

# Left-invariant metrics and submanifold geometry

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The 7th International Workshop on Differential Geometry  
“Dedicated to Professor Tian Gang for his 60th birthday”  
Karatsu, 24/March/2017

# Abstract (1/2)

## Background

- Left-invariant (Riemannian) metrics on Lie group:
- $\exists$  many “nice” such metrics, e.g., Einstein, Ricci soliton, ...

## Our Framework

- A left-invariant metric  $\langle \cdot, \cdot \rangle$  defines a submanifold  $[\langle \cdot, \cdot \rangle] \subset \widetilde{\mathfrak{M}}$   
( $\widetilde{\mathfrak{M}}$  is some noncpt Riemannian symmetric space)
- Expectation: a “nice” metric corresponds to a “nice” submfd.

## Abstract

- The above expectation is true for several cases.

# Abstract (2/2)

## Contents

- Introduction (to our framework)
- Case 1: Easy cases
- Case 2: Low-dim. solvable Lie groups
- Case 3: Some general cases
- Summary

# Intro (1/6)

## Our Framework (recall)

- $\langle \cdot, \cdot \rangle$  : a left-inv. metric on  $G \longrightarrow [\langle \cdot, \cdot \rangle]$  : a submfd in  $\widetilde{\mathfrak{M}}$ .

## Basic Fact

$\exists$  1-1 correspondence between

- a left-inv. (Riemannian) metric on  $G$ ,
- a (positive definite) inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g} := \text{Lie}(G)$ .

## Def. (the ambient space)

The **space of left-inv. metrics** on  $G$  is defined by

- $\widetilde{\mathfrak{M}} := \{ \langle \cdot, \cdot \rangle : \text{an inner product on } \mathfrak{g} \}$ .

# Intro (2/6)

## Recall

- $\widetilde{\mathfrak{M}} := \{ \langle \cdot, \cdot \rangle : \text{an inner product on } \mathfrak{g} \}$ .

## Prop. (well-known)

If  $\dim G = n$ , then

- $GL_n(\mathbb{R}) \curvearrowright \widetilde{\mathfrak{M}}$  by  $g \cdot \langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$ ;
- $\widetilde{\mathfrak{M}} \cong GL_n(\mathbb{R})/O(n)$ ;
- $\widetilde{\mathfrak{M}}$  is a noncompact Riemannian symmetric space.

## Note

- Question: a nice left-inv. metric  $\leftrightarrow$  a nice point on  $\widetilde{\mathfrak{M}}$  ...?
- This is not good, since every point on  $\widetilde{\mathfrak{M}}$  looks the same.

# Intro (3/6)

## Def.

$\langle, \rangle_1 \sim \langle, \rangle_2$  (**isometric up to scalar**) if

- $\exists \varphi \in \text{Aut}(\mathfrak{g}), \exists c > 0 : c\varphi.\langle, \rangle_1 = \langle, \rangle_2.$

## Note

$\langle, \rangle_1 \sim \langle, \rangle_2$

$\Rightarrow$  all Riemannian geometric properties of them are the same.

## Def. (the submfd)

We define the **corresponding submfd** of  $\langle, \rangle$  by

- $[\langle, \rangle] :=$  “the isometry and scaling class of  $\langle, \rangle$ ” ( $\subset \widetilde{\mathfrak{M}}$ ).

## Intro (4/6)

### Recall

- $\langle \cdot, \cdot \rangle_1 \sim \langle \cdot, \cdot \rangle_2 \Leftrightarrow \exists \varphi \in \text{Aut}(\mathfrak{g}), \exists c > 0 : c\varphi.\langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_2$ .
- $[\langle \cdot, \cdot \rangle] :=$  “the isometry and scaling class of  $\langle \cdot, \cdot \rangle$ ” ( $\subset \widetilde{\mathfrak{M}}$ ).

### Prop. (cf. Kodama-Takahara-T. 2011)

- $[\langle \cdot, \cdot \rangle] = \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle$ ,  
(Recall:  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \subset \text{GL}_n(\mathbb{R}) \curvearrowright \widetilde{\mathfrak{M}} = \text{GL}_n(\mathbb{R})/O(n)$ )

### We got:

- $\widetilde{\mathfrak{M}} \cong \text{GL}_n(\mathbb{R})/O(n)$  : a noncpt Riem. symmetric space.
- $\widetilde{\mathfrak{M}} \supset [\langle \cdot, \cdot \rangle]$  : a homogeneous submanifold.

# Intro (5/6)

## Note

- Question: a nice left-inv. metric  $\leftrightarrow$  a nice submfd  $[\langle, \rangle]$  in  $\widetilde{\mathfrak{M}}$  ?
- Note:  $[\langle, \rangle_1]$  and  $[\langle, \rangle_2]$  are different in general.

## Note (why this framework would be interesting)

- This connects two different areas:
  - “geometry of left-inv. metrics” vs “submfd geometry”.
- Both have been studied actively in these years.



# Intro (6/6)

## Note (Both have been studied actively)

Geometry of left-inv. metrics:

- Milnor (Adv. Math. 1976), Heber (Invent. Math. 1998), Lauret (Math. Ann. 2001, Crelle 2011), T. (Math. Ann. 2011), and many others ...

Homog. submfds (isometric actions) in noncpt symmetric spaces:

- Cartan (1930's), Berndt-Brück (Crelle 2001), Berndt-T. (JDG 2003, Crelle 2013), Berndt-DíazRamos-T. (JDG 2010), and many others ...

## We hope to

- characterize nice left-inv. metrics in terms of submfds...
- obtain nice submfds (isom. actions) from left-inv. metrics...

## Case 1: Easy cases (1/5)

### Recall (our expectation)

- A “nice” metric  $\langle, \rangle$  corresponds to a “nice” submfd  $[\langle, \rangle]$ .

### Ex. (Sec $\equiv 0$ )

For  $\mathfrak{g} = \mathbb{R}^n$  (abelian),

- $\forall \langle, \rangle$  is flat.
- $\forall \langle, \rangle$ , one has  $[\langle, \rangle] = \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle = \widetilde{\mathfrak{M}}$ .  
(i.e.,  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is transitive)

## Case 1: Easy cases (2/5)

Ex. (Nice metric; including the case  $\text{Sec} \equiv c > 0$ )

Let  $\mathfrak{g}$  be compact simple, and  $\langle, \rangle_K$  the Killing metric. Then

- $\langle, \rangle_K$  is Einstein,  $\text{Sec} \geq 0$ .
- $[\langle, \rangle_K] = \mathbb{R}^\times \text{Aut}(\mathfrak{g}) \cdot \langle, \rangle_K = \mathbb{R}^\times \cdot \langle, \rangle_K \cong \mathbb{R}$  : geodesic.  
(since it is bi-inv.; other orbits have larger dimensions)

### Note

$[\langle, \rangle]$  contains information of

- how large the symmetry of  $\langle, \rangle$  is ...  
(large symmetry  $\leftrightarrow \text{Aut}(\mathfrak{g})_{\langle, \rangle}$  is large  $\leftrightarrow [\langle, \rangle]$  is small)
- also the “position” of  $\langle, \rangle$ .

## Case 1: Easy cases (3/5)

Prop. (Nice action; Lauret 2003, Kodama-Takahara-T. 2011)

The action  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is transitive iff

- $\mathfrak{g} = \mathbb{R}^n$ ,  $\mathfrak{g}_{\mathbb{R}H^n}$ , or  $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ .

Ex. ( $\text{Sec} \equiv c < 0$ ; Milnor 1976, Lauret 2003, KTT 2011)

For  $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}H^n} = \text{span}\{e_1, \dots, e_n\}$  with  $[e_1, e_j] = e_j$ ,

- $\forall \langle, \rangle$  is const. negative sectional curvature;
- $\forall \langle, \rangle$ , one has  $[\langle, \rangle] = \widetilde{\mathfrak{M}}$ .

## Case 1: Easy cases (4/5)

### Recall

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is transitive  $\Leftrightarrow \mathfrak{g} = \mathbb{R}^n, \mathfrak{g}_{\mathbb{R}H^n}, \mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ .

### Note

- $\mathfrak{h}^3$  : 3-dim. Heisenberg;
- $\forall \langle, \rangle$  on  $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$  is a non-Einstein (algebraic) Ricci soliton.

### Def.

$(\mathfrak{g}, \langle, \rangle)$  is an **algebraic Ricci soliton (ARS)** if

- $\exists c \in \mathbb{R}, \exists D \in \text{Der}(\mathfrak{g}) : \text{Ric}_{\langle, \rangle} = c \cdot \text{id} + D$ .

## Case 1: Easy cases (5/5)

### Recall

- $(g, \langle, \rangle) : \mathbf{ARS} \Leftrightarrow \exists c \in \mathbb{R}, \exists D \in \text{Der}(g) : \text{Ric}_{\langle, \rangle} = c \cdot \text{id} + D.$

### Fact (Lauret 2001, 2011, Jablonski 2014)

ARS is “almost” equivalent to Ricci soliton;

- $(g, \langle, \rangle) : \text{ARS}$   
 $\Rightarrow$  the corresponding simply-conn.  $(G, \langle, \rangle)$  is Ricci soliton.
- $(G, \langle, \rangle)$  is left-inv. Ricci soliton  
 $\Rightarrow$  it is “isometric” to ARS (as Riem. mfd).

### Note

A (complete) Riemannian manifold is said to be **Ricci soliton** if

- $\exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(M) : \text{Ric}_g = c \cdot \text{id} + \mathcal{L}_X g.$

## Case 2: Low-dim. solvable Lie groups (1/6)

### Recall (our expectation)

- A “nice” metric  $\langle, \rangle$  corresponds to a “nice” submfd  $[\langle, \rangle]$ .

### Thm. (Hashinaga-T.)

Let  $\mathfrak{g}$  be a 3-dim. solvable Lie algebra. Then

- $\langle, \rangle$  is ARS  $\Leftrightarrow [\langle, \rangle]$  is minimal.

### Idea of Proof

Study them case-by-case... In fact,

- One knows the classification of 3-dim. solvable Lie algebras.
- We can see  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is of cohomogeneity at most one.

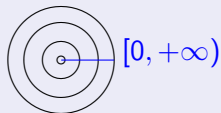
## Case 2: Low-dim. solvable Lie groups (2/6)

### More on 3-dim. case

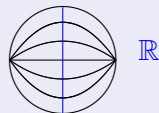
Let  $\mathfrak{g}$  be a 3-dim. solvable Lie algebra with  $[\langle, \rangle] \neq \widetilde{\mathfrak{M}}$ .

- $\exists$  3 families of such  $\mathfrak{g}$ .
- Consider  $H := (\mathbb{R}^\times \text{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ .
- Then, for each  $\mathfrak{g}$ ,  $H \curvearrowright \widetilde{\mathfrak{M}}$  satisfies one of the following:

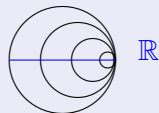
### Picture



type (K)



type (A)

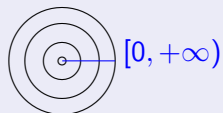


type (N)

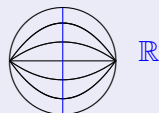


## Case 2: Low-dim. solvable Lie groups (3/6)

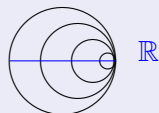
### Picture (recall)



type (K)



type (A)



type (N)

### Note

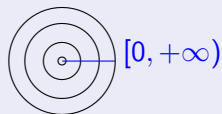
(K)  $\exists 1$  singular orbit;

(A)  $\nexists$  singular orbit,  $\exists 1$  minimal orbit;

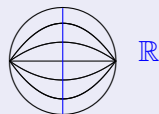
(N)  $\nexists$  singular orbit, all orbits are congruent to each other.

## Case 2: Low-dim. solvable Lie groups (4/6)

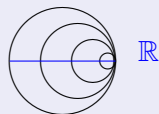
### Picture (recall)



type (K)



type (A)



type (N)

### Note

Let  $SL(2, \mathbb{R}) = KAN$  be the Iwasawa dec. Then

- $K, A, N \curvearrowright \mathbb{RH}^2$  are of type (K), (A), (N), respectively.

## Case 2: Low-dim. solvable Lie groups (5/6)

Thm. (recall, 3-dim. solvable case)

- $\langle, \rangle$  is ARS  $\Leftrightarrow$   $[\langle, \rangle]$  is minimal.

Note (for higher dim. case; good news)

We know that

- $\exists$  several  $\mathfrak{g}$  satisfying the above " $\Leftrightarrow$ ".

Note (for 4-dim. case; bad news)

Hashinaga (2014) proved that

- $\exists \mathfrak{g}$  : the above " $\Leftarrow$ " does not hold.
- $\exists \mathfrak{g}$  : the above " $\Rightarrow$ " does not hold.

## Case 2: Low-dim. solvable Lie groups (6/6)

### Recall

Our expectation is:

- a “nice” metric  $\langle, \rangle$  corresponds to a “nice” submfd  $[\langle, \rangle]$ .

### Note

Our studies imply that

- the minimality of  $[\langle, \rangle]$  is not enough, in general.

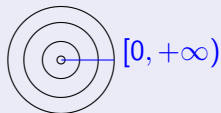
## Case 3: Some general cases (1/5)

### Recall

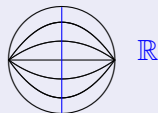
$\mathfrak{g}$  : 3-dim. solvable,  $(\mathbb{R}^\times \text{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$  of type (K)

$\Rightarrow [\langle, \rangle]$  is a singular orbit iff  $\langle, \rangle$  is ARS.

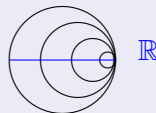
### Picture (recall)



type (K)



type (A)



type (N)

## Case 3: Some general cases (2/5)

### Claim

- For actions of type  $(K)$ , we can generalize it.

### Thm. (Taketomi)

Let  $\mathfrak{g}$  be any Lie algebra, and assume that

- $[\langle, \rangle] := \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$  is an isolated orbit of  $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ .

Then  $\langle, \rangle$  is ARS.

### Cor.

Let  $\mathfrak{g}$  be any Lie algebra, and assume that

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is a cohomogeneity one action of type  $(K)$ ,
- $[\langle, \rangle]$  is a singular orbit.

Then  $\langle, \rangle$  is ARS (since it is isolated).

## Case 3: Some general cases (3/5)

### Recall (Assumption by Taketomi)

- $[\langle, \rangle] := \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$  is an isolated orbit of  $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ .

### Idea of Proof $+\alpha$

- Fact: isolated  $\Leftrightarrow$  an invariant normal vector must be zero.
- Therefore, isolated  $\Rightarrow$  minimal.  
(since the mean curv. vector is an invariant normal vector)
- Similarly, isolated  $\Rightarrow \text{ric}_p^\perp = 0$  ( $\forall p \in [\langle, \rangle]$ ).  
(Hence ric is tangential to  $[\langle, \rangle]$  at any point)

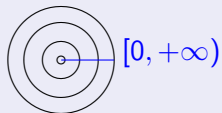
## Case 3: Some general cases (4/5)

### Recall

$\mathfrak{g}$  : 3-dim. solvable,  $(\mathbb{R}^\times \text{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$  of type (N)

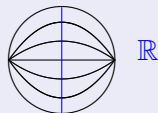
$\Rightarrow \mathcal{A}\langle, \rangle$  which is ARS.

### Picture (recall)



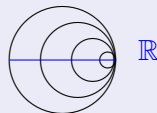
$[0, +\infty)$

type (K)



$\mathbb{R}$

type (A)



$\mathbb{R}$

type (N)



## Case 3: Some general cases (5/5)

### Conjecture (Taketomi-T., to appear)

All orbits of  $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$  are congruent to each other

$\Rightarrow \exists \langle, \rangle$  which is ARS.

### Note

The assumption of the conjecture means that

- all orbits are looks the same (i.e., no distinguished orbit)...

### Prop. (Taketomi-T., to appear)

$\forall n \geq 3, \exists \mathfrak{g} : \text{Lie algebra of dim. } n :$

- all orbits of  $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$  are congruent to each other, and
- $\exists \langle, \rangle$  on  $\mathfrak{g}$  which is ARS.

# Summary (1/4)

## Our Framework/Expectation

- $\langle, \rangle$  on  $\mathfrak{g}$  defines a submfd  $[\langle, \rangle] \subset \widetilde{\mathfrak{M}}$ .
- Does a “nice”  $\langle, \rangle$  corresponds to a “nice” submfd  $[\langle, \rangle]$ ...?

## Our Results

For 3-dim. solvable case,

- there is a very nice correspondence.

For 4-dim. solvable case,

- not so nice as the 3-dim. case...

For general cases,

- Taketomi obtained a sufficient condition for  $\langle, \rangle$  to be ARS;
- We conjecture an obstruction for the existence of ARS.

## Summary (2/4)

### Related Topics

- (Taketomi 2015)  
Constructed examples of  $\mathfrak{g}$  with  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  hyperpolar.
- (Kubo-Onda-Taketomi-T. 2016)  
A study on left-inv. pseudo-Riem. metrics.
- (in progress)  
A similar framework for left-inv. complex/symplectic str.

### Problems (1/3)

Can we characterize ARS in terms of  $[\langle, \rangle] \dots ?$

- Certainly, the minimality is not enough.
- Taketomi's sufficient condition cannot be a necessary cond.
- So, what else?
- First of all, study a nice class of Lie algebras.

## Summary (3/4)

### Problems (2/3)

Find special classes: can we classify

- $\mathfrak{g}$  such that  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$  is special  
(e.g., cohomogeneity one, hyperpolar, polar, ...)
- $(\mathfrak{g}, \langle, \rangle)$  such that  $\mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$  is special  
(e.g., totally geodesic, minimal, austere, ...)

### Problems (3/3)

For the existence of a left-inv. “nice” metric (on a given  $\mathfrak{g}$ ),

- a necessary and sufficient condition seems to be very hard;
- so, can we get an obstruction?  
— our conjecture is one possibility, but the condition is not easy to check.

## Summary (4/4)

### Ref. (just for our papers)

- Hashinaga, T.: Hiroshima Math. J. (2014)
- Hashinaga, T., Tamaru, H.: arXiv:1501.05513
- Hashinaga, T., Tamaru, H., Terada, K.: J. Math. Soc. Japan (2016)
- Kodama, H., Takahara, A., Tamaru, H.: Manuscripta Math. (2011)
- Kubo, A., Onda, K., Taketomi, Y., Tamaru, H.: Hiroshima Math. J. (2016)
- Taketomi, Y.: Topology Appl. (2015)
- Taketomi, Y.: Hiroshima Math. J., to appear.
- Taketomi, Y., Tamaru, H.: Transf. Groups, to appear

Thank you very much!