Quandles and discrete symmetric spaces — flatness and commutativity

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Introduction - (1/7)

Abstract

(起) Quandles are algebraic systems, originated in knot theory.

(承) Symetric spaces are quandles.

(転) Construct a theory of quandles = "discrete symmetric spaces".

(結) In this talk, we mention some results related to "flatness".

Contents

Introduction - (2/7)

Def. (cf. Joyce 1982)

Let X be a set, and $s : X \to \mathrm{Map}(X, X) : x \mapsto s_x$ be a map. Then (X, s) is **quandle** if

(S1) $\forall x \in X$, $s_x(x) = x$.

(S2) $\forall x \in X$, s_x is bijective.

(S3) $\forall x, y \in X$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Note

- The original formulation is given by $* : X \times X \to X$,
- The correspondence is $s_x(y) = y * x$.

Introduction - (3/7)

Fact (motivation from knot theory)

- Quandles give some knot invariants.

Fact (motivation from differential geometry)

- Any connected Riemannian symmetric space is a quandle.

Note

Our viewpoint is:

- quandles = "discrete symmetric spaces",
- although it also contains "3-symmetic spaces"...

We would like to construct their structure theory.

Introduction - (4/7)

Ex.

The **trivial quandle**:

- $s_x := \mathrm{id}_X$ $(\forall x \in X)$.

The **dihedral quandle**:

- $D_n := \{p_1, \ldots, p_n : n\text{-equal dividing pts on } S^1\}$.

The **tetrahedral quandle**:

- $X := \{\text{verteces of tetrahedron}\}$ with s some 120°-rotations.

Introduction - (5/7)

Def.

$f : (X, s^X) \to (Y, s^Y)$ is a **homomorphism** if

- $\forall x \in X,\ f \circ s_x = s_{f(x)} \circ f$.

Def.

The **automorphism group** of (X, s) is

- $\mathrm{Aut}(X, s) := \{f : X \to X : \text{auto. (i.e., bijective homo.)}\}$.

(X, s) is **homogeneous** if

- $\mathrm{Aut}(X, s) \curvearrowright X$ is transitive,

Ex.

The follwing quandles are homogeneous:

- trivial quandles, dihedral quandles, the tetrahedral quandle.

Introduction - (6/7)

Def.

The **inner automorphism group** of (X, s) is

- $\mathrm{Inn}(X, s) := \langle \{ s_x \mid x \in X \} \rangle$.

(X, s) is **connected** if

- $\mathrm{Inn}(X, s) \curvearrowright X$ is transitive.

Rem.

- $\mathrm{Inn}(X, s) \subset \mathrm{Aut}(X, s)$. Hence, connected $\Rightarrow$ homogeneous.

Ex.

- trivial quandles are disconnected (unless $\# X = 1$),
- D_n is connected $\Leftrightarrow$ n is odd.

Introduction - (7/7)

Def. (T. 2013)

(X, s) is **two-point homogeneous** if

- $\mathrm{Inn}(X, s) \curvearrowright (X \times X \setminus \mathrm{diag}(X))$ transitively.

Results

Two-point homogeneous finite quandles have been classified:

- $\# X$ is prime: T. 2013;
- $\# X$ is small: Kamada-T.-Wada 2016;
- $\# X$ is not prime power: Vendramin (to appear);
- $\# X$ is prime power: Wada 2015.

Today

- We will talk the next topic, on the “flatness”.

Topic 1 - flat connected finite quandles (1/6)

Motivation

- "Maximal flats" in symmetric spaces play fundamental roles.
- We would like to have an anolougus notion for quandles.

Result of this section

- We define the notion of "flatness" for quandles.
- Thm.: flat connected finite quandles $\Rightarrow$ "discrete tori".

Def. (Ishihara-T. 2016)

A quandle (X, s) is **flat** if

- $G^0(X, s) := \langle \{ s_x \circ s_y \mid x, y \in X \} \rangle$ is abelian.

Topic 1 - flat connected finite quandles (2/6)

Fact

A Riemannian symmetric space M is flat (i.e., $\mathrm{curv} \equiv 0$) iff

- $G^0(M) := \langle \{ s_x \circ s_y \mid x, y \in M \} \rangle$ is abelian.

Ex.

For a circle S^1,

- $\mathrm{Isom}(S^1) = \mathrm{O}(2)$ is not abelian,
- $G^0(S^1) = \mathrm{SO}(2)$ is abelian.

Rem. (Jedlicka-Pilitowska-Stanovsky-ZamojskaDzienio 2015)

A quandle (X, s) is **medial** if

- $\langle \{ s_x \circ s_y^{-1} \mid x, y \in M \} \rangle$ is abelian.

Topic 1 - flat connected finite quandles (3/6)

Recall

- D_n : a dihedral quandle of order n.
- D_n is connected $\Leftrightarrow$ n is odd.

Thm. (Ishihara-T. 2016)

(X, s) is a flat connected finite quandle iff

- $X \cong D_{n_1} \times \cdots \times D_{n_k}$, where $n_1, \ldots, n_k$ are odd.

Topic 1 - flat connected finite quandles (4/6)

What are interesting (1):

- We call $D_{n_1} \times \cdots \times D_{n_k}$ a "dicrete torus".
- Our result is a "discrete verion" of
  Fact: a cpt connected Riem. symmetric space is flat $\Leftrightarrow$ torus.

What are interesting (2):

- (X, s) : flat connected finite $\Rightarrow$ involutive (i.e., $s_x^2 = \mathrm{id}$).
- This is not true for flat "homogeneous" finite quandles...

Topic 1 - flat connected finite quandles (5/6)

Idea of Proof

We refer to the theory of symmetric spaces:

(1) In the theory of symmetric spaces, there is a notion of "symmetric pairs" (G, K, σ).

(2) Analogously, for homogeneous quandles, there is a notion of "quandle triplet" (G, K, σ).

(3) If a quandle (X, s) is connected, then we can take $G := G^0(X, s)$.

(4) Since (X, s) is flat and finite, G is a finite abelian group.

(5) We can analyze possibilities for K and σ.

Topic 1 - flat connected finite quandles (6/6)

Comments (Singh 2016 (JKTR))

- Flat connected (infinite) quandles are classified.

Topic 2 - flat homogeneous finite quandles (1/7)

Motivation

- Recall: a quandle is connected $\Rightarrow$ homogeneous.
- a discrete torus with even cardinality
  $\Rightarrow$ flat homogeneous (disconnected) finite.
- Are there other such examples?

Result of this section

- We construct such examples from "vertex-transitive graph".
- Some of them also relate to "oriented real Grassmannians".

Topic 2 - flat homogeneous finite quandles (2/7)

Ex.

Let $A^n := \{\pm e_1, \dots, \pm e_n\} \subset S^{n-1}$. Then

- A^n is a subquandle,
- A^n is flat, homogeneous, disconnected.

Idea of Proof

Flat:

- $s_{e_1} = \mathrm{diag}(1, -1, \dots, -1)$.
- Similarly, all $s_{\pm e_i}$ can be realized by diagonal matrices.
- Hence, $\mathrm{Inn}(A^n)$ itself is abelian.

Disconnected:

- $\forall x \in A^n$, s_x preserves $\{\pm e_1\}, \{\pm e_2\}, \dots, \{\pm e_n\}$.

Topic 2 - flat homogeneous finite quandles (3/7)

Ex.

Let $A(k,n) := \{\pm e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\} \subset G_k(\mathbb{R}^n)^{\sim}$. Then

- $A(k,n)$ is a subquandle,
- $A(k,n)$ is flat, homogeneous, disconnected.

Idea of Proof

Flat:

- $\forall x \in A(k,n)$, s_x can be realized by diagonal matrices.

Disconnected:

- $\forall x \in A(k,n)$, s_x preserves $\{\pm e_1 \wedge \cdots \wedge e_k\}, \ldots$.

Topic 2 - flat homogeneous finite quandles (4/7)

Observation

For $A(2,4) \subset G_2(\mathbb{R}^4)^{\sim}$ (for simplicity), put $(ij) := e_i \wedge e_j$. Then

- $\{\pm(12)\} \sqcup \{\pm(13)\} \sqcup \{\pm(14)\} \sqcup \{\pm(23)\} \sqcup \{\pm(24)\} \sqcup \{\pm(34)\}$ is the $\mathrm{Inn}(A(2,4))$-orbit decomposition,
- $s_{(12)} \curvearrowright \{\pm(13)\}$: nontrivial,
- $s_{(12)} \curvearrowright \{\pm(34)\}$: trivial.

Idea for a generalization

The above defines a graph:

- $V := \{\mathrm{Inn}(A(2,4))\text{-orbits}\}$.
- Define $\{\pm(ij)\} \sim \{\pm(kl)\}$ if $s_{(ij)} \curvearrowright \{\pm(kl)\}$ nontrivially.

Conversely, we can define a quandle for a graph.

Topic 2 - flat homogeneous finite quandles (5/7)

Prop. (Furuki-T.)

Let $G = (V, E)$ be a graph.
Then $Q_G := (V \times \mathbb{Z}_2, s)$ is a quandle, where

- $s_{(v,a)}(w, b) := (w, b + e(v, w))$,
  with $e(v, w) := 1$ (if $v \sim w$), and $e(v, w) := 0$ (otherwise).

Ex

- G : empty graph ($E = \emptyset$) $\Rightarrow$ Q_G : trivial quandle.
- G : complete graph (with $\# V = n$) $\Rightarrow$ $Q_G \cong A^n$ ($\subset S^{n-1}$).

Topic 2 - flat homogeneous finite quandles (6/7)

Thm. (Furuki-T.)

- Q_G is always flat, disconnected.
- Q_G is homogeneous $\Leftrightarrow$ G is vertex-transitive.

Note

- $\exists$ many flat homogeneous (disconnected) finite quandles.
- $A(k,n)$ ($\subset G_k(\mathbb{R}^n)^{\sim}$) is isomorphic to Q_G for some G.

Topic 2 - flat homogeneous finite quandles (7/7)

Plan (vs. symmetric spaces)

- Draw the graph G such that $Q_G \cong A(k, n)$ ... (complecated)
- $\exists$ such subquandles in other symmetric spaces?

Plan (vs. quandle theory)

- Classify flat homogeneous finite quandles.
- In progress (1): construction from "oriented graphs".
- In progress (2): construction from graphs with attaching $\mathbb{Z}_3$...

Topic 3 - some commutativity of quandles (1/4)

Motivation

- $A^n \subset S^{n-1}$, $A(k,n) \subset G_k(\mathbb{R}^n)^\sim$ are interesting.
- We would like to characterize them!

Results (in progress)

- It would be good to consider "maximal commutative subsets".
- This probably relates to "antipodal sets".

Topic 3 - some commutativity of quandles (2/4)

Def.

A subset A in a quandle (X, s) is ***s*-commutative** if

- $\forall a, b \in A,\ s_a \circ s_b = s_b \circ s_a$.

Note

- We are interested in "maximal s-commutative subsets".
- This is a temporal name ...

Prop. (cf. Nagashiki)

- antipodal (i.e., $s_a(b) = b$) $\Rightarrow$ s-commutative.

  $(\because\ s_a \circ s_b = s_{s_a(b)} \circ s_a)$

- maximal s-commutative $\Rightarrow$ subquandle.

Topic 3 - some commutativity of quandles (3/4)

Prop. (cf. Nagashiki)

- $A \subset S^n$ with $n \geq 1$ is maximal s-commutative
  $\Leftrightarrow A \cong A^{n-1}$ (defined above) by $\mathrm{Aut}(S^n)$.
- $A \subset \mathbb{R}\mathrm{P}^n$ with $n \geq 2$ is maximal s-commutative
  $\Leftrightarrow A$ is maximal (great) antipodal.

Natural Question

- How about the case of $G_k(\mathbb{R}^n)$, $G_k(\mathbb{R}^n)^{\sim}$, ... ?

Topic 3 - some commutativity of quandles (4/4)

- MsC := maximal s-commutative.

Plan (vs. symmetric spaces)

- Determine MsC subsets in (some) symmetric spaces.
- When MsC is homogeneous? unique? antipodal?
- Can we apply MsC to the studies on antipodal sets?

Plan (vs. quandle theory)

- $\exists$ nice (intrinsic) properties of MsC subsets?
- When MsC is homogeneous? unique? antipodal?
- Establish the "covering theory" of quandles.

References (only from our seminar)

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Thank you!