Flat quandles and finite subsets in symmetric spaces

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Abstract

Slogan

- Which subset "approximates" a symmetric space?

Contents

Introduction - (1/6)

In this section, we recall "quandles".

Def.

Let X be a set, and consider

- $s : X \times X \to X : (y, x) \mapsto s_x(y)$.

Then (X, s) is a **quandle** (or **symmetric space**) if

(S1) $\forall x \in X$, $s_x(x) = x$.

(S2) $\forall x \in X$, s_x is bijective (or $s_x^2 = \mathrm{id}$).

(S3) $\forall x, y \in X$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Note

- The notion of "quandles" is originated in knot theory (Joyce (1982), Matveev (1982)).

Introduction - (2/6)

some remarks on quandles:

Note

- The above symmetric space is also called
  - **kei (圭)** by Takasaki (1943),
  - **symmetric set** by Nobusawa (1970's), or
  - **involutory quandle**.

Fact (our motivation)

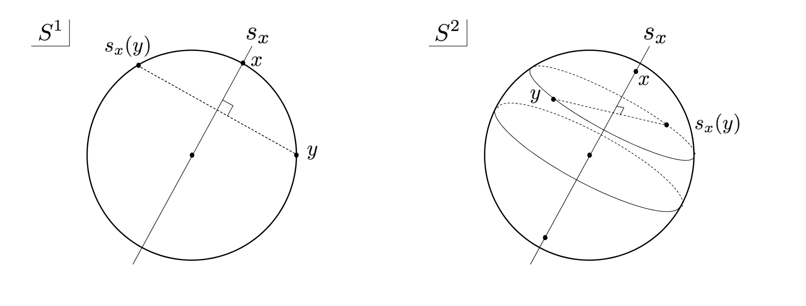
- Any connected Riemannian symmetric space is a quandle.

Introduction - (3/6)

Ex. (sphere)

The unit sphere S^n is a symmetric space with

- $s_x(y) = 2\langle x, y\rangle x - y$.



(Thanks to Y. Tada)

Introduction - (4/6)

Recall

- For Riemannian symmetric spaces,
some submanifolds ("maximal flats") play fundamental roles.

Naive Problem

- Find subsets of quandles, which approximate (reflect some properties of) the ambient quandles.

Introduction - (5/6)

Def. (Chen-Nagano (1988))

A subset C in a quandle (X, s) is **antipodal** if

- $s_x(y) = y \ (\forall x, y \in C)$.

Def. (Nagashiki et al. (in progress))

A subset C in a quandle (X, s) is ***s*-commutative** if

- $s_x \circ s_y = s_y \circ s_x \ (\forall x, y \in C)$.

Prop.

- antipodal $\Rightarrow$ s-commutative;
- maximal s-commutative $\Rightarrow$ subquandle.

Introduction - (6/6)

Ex.

For a circle S^1,

- $\{\pm e_1\}$ is maximal antipodal;
- $\{\pm e_1, \pm e_2\}$ is MsC (maximal s-commutative).

Contents (recall)

- Result 1: Grassmannian case;
- Result 2: Group-type case.

Result 1: Grassmannian case (1/8)

a motivation for "s-commutative":

Def. (Ishihara-T. (2016))

A quandle (X, s) is **flat** if

- $G^0(X, s) := \langle \{ s_x \circ s_y \mid x, y \in X \} \rangle$ is abelian.

Note

- Similar to the theory of symmetric space, "maximal flat subquandles" play nice roles?
- s-commutative $\Rightarrow$ flat.

Result 1: Grassmannian case (2/8)

Prop.

The following $A(1, n)$ is a flat subquandle in S^{n-1}:

- $A(1, n) := \{\pm e_1, \dots, \pm e_n\}$.

Prop.

For S^{n-1},

- $A(1, n) := \{\pm e_1, \dots, \pm e_n\}$ is a MsC subset;
- any MsC subsets are congruent to $A(1, n)$ by $\mathrm{O}(n)$.

Note

- $s_{e_i}(\pm e_j) = \mp e_j$ for $i \neq j$;
- hence, each $s_{\pm e_i}$ is a diagonal matrices.

Result 1: Grassmannian case (3/8)

Def.

The **real Grassmannian** $(G_k(\mathbb{R}^n), s)$ is define by

- $G_k(\mathbb{R}^n) := \{V : k\text{-dim. linear subspace in } \mathbb{R}^n\}$;
- $s_V(W) :=$ "the reflection of W wrt V".

Prop.

For $G_k(\mathbb{R}^n)$,

- Put $(i_1, \ldots, i_k) := \mathrm{Span}\{e_{i_1}, \ldots, e_{i_k}\}$.
- Then $A(k,n)' := \{(i_1, \ldots, i_k) \mid i_1 < \cdots < i_k\}$ is a subquandle.

Ex.

- $s_{(1,2)}(1,3) = \mathrm{Span}\{e_1, -e_3\} = (1,3)$.

Result 1: Grassmannian case (4/8)

Fact (Chen-Nagano, Tanaka-Tasaki)

- $A(k,n)'$ is a maximal antipodal subset in $G_k(\mathbb{R}^n)$;
- it is unique up to congruence by $\mathrm{O}(n)$.

Prop.

- If $n \neq 2k$, then $A(k,n)'$ is a MsC subset in $G_k(\mathbb{R}^n)$;
  (it is unique up to congruence by $\mathrm{O}(n)$)
- If $n = 2k$, then $A(k,n)'$ is s-commutative, but not MsC.

Result 1: Grassmannian case (5/8)

Def.

The **real oriented Grassmannian** $(G_k(\mathbb{R}^n)^\sim, s)$ is define by

- $G_k(\mathbb{R}^n)^\sim := \{(V, \sigma) \mid V \in G_k(\mathbb{R}^n),\ \sigma : \text{orientation}\}$
  (an orientation is $\sigma \in \{\text{bases of } V\}/\mathrm{GL}(k, \mathbb{R})^+$);
- $s_{(V,\sigma)}(W, \tau) :=$ "the reflection of (W, τ) wrt V".

Note

- $G_1(\mathbb{R}^n)^\sim \cong S^{n-1}$.

Result 1: Grassmannian case (6/8)

Prop.

For $G_k(\mathbb{R}^n)^{\sim}$,

- Put $\pm(i_1, \ldots, i_k) := (\mathrm{Span}\{e_{i_1}, \ldots, e_{i_k}\}, [(e_{i_1}, \ldots, e_{i_k})])$.
- $A(k, n) := \{\pm(i_1, \ldots, i_k) \mid i_1 < \cdots < i_k\}$ is a subquandle.

Ex.

- $s_{(1,2)}(1, 3) = (\mathrm{Span}\{e_1, -e_3\}, [(e_1, -e_3)]) = -(1, 3)$.
- Hence $A(k, n)$ is not antipodal.

Result 1: Grassmannian case (7/8)

Thm. (Nagashiki et al.)

For $G_k(\mathbb{R}^n)^\sim$ with $n \neq 2k$, we have

- $A(k,n) := \{\pm(i_1, \dots, i_k)\}$ is a MsC subset;
- it is unique up to congruence by $\mathrm{O}(n)$.

Remark

For $G_k(\mathbb{R}^n)^\sim$ with $n = 2k$, we have

- $A(k,n) := \{\pm(i_1, \dots, i_k)\}$ is s-commutative, but not maximal.

For $G_2(\mathbb{R}^4)^\sim$, the union of the following is MsC:

- $A(2,4) = \pm\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$;
- $\pm\{(1 \pm 2, 3 \pm 4),(1 \pm 3, 2 \pm 4),(1 \pm 4, 2 \pm 3)\}$,

  where

  $\pm(i \pm j, k \pm l) := (\mathrm{Span}\{e_i \pm e_j, e_k \pm e_l\}, \pm[(e_i \pm e_j, e_k \pm e_l)])$.

Result 1: Grassmannian case (8/8)

Comments

- Classification of max. antipodal subsets in $G_k(\mathbb{R}^n)^\sim$ is open (however we could do it for MsC subsets when $n \neq 2k$);
- Some strange things happen when $n = 2k$ (it relates to the example in $G_2(\mathbb{C}^4)$ by Kurihara-Okuda?)

Result 2: Group type case (1/5)

General Problem

- Classify MsC subsets in a symmetric space (X, s).

Note

- Recall: a group G is a symmetric space by

$$s_g(h) := gh^{-1}g.$$

- Such (G, s) is called a symmetric space **of group type**.
- We study MsC subsets in compact classical groups G.

Result 2: Group type case (2/5)

Prop.

Let (G, s) be a symmetric space of group type. Then

- $G \times G \subset \mathrm{Aut}(G, s)$.

  (namely, the left and right actions are automorphisms)

"Thm." (Akase-T.-Tanaka-Tasaki)

We classified MsC subsets in the following (G, s) up to congruence by $G \times G$:

- $G = \mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Spin}(n)$.

Result 2: Group type case (3/5)

Thm. (for $\mathrm{O}(n)$)

Let $G := \mathrm{O}(n)$. Then, up to $G \times G$,

- $n = 2^m$ (with $m \in \mathbb{Z}_{\geq 1}$) $\Rightarrow \exists$ m MsC subsets;
- $n = 2^m \cdot \ell$ (with ℓ odd ($\neq 1$)) $\Rightarrow \exists$ $m + 1$ MsC subsets.

Ex.

When n is odd,

- $\Delta_n := \{\mathrm{diag}(\pm 1, \ldots, \pm 1)\}$ is a (unique) MsC subset;
- this is also a (unique) maximal antipodal subset.

Ex.

In $G := \mathrm{O}(2)$,

- $D_2 := \Delta_2 \cup \left\{ \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}$ is a (unique) MsC subset.

Result 2: Group type case (4/5)

Recall (for $\mathrm{O}(n)$)

- $n = 2^m$ (with $m \in \mathbb{Z}_{\geq 1}$) $\Rightarrow \exists$ m MsC subsets;
- $n = 2^m \cdot \ell$ (with ℓ odd ($\neq 1$)) $\Rightarrow \exists$ $m + 1$ MsC subsets.

Ex.

Let $G := \mathrm{O}(6)$. Then, up to $G \times G$,

- all MsC subsets are Δ_6, $D_2 \otimes \Delta_3$.

Ex.

Let $G := \mathrm{O}(8)$. Then, up to $G \times G$,

- all MsC subsets are Δ_8, $D_2 \otimes \Delta_4$, $D_2 \otimes D_2 \otimes D_2$.

Result 2: Group type case (5/5)

Thm. (for $\mathrm{Spin}(n)$)

Let $G := \mathrm{Spin}(n)$. Then, up to $G \times G$,

- $n \notin 4\mathbb{N}$ $\Rightarrow \exists 1$ MsC subset;
- $n = 4$ $\Rightarrow \exists 1$ MsC subset;
- $n = 4 \cdot 2^m$ (with $m \in \mathbb{Z}_{\geq 1}$) $\Rightarrow \exists\ m+2$ MsC subset;
- $n = 4 \cdot 2^m \cdot \ell$ ($m \in \mathbb{Z}_{\geq 0}$, ℓ odd ($\neq 1$)) $\Rightarrow \exists\ m+3$ MsC subset.

Ex.

Let $G := \mathrm{Spin}(3)$ ($\cong \mathrm{Sp}(1)$). Then, up to $G \times G$,

- $\{\pm 1, \pm i, \pm j, \pm k\}$ is a (unique) MsC subset.

Summary (1/2)

Motivation

- Which subset "approximates" the symmetric space?
- Candidates: maximal antipodal subsets, and MsC subsets.

Results

- We (almost) classified MsC subsets in Grassmannians and classical groups.
- It provides nice examples of subquandles/flat quandles.
- For some cases, it is easier than maximal antipodal subsets.
- On the other hand, it is far from the "uniqueness".

Summary (2/2)

Problems

- Classify maximal s-commutative subsets in other symmetric spaces (or quandles).
- How MsC subsets approximate (reflect some properties of) the ambient symmetric spaces?

Thank you!