm{Lie}(G)$.

Def. (the ambient space)

The **space of left-inv. metrics** on G is defined by

- $\widetilde{\mathfrak{M}} := \{ \langle , \rangle$: an inner product on $\mathfrak{g} \}$.

Intro (2/5)

Recall

- $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\}$.

Prop. (well-known)

If $\dim G = n$, then

- $\widetilde{\mathfrak{M}} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$

  where $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \widetilde{\mathfrak{M}}$ by $g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$;
- Hence $\widetilde{\mathfrak{M}}$ is a noncompact Riemannian symmetric space.

Note

- Finding a nice left-inv. metric on G

  $\leftrightarrow$ Finding a nice point on $\widetilde{\mathfrak{M}}$...?

...but every point on $\widetilde{\mathfrak{M}}$ looks the same.

Intro (3/5)

Def.

Let $\langle,\rangle_1, \langle,\rangle_2 \in \widetilde{\mathfrak{M}}$. We say $\langle,\rangle_1 \sim \langle,\rangle_2$ (**isometric up to scalar**)

$:\Leftrightarrow \ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \exists c > 0 : c \cdot \varphi.\langle,\rangle_1 = \langle,\rangle_2$.

Note

$\langle,\rangle_1 \sim \langle,\rangle_2$

$\Rightarrow$ all Riemannian geometric properties of them are the same.

Def. (the submfd)

We define the **corresponding submfd** of $\langle,\rangle$ by

- $\widetilde{\mathfrak{M}} \supset [\langle,\rangle] :=$ "the isometry and scaling class of $\langle,\rangle$"
  $= \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle,\rangle$.

Intro (4/5)

We got:

- $\widetilde{\mathfrak{M}} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$: a noncpt Riem. symmetric space.
- $\widetilde{\mathfrak{M}} \supset [\langle , \rangle]$: a homogeneous submanifold.

Question

- $\langle , \rangle$ is a "nice" left-inv. metric
  $\leftrightarrow$ $[\langle , \rangle]$ is a "nice" submfd $[\langle , \rangle]$ in $\widetilde{\mathfrak{M}}$...?

Note

- $[\langle , \rangle_1]$ and $[\langle , \rangle_2]$ are different in general.
  (Even the dimensions can be different)

Intro (5/5)

Expectation

- Some nice interplay between
  - geometry of left-inv. metrics on Lie groups, and
  - submfd geometry (in noncompact symmetric spaces).

We hope that

- characterize nice left-inv. metrics in terms of submfds...
- obtain nice submfds (isom. actions) from left-inv. metrics...

Preliminaries (1/5)

Note

We here recall general facts on

- $M = G/K$: a noncpt Riemannian symmetric space;
- $H \curvearrowright M$: an isometric action.

Def.

For $H \curvearrowright M$,

- an orbit $H.p$ is **regular** if it is of maximal dimension;
- other orbits are **singular**;
- the **cohomogeneity** is the codimension of regular orbits.

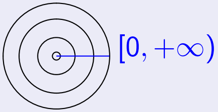
Preliminaries (2/5)

Ex.

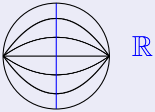
Let $\mathrm{SL}_2(\mathbb{R}) = KAN$ be the Iwasawa decomposition.
Then K, A, $N \curvearrowright \mathbb{R}\mathrm{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ are of cohom. one, and

- (K) $\exists 1$ singular orbit;
- (A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;
- (N) $\nexists$ singular orbit, all orbits are congruent to each other.

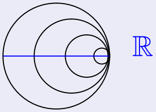
Picture



type (K)



type (A)



type (N)

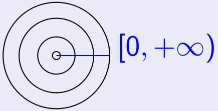
Preliminaries (3/5)

Thm. (Berndt-T. 2003, 2013)

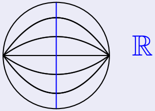
M : an irreducible symmetric space of noncpt type;
$H \curvearrowright M$: of cohomo. one (with H connected).
Then it satisfies one of the following:

- (K) $\exists 1$ singular orbit;
- (A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;
- (N) $\nexists$ singular orbit, all orbits are congruent to each other.

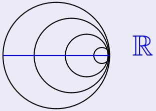
Picture



type (K)



type (A)



type (N)

Preliminaries (4/5)

Note

The above picture is just for a reference, since

- $\widetilde{\mathfrak{M}} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is not irreducible;
- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is not connected in general;
- the cohomogeneity of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ can be high.

Note ($\langle , \rangle$ vs $[\langle , \rangle]$)

- $[\langle , \rangle]$ is small
  "$\Leftrightarrow$" the stabilizer $\mathrm{Aut}(\mathfrak{g})_{\langle , \rangle}$ is large
  "$\Leftrightarrow$" $\langle , \rangle$ has a large symmetry.
- Recall that $G \ltimes \mathrm{Aut}(\mathfrak{g})_{\langle , \rangle} \subset \mathrm{Isom}(G, \langle , \rangle)$.

Preliminaries (5/5)

Ex. (typical example)

Let $\mathfrak{g}$ be compact simple, and $\langle , \rangle_{\mathrm{K}}$ the Killing metric. Then

- $\langle , \rangle_{\mathrm{K}}$ is Einstein, $\mathrm{Sec} \geq 0$.
- $[\langle , \rangle_{\mathrm{K}}] = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle_{\mathrm{K}} = \mathbb{R}^{\times}.\langle , \rangle_{\mathrm{K}} \cong \mathbb{R}$: geodesic.

  (since it is bi-inv.; other orbits have larger dimensions)

Note

- Recall: $[\langle , \rangle]$ is small "$\Leftrightarrow$" $\langle , \rangle$ has a large symmetry.
- $[\langle , \rangle]$ also contains some more information of $\langle , \rangle$...

  (Even if $\mathrm{Aut}(\mathfrak{g})_{\langle , \rangle}$ are the same, $[\langle , \rangle]$ can be different.)

Case 1: Low-dim. solvable Lie groups (1/6)

Recall (our expectation)

- A "nice" metric $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Preliminaries (Lauret 2003, essentially)

$\mathfrak{g} = \mathbb{R}^n, \mathfrak{g}_{\mathbb{RH}^n}, \mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ iff

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: transitive.
  (i.e., left-inv. metric is unique up to isometry and scaling)

Note: $\mathrm{GL}_n(\mathbb{R}) \supset \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is parabolic in these cases.

Preliminaries (Kodama-Takahara-T. 2011)

$\exists$ several Lie algebras (including 3-dim. solvable) :

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ has cohomogeneity 1.

Case 1: Low-dim. solvable Lie groups (2/6)

Thm. (Hashinaga-T. 2017)

Let $\mathfrak{g}$ be a 3-dim. solvable Lie algebra. Then

- $\langle , \rangle$ is ARS $\Leftrightarrow$ $[\langle , \rangle]$ is minimal.

Def.

$(\mathfrak{g}, \langle , \rangle)$ is an **algebraic Ricci soliton** (**ARS**) if

- $\exists c \in \mathbb{R}$, $D \in \mathrm{Der}(\mathfrak{g})$: $\mathrm{Ric} = c \cdot \mathrm{id} + D$.

Idea of Proof

Study them case-by-case... In fact,

- One knows the classification of 3-dim. solvable Lie algebras.
- Classification of ARS has been known (Lauret 2011).

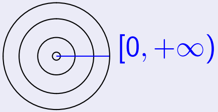
Case 1: Low-dim. solvable Lie groups (3/6)

More on 3-dim. case

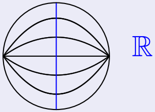
Let $\mathfrak{g}$ be a 3-dim. solvable Lie algebra with $[\langle , \rangle] \neq \widetilde{\mathfrak{M}}$.

- $\exists$ 3 families of such $\mathfrak{g}$.
- $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohom. one.
- $H := (\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ is of type (K), (A), or (N).

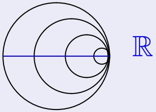
Picture (recall)



type (K)



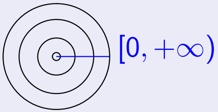
type (A)



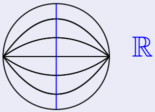
type (N)

Case 1: Low-dim. solvable Lie groups (4/6)

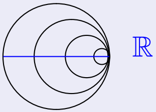
Picture (recall)



type (K)



type (A)



type (N)

Note

For $\mathfrak{g}$: 3-dim. solvable with $[\langle , \rangle] \neq \widetilde{\mathfrak{M}}$, we have

- (K) the unique singular orbit is precisely ARS (Einstein);
- (A) the unique minimal orbit is precisely ARS;
- (N) $\nexists$ ARS.

Case 1: Low-dim. solvable Lie groups (5/6)

Thm. (recall, 3-dim. solvable case)

- $\langle , \rangle$ is ARS $\Leftrightarrow$ $[\langle , \rangle]$ is minimal.

Note (for higher dim. case; good news)

We know that

- $\exists$ several $\mathfrak{g}$ satisfying the above "$\Leftrightarrow$".

Note (for 4-dim. case; bad news)

Hashinaga (2014) proved that

- $\exists\ \mathfrak{g}$: the above "$\Leftarrow$" does not hold.
- $\exists\ \mathfrak{g}$: the above "$\Rightarrow$" does not hold.

Case 1: Low-dim. solvable Lie groups (6/6)

Recall

Our expectation is:

- a "nice" metric $\langle , \rangle$ corresponds to a "nice" submfd $[\langle , \rangle]$.

Note

From our studies,

- $\exists$ a nice correspondence for 3-dim. solvable case;
- however, the minimality of $[\langle , \rangle]$ is not enough in general.
  (we need more global information?)

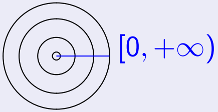
Case 2: Some general cases (1/5)

Recall

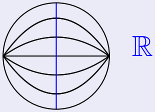
$\mathfrak{g}$: 3-dim. solvable, $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ of type (K)

$\Rightarrow$ $[\langle,\rangle]$ is a singular orbit iff $\langle,\rangle$ is ARS.

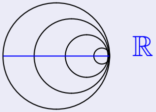
Picture (recall)



type (K)



type (A)



type (N)

Case 2: Some general cases (2/5)

Claim

- For actions of type (K), we can generalize it.
- Note: its singular orbit is an isolated orbit.

Thm. (Taketomi 2016)

Let $\mathfrak{g}$ be **any** Lie algebra, and $\langle , \rangle$ on $\mathfrak{g}$. Then

- $[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is an isolated orbit $\Rightarrow$ $\langle , \rangle$ is ARS.

Case 2: Some general cases (3/5)

Idea of Proof

- Assume $[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is isolated.
- Then $\forall$ normal vector at $\langle , \rangle$ is not fixed by $\mathrm{Aut}(\mathfrak{g})_{\langle , \rangle}$.
- This shows $\mathrm{ric}^{\perp}_{\langle , \rangle} = 0$.

  (Hence $\mathrm{ric}_{\langle , \rangle}$ is tangential to $[\langle , \rangle]$)

Note

- The converse does not hold in general.
- Recall: isolated $\Rightarrow$ minimal.

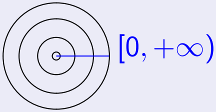
Case 2: Some general cases (4/5)

Recall

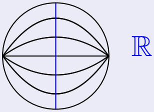
$\mathfrak{g}$: 3-dim. solvable, $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ of type (N)

$\Rightarrow$ $\nexists\langle , \rangle$ which is ARS.

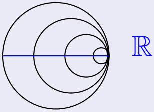
Picture (recall)



type (K)



type (A)



type (N)

Case 2: Some general cases (5/5)

Conjecture

All orbits of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are congruent to each other (and $\neq \widetilde{\mathfrak{M}}$)

$\Rightarrow \nexists\langle , \rangle$ which is ARS.

Note

The assumption of the conjecture means that

- all orbits are looks the same (i.e., no distinguished orbit)...

Prop. (Taketomi-T., to appear)

$\forall n \geq 3$, $\exists \mathfrak{g}$: Lie algebra of dim. n :

- all orbits of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are congruent to each other, and
- $\nexists\langle , \rangle$ on $\mathfrak{g}$ which is ARS.

Summary (1/4)

Our Framework/Expectation

- $\mathfrak{g}$ defines an action $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$;
- $\langle , \rangle$ on $\mathfrak{g}$ defines a submfd $[\langle , \rangle] \subset \widetilde{\mathfrak{M}}$;
- Does a "nice" $\langle , \rangle$ correspond to a "nice" submfd $[\langle , \rangle]$...?

Our Results

For 3-dim. solvable case,

- $\exists$ a very nice correspondence.

For 4-dim. solvable case,

- not so nice as the 3-dim. case...

For general cases,

- $\exists$ a sufficient condition for $\langle , \rangle$ to be ARS;
- $\exists$ a conjecture of .an obstruction for the existence of ARS.

Summary (2/4)

Related Topics

Taketomi 2015:

- Constructed examples of $\mathfrak{g}$ s.t. $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: hyperpolar.

Kubo-Onda-Taketomi-T. 2016:

- A study on left-inv. pseudo-Riem. metrics, in terms of $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p, q)$.

In progress:

- A similar framework for left-inv. symplectic structures...

Summary (3/4)

Problems

Find special classes: can we classify

- $\mathfrak{g}$ such that $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is special

  (e.g., cohomogeneity one, hyperpolar, polar, ...)
- $(\mathfrak{g}, \langle , \rangle)$ such that $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is special

  (e.g., totally geodesic, minimal, austere, ...)

Problems

Construct invariants (or obstructions) of Lie algebras:

- properties of $[\langle , \rangle]$ are invariants of an inner product.
- properties of $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ are invariants of $\mathfrak{g}$.

  — it would be very nice if one could get an obstruction for $\mathfrak{g}$ to admit nice metrics.

Summary (4/4)

Thank you very much!

Ref. (just for our papers)

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