Ricci soliton and contact homogeneous hypersurfaces in noncompact symmetric spaces

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Preface (1/3)

Background

Let us consider

- M : a "nice" Riemannian manifold,
- $M \supset M'$: a "nice" submanifold.

Then M' often provides examples with "nice" intrinsic properties.

Typical Example

Assume that

- M : a Kähler manifold,
- $M \supset M'$: a real hypersurface.

Then M' naturally has an almost contact metric structure.

Preface (2/3)

This Talk

We concentrate on

- M : irreducible symmetric spaces of noncompact type,
- $M \supset M'$: "Lie hypersurfaces".

We mention that they provide several nice examples.

Contents

- Sec. 1: Lie hypersurfaces
- Sec. 2: Case of $\mathbb{C}\mathrm{H}^n$
- Sec. 3: Case of $G_2^*(\mathbb{R}^n)$

Preface (3/3)

Note

This talk is based on several joint works with

- Jong Taek Cho (Chonnam National University)
- Takahiro Hashinaga
  (National Inst. Tech., Kitakyushu College)
- Akira Kubo (Hiroshima Shudo University)
- Yuichiro Taketomi (Hiroshima University)

Sec. 1: Lie hypersurfaces (1/9)

Def

Let $H \curvearrowright M$ be an isometric action. Then

- a **regular orbit** is a maximal dimensional orbit;
- other orbits are **singular**;
- it is of **cohomogeneity one** if $\mathrm{codim}(\text{regular orbit}) = 1$.

Def

$M \supset M'$ is a **Lie hypersurface** if

- $\exists H \curvearrowright M$: cohomogeneity one without singular orbit s.t. $M' = H.p$ (an orbit).

Ex.

- $\mathbb{R}^{n-1} \subset \mathbb{R}^n$; $\mathbb{R}\mathrm{H}^{n-1} \subset \mathbb{R}\mathrm{H}^n$; "horosphere" $\subset \mathbb{R}\mathrm{H}^n$.

Sec. 1: Lie hypersurfaces (2/9)

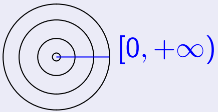
Ex.

Let us consider

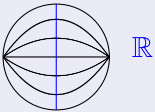
- $\mathbb{R}\mathrm{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$,
- $\mathrm{SL}(2,\mathbb{R}) = KAN$: an Iwasawa decomposition.

Then $K, A, N \curvearrowright \mathbb{R}\mathrm{H}^2$ look like as follows, and

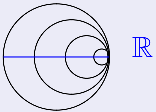
- orbits of A and N are Lie hypersurfaces.



type (K)



type (A)



type (N)

Sec. 1: Lie hypersurfaces (3/9)

Thm. (Berndt-T.: JDG 2003, Crelle 2013)

Let

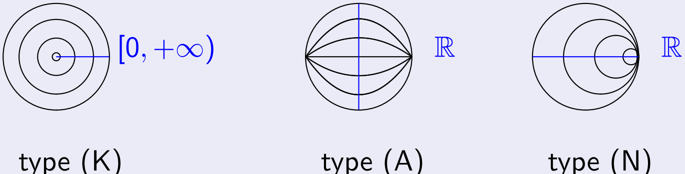
- M : an irreducible Riem. symmetric space of noncpt type,
- $H \curvearrowright M$: of cohomogeneity one, with H being connected.

Then it satisfies one of:

(K) $\exists 1$ singular orbit;

(A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;

(N) $\nexists$ singular orbit, all orbits are congruent to each other.

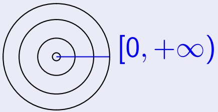


Sec. 1: Lie hypersurfaces (4/9)

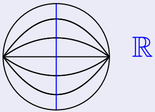
Note

TFAE:

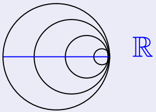
- M' is a Lie hypersurface;
- M' is an orbit of an action of type (A) or (N);
- M' is a homogeneous real hypersurface without focal submfds.



type (K)



type (A)



type (N)

Sec. 1: Lie hypersurfaces (5/9)

Some Motivation

- Lie hypersurfaces reflect some specialities of noncpt setting.
- Lie hypersurfaces are "solvmfds"

  — good candidates for Einstein/Ricci soliton.
- We know the classification of Lie hypersurfaces

  — easier to study (than generic homogeneous hypersurfaces).

Sec. 1: Lie hypersurfaces (6/9)

- M : an irr. Riem. symmetric space of noncpt type.
- C1-action := cohomogeneity one action.

Note

- All C1-actions of type (A) and (N) (i.e., those without singular orbit) can be constructed explicitly.

Thm. (Berndt-T.: JDG 2003)

Let $r := \mathrm{rank}(M)$. Then

- $\{$C1-actions of type (A)$\}/$equiv $= \{1, \ldots, r\}/(*)$;
- $\{$C1-actions of type (N)$\}/$equiv $= \mathbb{R}\mathrm{P}^{r-1}/(*)$.

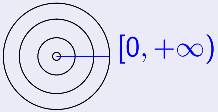
Note: $(*)$ is a finite group (determined by M).

Sec. 1: Lie hypersurfaces (7/9)

Ex.

When $\mathrm{rank}(M) = 1$ (i.e., $M = \mathbb{R}\mathrm{H}^n, \mathbb{C}\mathrm{H}^n, \mathbb{H}\mathrm{H}^n, \mathbb{O}\mathrm{H}^2$):

- $\exists 1$ type (A) action:
  — for $\mathbb{R}\mathrm{H}^n$, the minimal orbit is the tot. geod. $\mathbb{R}\mathrm{H}^{n-1}$;
  — for $\mathbb{C}\mathrm{H}^n$, the minimal orbit is ruled minimal.
- $\exists 1$ type (N) action:
  — orbits are horospheres.



type (K) type (A) type (N)

Sec. 1: Lie hypersurfaces (8/9)

Recall

- (N) $\nexists$ singular orbit, all orbits are congruent to each other.
- $\{$C1-actions of type (N)$\}$/equiv $= \mathbb{R}\mathrm{P}^{r-1}/(*)$.

Prop. (Heber: Invent. Math. 1998)

Assume $\mathrm{rank}(M) \geq 2$. Then

- $\exists H \curvearrowright M$ of type (N) : its orbits are Einstein.

Prop. (Lauret: Crelle 2011, cf. CHKTT)

- $\forall H \curvearrowright M$ of type (N), its orbits are Ricci soliton.

Sec. 1: Lie hypersurfaces (9/9)

Recall (Lauret: Crelle 2011, cf. CHKTT)

- $\forall H \curvearrowright M$ of type (N), its orbits are Ricci soliton.

Idea of Proof

- $(\mathfrak{g}, \langle , \rangle)$ is **algebraic Ricci soliton** (**ARS**) if

$$\mathrm{Ric} = c \cdot \mathrm{id} + D \quad (\exists c \in \mathbb{R}, \ \exists D \in \mathrm{Der}(\mathfrak{g})).$$

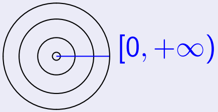
- Fact: ARS $\Rightarrow (G, \langle , \rangle)$ is Ricci soliton (G : simply-connected).
- One can show that the above orbits are ARS.
- Note: These Ricci solitons are nongradient.

Sec. 2: Case of $\mathbb{C}\mathrm{H}^n$ (1/3)

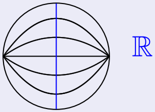
Note

Lie hypersurfaces in $\mathbb{R}\mathrm{H}^n$ are well-understood:

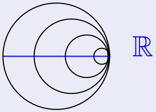
- type (A): the orbits are $\mathbb{R}\mathrm{H}^{n-1}$;
- type (N): the orbits are $\mathbb{R}^{n-1}$.



type (K)



type (A)



type (N)

Sec. 2: Case of $\mathbb{C}\mathrm{H}^n$ (2/3)

Note

- The first nontrivial case is $\mathbb{C}\mathrm{H}^n$.

Fact

- $\nexists$ Einstein real hypersurfaces in $\mathbb{C}\mathrm{H}^n$;
- Orbits of the (N)-type action (horosphere) is Ricci soliton.

Thm. (Hashinaga-Kubo-T.: Tohoku 2016)

- When $n > 2$, $\exists 1$ Ricci soliton Lie hypersurface (horosphere);
- When $n = 2$, $\exists$ exactly two Ricci soliton Lie hypersurfaces.
  (horosphere $+$ the ruled minimal real hypersurface)

Sec. 2: Case of $\mathbb{C}\mathrm{H}^n$ (3/3)

Why this is interesting

- The result depends on whether $n = 2$ or $n > 2$.
- It is relevant to the following.

Thm. (Cho-Kimura: Math. Nachr. 2011)

- $\nexists$ cpt Hopf hypersurfaces in $\mathbb{C}\mathrm{H}^n$ which are Ricci soliton.
- $\nexists$ ruled hypersurfaces in $\mathbb{C}\mathrm{H}^n$ which are gradient Ricci soliton.

Note

The assumptions above cannot be removed, namely,

- the horosphere is **noncpt** Hopf Ricci soliton;
- the ruled minimal hypersurface in $\mathbb{C}\mathrm{H}^2$ is **nongradient** Ricci soliton.

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (1/6)

Note

- $G_2^*(\mathbb{R}^{n+2})$: the noncpt Grassmannian;
- $G_2^*(\mathbb{R}^{n+2}) = \mathrm{SO}^0(2, n)/\mathrm{SO}(2)\mathrm{SO}(n)$;
- It is a Hermitian symmetric space of rank two.

Note

- $\exists$ many Lie hypersurfaces in $G_2^*(\mathbb{R}^{n+2})$, since the rank is two.

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (2/6)

Question (Cho: personally, 2014)

What is the following Lie algebra $\mathfrak{g}_{0,2}$?

- $\mathfrak{g}_{0,2} := \mathrm{span}_{\mathbb{R}}\{\xi, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$,

$$
\begin{aligned}
[\xi, Y_i] &= 2X_i && (i \geq 1)\\
[Y_2, Y_i] &= 2Y_i && (i \neq 2)\\
[X_2, Y_2] &= 2\xi && \\
[X_2, Y_i] &= 2X_i && (i \neq 2)\\
[X_i, Y_i] &= -2X_2 + 2\xi && (i \neq 2)
\end{aligned}
$$

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (3/6)

Note

- $\mathfrak{g}_{0,2}$ is a special case of $\mathfrak{g}_{\alpha,\beta}$;
- $\mathfrak{g}_{\alpha,\beta}$ is introduced by Boeckx (2000);
- $\mathfrak{g}_{\alpha,\beta}$ has a left-invariant (κ, μ)-contact metric structure.

Def. (Blair-Koufogiorgos-Papantoniou: 1995)

A contact metric mfd $(M; \eta, \xi, \varphi, g)$ is a **(κ, μ)-space** $(\kappa, \mu \in \mathbb{R})$

$:\Leftrightarrow$ $\forall X, Y \in \mathfrak{X}(M)$,

$$R(X, Y)\xi = (\kappa \,\mathrm{id} + (\mu/2)\mathcal{L}_\xi \varphi)(\eta(Y)X - \eta(X)Y).$$

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (4/6)

Note

- A contact metric mfd is Sasakian iff (κ, μ)-space with $\kappa = 1$.

Thm. (Ghosh-Sharma: 2014)

A non-Sasakian (κ, μ)-space is

- gradient Ricci soliton $\Leftrightarrow$ $(0, 0)$-space;
- nongradient Ricci soliton $\Rightarrow$ $(0, 4)$-space (iff $G_{0,2}$).

Rem.

$(0, 4)$-space with dimension n is

- $n = 3 \Rightarrow \mathrm{Sol}$, nongradient Ricci soliton;
- $n \geq 5 \Rightarrow$?

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (5/6)

Recall ($\mathfrak{g}_{0,2}$)

$$\begin{aligned}
[\xi, Y_i] &= 2X_i && (i \geq 1) \\
[Y_2, Y_i] &= 2Y_i && (i \neq 2) \\
[X_2, Y_2] &= 2\xi && \\
[X_2, Y_i] &= 2X_i && (i \neq 2) \\
[X_i, Y_i] &= -2X_2 + 2\xi && (i \neq 2)
\end{aligned}$$

Test

- $\dim \mathfrak{g}_{0,2} = 2n + 1, \quad \mathfrak{n} := [\mathfrak{g}_{0,2}, \mathfrak{g}_{0,2}]$: $2n$-dim.
- $\dim[\mathfrak{n}, \mathfrak{n}] = n, \quad \dim[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 1.$
- Conclusion: $\mathfrak{g}_{0,2}$ is solvable, $\mathfrak{n}$ is of 3-step nilpotent.

$\longrightarrow$ this looks a horosphere in $G_2^*(\mathbb{R}^{n+3})$...!

Sec. 3: Case of $G_2^*(\mathbb{R}^n)$ (6/6)

Thm. (Cho-Hashinaga-Kubo-Taketomi-T.: preprint)

- $G_{0,2}$ with dim. $n \geq 5$ is isomorphic to an orbit of a type (N) action (i.e., horosphere) of $G_2^*(\mathbb{R}^{n+3})$ as contact metric mfds.

Cor.

- $G_{0,2}$ is (nongradient) Ricci soliton, in any dimension.

Summary (1/3)

Our Aim

- Lie hypersurfaces in symmetric spaces of noncpt type
  — would provide interesting mfds.

Our Results

- Ricci soliton Lie hypersurfaces in $\mathbb{C}\mathrm{H}^n$;
- $(0,4)$-contact metric mfds vs Lie hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$.

Summary (2/3)

Problem (1)

- Study Lie hypersurfaces in other symmetric spaces M.
- When M is Hermitian, $\exists$ contact Ricci soliton?

Problem (2)

- Other (κ, μ)-contact metric mfds $(G_{\alpha,\beta})$ can be realized as a hypersurface?
- Not Lie hypersurfaces, but other homogeneous ones.

Summary (3/3)

Ref. (just for our papers)

- Berndt, J., Tamaru, H.: *Homogeneous codimension one foliations on noncompact symmetric spaces*. J. Differential Geom. 2003.
- Cho, J. T., Hashinaga, T., Kubo, A., Taketomi, Y., Tamaru, H.: *Realizations of some contact metric manifolds as Ricci soliton real hypersurfaces*. ArXiv:1702.07256.
- Cho, J. T., Hashinaga, T., Kubo, A., Taketomi, Y., Tamaru, H.: *The solvable models of noncompact real two-plane Grassmannians and some applications*. Springer Proc. Math. Stat., to appear.
- Hashinaga, T., Kubo, A., Tamaru, H.: *Homogeneous Ricci soliton hypersurfaces in the complex hyperbolic spaces*. Tohoku Math. J. 2016.
- Kubo, A., Tamaru, H.: *A sufficient condition for congruency of orbits of Lie groups and some applications*. Geom. Dedicata 2013.

Thank you very much!