Quandles and discrete symmetric spaces — flatness and commutativity

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Abstract

Quandles are algebraic systems, originated in knot theory.
Symmetric spaces are quandles.
Construct a theory of quandles = “discrete symmetric spaces”.
In this talk, we mention some results related to “flatness”.

Contents

§1: Introduction to quandles
§2: Topic 1 - flat connected finite quandles
§3: Topic 2 - flat homogeneous finite quandles
§4: Topic 3 - some commutativity of quandles
Def. (cf. Joyce 1982)

Let $X$ be a set, and $s : X \to \text{Map}(X, X) : x \mapsto s_x$ be a map. Then $(X, s)$ is **quandle** if

(S1) $\forall x \in X$, $s_x(x) = x$.

(S2) $\forall x \in X$, $s_x$ is bijective.

(S3) $\forall x, y \in X$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

**Note**

- The original formulation is given by $* : X \times X \to X$,
- The correspondence is $s_x(y) = y * x$. 
Fact (motivation from knot theory)

- Quandles give some knot invariants.

Fact (motivation from differential geometry)

- Any connected Riemannian symmetric space is a quandle.

Note

Our viewpoint is:

- quandles = “discrete symmetric spaces”,
- although it also contains “3-symmetric spaces”...

We would like to construct their structure theory.
Ex.

The **trivial quandle**:  
- $s_x := \text{id}_X \ (\forall x \in X)$.

The **dihedral quandle**:  
- $D_n := \{p_1, \ldots, p_n : n\text{-equal dividing pts on } S^1\}$.

The **tetrahedral quandle**:  
- $X := \{\text{vertices of tetrahedron}\} \text{ with } s \text{ some } 120^\circ\text{-rotations.}$
Def.

\( f : (X, s^X) \rightarrow (Y, s^Y) \) is a **homomorphism** if

- \( \forall x \in X, \ f \circ s_x = s_{f(x)} \circ f \).

Def.

The **automorphism group** of \((X, s)\) is

- \( \text{Aut}(X, s) := \{ f : X \rightarrow X : \text{auto. (i.e., bijective homo.)} \} \).

\((X, s)\) is **homogeneous** if

- \( \text{Aut}(X, s) \curvearrowright X \) is transitive,

Ex.

The following quandles are homogeneous:
- trivial quandles, dihedral quandles, the tetrahedral quandle.
Def.

The **inner automorphism group** of \((X, s)\) is

- \(\text{Inn}(X, s) := \langle \{s_x \mid x \in X\} \rangle\).

\((X, s)\) is **connected** if

- \(\text{Inn}(X, s) \acts X\) is transitive.

Rem.

- \(\text{Inn}(X, s) \subseteq \text{Aut}(X, s)\). Hence, connected \(\Rightarrow\) homogeneous.

Ex.

- trivial quandles are disconnected (unless \(\#X = 1\)),
- \(D_n\) is connected \(\iff n\) is odd.
Introduction - (7/7)

Def. (T. 2013)

$(X, s)$ is **two-point homogeneous** if

- $\text{Inn}(X, s) \simeq (X \times X \setminus \text{diag}(X))$ transitively.

Results

Two-point homogeneous finite quandles have been classified:

- $\#X$ is prime: T. 2013;
- $\#X$ is small: Kamada-T.-Wada 2016;
- $\#X$ is not prime power: Vendramin (to appear);
- $\#X$ is prime power: Wada 2015.

Today

- We will talk the next topic, on the “flatness”.

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**Def. (T. 2013)**

$(X, s)$ is **two-point homogeneous** if

- $\text{Inn}(X, s) \simeq (X \times X \setminus \text{diag}(X))$ transitively.
Motivation

- “Maximal flats” in symmetric spaces play fundamental roles.
- We would like to have an analogous notion for quandles.

Result of this section

- We define the notion of “flatness” for quandles.
- Thm.: flat connected finite quandles $\Rightarrow$ “discrete tori”.

Def. (Ishihara-T. 2016)

A quandle $(X, s)$ is **flat** if

- $G^0(X, s) := \langle \{s_x \circ s_y \mid x, y \in X\} \rangle$ is abelian.
Fact

A Riemannian symmetric space $M$ is flat (i.e., $\text{curv} \equiv 0$) iff

- $G^0(M) := \langle \{s_x \circ s_y \mid x, y \in M\} \rangle$ is abelian.

Ex.

For a circle $S^1$,

- $\text{Isom}(S^1) = O(2)$ is not abelian,
- $G^0(S^1) = \text{SO}(2)$ is abelian.

Rem. (Jedlicka-Pilitowska-Stanovský-Zamojska-Dzienio 2015)

A quandle $(X, s)$ is **medial** if

- $\langle \{s_x \circ s_y^{-1} \mid x, y \in M\} \rangle$ is abelian.
Recall

- $D_n$: a dihedral quandle of order $n$.
- $D_n$ is connected $\iff n$ is odd.

Thm. (Ishihara-T. 2016)

$(X, s)$ is a flat connected finite quandle iff

- $X \cong D_{n_1} \times \cdots \times D_{n_k}$, where $n_1, \ldots, n_k$ are odd.
**Topic 1 - flat connected finite quandles (4/6)**

**What are interesting (1):**

- We call $D_{n_1} \times \cdots \times D_{n_k}$ a “discrete torus”.
- Our result is a “discrete version” of
  
  Fact: a cpt connected Riem. symmetric space is flat $\Leftrightarrow$ torus.

**What are interesting (2):**

- $(X, s)$: flat connected finite $\Rightarrow$ involutive (i.e., $s_x^2 = \text{id}$).
- This is not true for flat “homogeneous” finite quandles...
Idea of Proof

We refer to the theory of symmetric spaces:

(1) In the theory of symmetric spaces, there is a notion of “symmetric pairs” \((G, K, \sigma)\).

(2) Analogously, for homogeneous quandles, there is a notion of “quandle triplet” \((G, K, \sigma)\).

(3) If a quandle \((X, s)\) is connected, then we can take \(G := G^0(X, s)\).

(4) Since \((X, s)\) is flat and finite, \(G\) is a finite abelian group.

(5) We can analyze possibilities for \(K\) and \(\sigma\).
Comments (Singh 2016 (JKTR))

- Flat connected (infinite) quandles are classified.
Motivation

- Recall: a quandle is connected $\Rightarrow$ homogeneous.
- a discrete torus with even cardinality $\Rightarrow$ flat homogeneous (disconnected) finite.
- Are there other such examples?

Result of this section

- We construct such examples from “vertex-transitive graph”.
- Some of them also relate to “oriented real Grassmannians”.
Topic 2 - flat homogeneous finite quandles (2/7)

Ex.

Let $A^n := \{\pm e_1, \ldots, \pm e_n\} \subset S^{n-1}$. Then

- $A^n$ is a subquandle,
- $A^n$ is flat, homogeneous, disconnected.

Idea of Proof

Flat:

- $s_{e_1} = \text{diag}(1, -1, \ldots, -1)$.
- Similarly, all $s_{\pm e_i}$ can be realized by diagonal matrices.
- Hence, $\text{Inn}(A^n)$ itself is abelian.

Disconnected:

- $\forall x \in A^n$, $s_x$ preserves $\{\pm e_1\}, \{\pm e_2\}, \ldots, \{\pm e_n\}$. 
Ex.

Let \( A(k, n) := \{ \pm e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k \} \subset G_k(\mathbb{R}^n) \). Then

- \( A(k, n) \) is a subquandle,
- \( A(k, n) \) is flat, homogeneous, disconnected.

Idea of Proof

Flat:

- \( \forall x \in A(k, n), s_x \) can be realized by diagonal matrices.

Disconnected:

- \( \forall x \in A(k, n), s_x \) preserves \( \{ \pm e_1 \wedge \cdots \wedge e_k \} \).
### Topic 2 - flat homogeneous finite quandles (4/7)

#### Observation

For $A(2, 4) \subset G_2(\mathbb{R}^4)$ (for simplicity), put $(ij) := e_i \wedge e_j$. Then

- $\{\pm(12)\} \sqcup \{\pm(13)\} \sqcup \{\pm(14)\} \sqcup \{\pm(23)\} \sqcup \{\pm(24)\} \sqcup \{\pm(34)\}$
  is the $\text{Inn}(A(2, 4))$-orbit decomposition,

- $s_{(12)} \circlearrowleft \{\pm(13)\}$: nontrivial,
- $s_{(12)} \circlearrowleft \{\pm(34)\}$: trivial.

#### Idea for a generalization

The above defines a graph:

- $V := \{$Inn$(A(2, 4))$-orbits$\}$.
- Define $\{\pm(ij)\} \sim \{\pm(kl)\}$ if $s_{(ij)} \circlearrowleft \{\pm(kl)\}$ nontrivially.

Conversely, we can define a quandle for a graph.
### Topic 2 - flat homogeneous finite quandles (5/7)

#### Prop. (Furuki-T.)

Let $G = (V, E)$ be a graph. Then $Q_G := (V \times \mathbb{Z}_2, s)$ is a quandle, where

- $s_{(v,a)}(w, b) := (w, b + e(v, w))$,
  with $e(v, w) := 1$ (if $v \sim w$), and $e(v, w) := 0$ (otherwise).

#### Ex

- $G$ : empty graph ($E = \emptyset$) $\Rightarrow$ $Q_G$ : trivial quandle.
- $G$ : complete graph (with $\#V = n$) $\Rightarrow$ $Q_G \cong A^n$ ($\subset S^{n-1}$).
Topic 2 - flat homogeneous finite quandles (6/7)

Thm. (Furuki-T.)

- $Q_G$ is always flat, disconnected.
- $Q_G$ is homogeneous $\iff G$ is vertex-transitive.

Note

- $\exists$ many flat homogeneous (disconnected) finite quandles.
- $A(k, n) (\subset G_k(\mathbb{R}^n)\sim)$ is isomorphic to $Q_G$ for some $G$. 
Plan (vs. symmetric spaces)

- Draw the graph $G$ such that $Q_G \cong A(k, n)$ ... (complicated)
- $\exists$ such subquandles in other symmetric spaces?

Plan (vs. quandle theory)

- Classify flat homogeneous finite quandles.
- In progress (1): construction from “oriented graphs”.
- In progress (2): construction from graphs with attaching $\mathbb{Z}_3$...
# Topic 3 - some commutativity of quandles (1/4)

## Motivation

- $A^n \subset S^{n-1}$, $A(k, n) \subset G_k(\mathbb{R}^n)$ are interesting.
- We would like to characterize them!

## Results (in progress)

- It would be good to consider “maximal commutative subsets”.
- This probably relates to “antipodal sets”.
### Def.

A subset $A$ in a quandle $(X, s)$ is **$s$-commutative** if

- $\forall a, b \in A$, $s_a \circ s_b = s_b \circ s_a$.

### Note

- We are interested in “maximal $s$-commutative subsets”.
- This is a temporal name ...

### Prop. (cf. Nagashiki)

- antipodal (i.e., $s_a(b) = b$) $\Rightarrow$ $s$-commutative.
  
  $(\therefore s_a \circ s_b = s_{s_a(b)} \circ s_a)$

- maximal $s$-commutative $\Rightarrow$ subquandle.
Prop. (cf. Nagashiki)

- $A \subset S^n$ with $n \geq 1$ is maximal $s$-commutative
  $\iff A \cong A^{n-1}$ (defined above) by $\text{Aut}(S^n)$.
- $A \subset \mathbb{RP}^n$ with $n \geq 2$ is maximal $s$-commutative
  $\iff A$ is maximal (great) antipodal.

Natural Question

- How about the case of $G_k(\mathbb{R}^n)$, $G_k(\mathbb{R}^n)^\sim$, ... ?
Topic 3 - some commutativity of quandles (4/4)

- MsC := maximal s-commutative.

Plan (vs. symmetric spaces)

- Determine MsC subsets in (some) symmetric spaces.
- When MsC is homogeneous? unique? antipodal?
- Can we apply MsC to the studies on antipodal sets?

Plan (vs. quandle theory)

- ∃ nice (intrinsic) properties of MsC subsets?
- When MsC is homogeneous? unique? antipodal?
- Establish the “covering theory” of quandles.
References (only from our seminar)


Thank you!