Flat quandles, graphs, and subsets in symmetric spaces

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Abstract

Theme

- "quandles" can be regarded as "discrete symmetric spaces".

Today's Story

$$
\begin{tikzcd}
\boxed{\S1\text{: Symm. Sp.}} \arrow[r, shift left, "\S3,\ \S4"] & \boxed{\S2\text{: Quandles}} \arrow[l, shift left, "\S6"] \arrow[d, leftrightarrow, "\S5"'] \\
 & \boxed{\text{Graphs}}
\end{tikzcd}
$$

Sec. 1 - Symmetric Spaces (1/4)

Def.

A Riemannian manifold (M, g) is a **symmetric space** if

- $\forall p \in M$, the "geodesic symmetry" s_p at p is an isometry.

Note

The geodesic symmetry $s_p \in \mathrm{Isom}(M, g)$ means

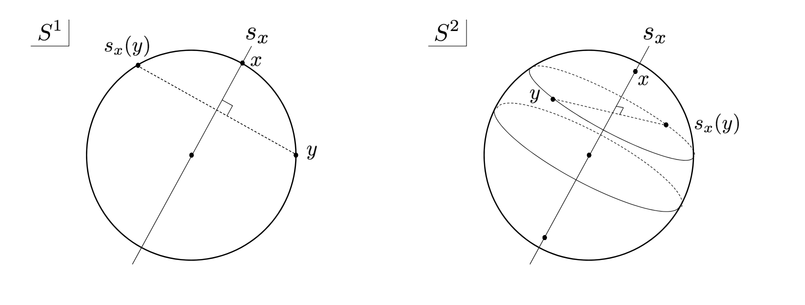
- $\forall \gamma : \mathbb{R} \to M$: geodesic with $\gamma(0) = p$, it satisfies $s_p(\gamma(t)) = \gamma(-t)$.

Sec. 1 - Symmetric Spaces (2/4)

Ex. (sphere)

The unit sphere S^n is a symmetric space with

- $s_x(y) = 2\langle x, y\rangle x - y$.



(Thanks to Y. Tada)

Sec. 1 - Symmetric Spaces (3/4)

Def.

The **real Grassmannian** $(G_k(\mathbb{R}^n), s)$ is define by

- $G_k(\mathbb{R}^n) := \{V : k\text{-dim. linear subspace in } \mathbb{R}^n\}$;
- $s_V(W) :=$ "the reflection of W wrt V".

Ex.

- For $G_2(\mathbb{R}^n)$,

$$
\begin{aligned}
s_{\mathrm{Span}\{e_1,e_2\}}(\mathrm{Span}\{e_1, e_3\}) &= \mathrm{Span}\{e_1, -e_3\} \\
&= \mathrm{Span}\{e_1, e_3\}.
\end{aligned}
$$

Sec. 1 - Symmetric Spaces (4/4)

Def.

The **real oriented Grassmannian** $(G_k(\mathbb{R}^n)^\sim, s)$ is define by

- $G_k(\mathbb{R}^n)^\sim := \{(V, \sigma) \mid V \in G_k(\mathbb{R}^n),\ \sigma : \text{orientation}\}$
  (an orientation is $\sigma \in \{\text{bases of } V\}/\mathrm{GL}(k, \mathbb{R})^+$);
- $s_{(V,\sigma)}(W, \tau) :=$ "the reflection of (W, τ) wrt V".

Ex.

- For simplicity, $(i, j) := (\mathrm{Span}\{e_i, e_j\}, [(e_i, e_j)]) \in G_2(\mathbb{R}^n)^\sim$.
- $s_{(1,2)}(1, 3) = (\mathrm{Span}\{e_1, -e_3\}, [(e_1, -e_3)]) = -(1, 3)$.

Note

- $G_1(\mathbb{R}^{n+1}) \cong \mathbb{R}\mathrm{P}^n,\quad G_1(\mathbb{R}^{n+1})^\sim \cong S^n$.

Sec. 2 - Quandles (1/3)

Def. (Joyce: 1982)

Let X be a set, and consider

- $s : X \times X \to X : (y, x) \mapsto s_x(y)$.

Then (X, s) is a **quandle** (or **symmetric set**) if

(S1) $\forall x \in X$, $s_x(x) = x$.

(S2) $\forall x \in X$, s_x is bijective (or $s_x^2 = \mathrm{id}$).

(S3) $\forall x, y \in X$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Fact

- Any connected Riemannian symmetric space is a quandle.

Sec. 2 - Quandles (2/3)

Ex.

The following $R_n := (X, s)$ is the **dihedral quandle**:

- X : the set consists of n-equal dividing points on S^1;
- $s_x := s_x^{S^1}|_X$.

Ex.

Any group G is a quandle by

- $s_g(h) := gh^{-1}g$;
- and also by $s'_g(h) := ghg^{-1}$. (a conjugation quandle)

Sec. 2 - Quandles (3/3)

Def.

$f : (X, s^X) \to (Y, s^Y)$ is a (quandle) **homomorphism** if

- $f \circ s_x = s_{f(x)} \circ f$ $(\forall x \in X)$.

Def.

- $\mathrm{Aut}(X, s) := \{f : X \to X$: bijective homo.$\}$;
- (X, s) is **homogeneous** if $\mathrm{Aut}(X, s) \curvearrowright X$ is transitive;
- $\mathrm{Inn}(X, s) := \langle \{s_x \mid x \in X\} \rangle$;
- (X, s) is **connected** if $\mathrm{Inn}(X, s) \curvearrowright X$ is transitive.

Ex.

- R_n (dihedral quandle) is always homogeneous;
- R_n is connected iff n is odd.

Sec. 3 - Flat Quandles (1/3)

Note

- Any group G is a quandle.
- Hence, it is hopeless to classify "finite" quandles ...

Naive Problem

- What are nice classes of quandles?

Note

- Among symmetric spaces, **flat** ones would be simplest.
- Note: flat $:\Leftrightarrow$ curvature $\equiv 0$.

Sec. 3 - Flat Quandles (2/3)

Def. (Ishihara-T.: 2016)

- A quandle (X, s) is **flat** if $G^0(X, s) := \langle \{ s_x \circ s_y \mid x, y \in X \} \rangle$ is abelian.

Note

- This is a characterizing condition for a Riem. symmetric space to be flat (i.e., curvature $\equiv 0$).

Ex.

- S^1 is flat. $(\because\ G^0(S^1, s) = \mathrm{SO}(2).)$
- R_n (dihedral quandle) is flat.

Sec. 3 - Flat Quandles (3/3)

Thm. (Ishihara-T.: 2016)

- (X, s) is flat finite connected iff
  it is isomorphic to a "discrete torus" with odd cardinality.

Note

- R_n is regarded as a "discrete S^1".
- A discrete torus is: $R_{n_1} \times \cdots \times R_{n_k}$.

Note

The above theorem is a discrete version of:

- A flat compact connected Riem. symmetric space is a torus.

Sec. 4 - Disconnected Examples (1/3)

Result of this section

- We construct more examples of **flat homogeneous** quandles (other than discrete tori).

Idea

- We consider some “subquandles” in $G_k(\mathbb{R}^n)^{\sim}$ (oriented real Grassmannians).

Sec. 4 - Disconnected Examples (2/3)

Def.

For $G_k(\mathbb{R}^n)^\sim$, we define

- Put $\pm(i_1, \ldots, i_k) := (\mathrm{Span}\{e_{i_1}, \ldots, e_{i_k}\}, \pm[(e_{i_1}, \ldots, e_{i_k})])$;
- $A(k, n) := \{\pm(i_1, \ldots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n\}$.

Thm. (Furuki-T.: preprint)

- $A(k, n)$ is a subquandle in $G_k(\mathbb{R}^n)^\sim$, which is flat, homogeneous, disconnected (and nontrivial).

Idea of Proof

- Flat: all $s_{\pm(i_1,\ldots,i_k)}$ can be expressed as a diagonal matrices.

Sec. 4 - Disconnected Examples (3/3)

Ex.

The case of $A(1,n)$:

- $A(1,n) = \{\pm(1), \pm(2), \dots, \pm(n)\} \subset G_1(\mathbb{R}^n)^\sim \cong S^{n-1}$;
- $s_{(i)} = s_{-(i)}$; $s_{(i)}(\pm(j)) = \mp(j)$ (if $i \neq j$); $s_{(i)}(\pm(i)) = \pm(i)$.

Ex.

The case of $A(2,4)$:

- $A(2,4) = \pm\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$;
- $s_{(1,2)}(1,3) = -(1,3)$; $s_{(1,2)}(3,4) = (3,4)$; $\cdots$

Note

- $s_{(i_1,\dots,i_k)}(j_1,\dots,j_k) = -(j_1,\dots,j_k)$
  $\Leftrightarrow$ "$k - \#(\{i_1,\dots,i_k\} \cap \{j_1,\dots,j_k\})$" is odd.

Sec. 5 - Graph Construction (1/4)

- $A(k,n)$ inspires a “graph construction” of flat quandles.

Recall

For $A(2,4)$,

- $[i,j] := \{\pm(i,j)\}$;
- $V := \{[1,2],[1,3],[1,4],[2,3],[2,4],[3,4]\}$;
- It holds $s_{(i,j)}(\pm(k,l)) = \pm(k,l) \Leftrightarrow s_{(k,l)}(\pm(i,j)) = \pm(i,j)$;

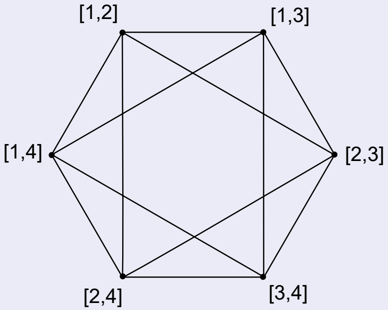
Sec. 5 - Graph Construction (2/4)

Note

For $A(2,4)$,

- join $[i,j]$ and $[k,l]$ if $s_{(i,j)}(k,l) = -(k,l)$.

Then we get



(Thanks to K. Furuki)

Sec. 5 - Graph Construction (3/4)

Def.

For a graph $G = (V, E)$, we define $Q_G = (X, s^G)$ by

- put $X := V \times \mathbb{Z}_2$;
- let $e : V \times V \to \mathbb{Z}_2$ be the adjacent function of G;
- define s^G by $s^G_{(v,a)}(w, b) := (w, b + e(v, w))$.

Thm. (Furuki-T.: preprint)

- Q_G is always a flat disconnected quandle;
- Q_G is homogeneous iff G is vertex-transitive.

Sec. 5 - Graph Construction (4/4)

Ex. (special cases)

- If $G = (V, E)$ with $E = \emptyset$, then Q_G is a trivial quandle;
- If $G = (V, E)$ is complete with $\#V = n$, then $Q_G \cong A(1, n)$.

Question

The following $G = (V, E)$ has a name?

- $V := \{v \subset \{1, \dots, n\} \mid \#v = k\}$;
- $(v, w) \in E :\Leftrightarrow$ "$k - \#(v \cap w)$" is odd.

Sec. 6 - Subsets in Symmetric Spaces (1/5)

Question

- What are $A(k, n)$ in $G_k(\mathbb{R}^n)^{\sim}$?

Def.

A subset C in a quandle (X, s) is ***s*-commutative** if

- $s_x \circ s_y = s_y \circ s_x \ (\forall x, y \in C)$.

Thm. (Nagashiki et al.: in progress)

If $2k \neq n$, then

- $A(k, n)$ is a maximal s-commutative subset in $G_k(\mathbb{R}^n)^{\sim}$;
- such subset in $G_k(\mathbb{R}^n)^{\sim}$ is essentially unique.

Sec. 6 - Subsets in Symmetric Spaces (2/5)

Note

For $x, y \in X$,

- $s_y \circ s_x = s_x \circ s_y \ (= s_{s_x(y)} \circ s_x) \ \Leftrightarrow \ s_{s_x(y)} = s_y$.

Note (why $2k \neq n$?)

Consider $G_k(\mathbb{R}^n)^{\sim}$. Then $s_{(V,\sigma)}$ and $s_{(W,\tau)}$ commute iff

- (when $n = 2k$ with k even)
  $$s_{(V,\sigma)}(W, \tau) = (W, \pm\tau) \text{ or } (W^{\perp}, *).$$
- (otherwise)
  $$s_{(V,\sigma)}(W, \tau) = (W, \pm\tau).$$

Sec. 6 - Subsets in Symmetric Spaces (3/5)

Problem (inspired by $A(k, n)$)

- For symmetric spaces or quandles, classify maximal s-commutative (**MsC**) subsets.

Why Interesting (1)

- MsC subsets would "approximate" the ambient spaces.
- This would be a discrete version of "maximal flats" in the theory of Riem. symmetric spaces...

Sec. 6 - Subsets in Symmetric Spaces (4/5)

Why Interesting (2)

- MsC subsets are related to "antipodal subsets".

Def. (Chen-Nagano: 1988))

A subset C in a quandle (X, s) is **antipodal** if

- $s_x(y) = y$ $(\forall x, y \in C)$.

Prop.

- antipodal $\Rightarrow$ s-commutative;
- maximal s-commutative $\Rightarrow$ subquandle.

Sec. 6 - Subsets in Symmetric Spaces (5/5)

Note

- Maximal antipodal subsets are sometimes hard to determine (cf. Tanaka-Tasaki), e.g., for $G_k(\mathbb{R}^n)^{\sim}$;
- contrary, we can determine MsC subsets in $G_k(\mathbb{R}^n)^{\sim}$ $(2k \neq n)$!

Note

- Akase-Tanaka-Tasaki-T. recently determined MsC subsets in a compact classical Lie groups (viewed as symmetric spaces);
- This shows the uniqueness of MsC subsets does not holds...

Summary

Today's Story

$$\begin{tikzcd} \boxed{\S1:\ \text{Symm. Sp.}} \arrow[r, shift left, "\S3,\ \S4"] & \boxed{\S2:\ \text{Quandles}} \arrow[l, shift left, "\S6"] \arrow[d, leftrightarrow, "\S5"'] \\ & \boxed{\text{Graphs}} \end{tikzcd}$$

- Thank you very much!