Left-invariant pseudo-Riemannian metrics on some Lie groups

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Introduction (1/5)

Originally

- We study isometric actions/submanifolds in symmetric spaces.

An application

- We study left-invariant Riemannian metrics on Lie groups.

Today's topic (a further application)

- Left-invariant **pseudo-Riemannian** metrics on Lie groups.

Introduction (2/5)

Main Result

- We “classify” left-invariant pseudo-Riemannian metrics on some particular Lie groups.

Contents:

Introduction (3/5)

Def.:

- A (pseudo-)Riem. metric on a Lie group G is **left-invariant** if all left-translation ($L_a(g) := ag$) are isometric.

Known Results:

$\exists$ left-invariant metrics which are

- Einstein;
- negative/nonpositive sectional curvature;
- Ricci soliton, ...

Introduction (4/5)

Nice Things:

- $\exists$ one-to-one correspondence between left-invariant metrics on a Lie group G and inner products on $\mathfrak{g}$ $(:= \mathrm{Lie}(G))$.
- All curvatures can be calculated in terms of $(\mathfrak{g}, \langle , \rangle)$.

However:

- The existence/nonexistence problems of “nice” left-invariant metrics are far from being well-understood...
- A famous open problem: does $\mathrm{SL}(3, \mathbb{R})$ admit left-invariant Riemannian metrics which are Einstein?

Introduction (5/5)

Why difficult:

- If $n = \dim G = \dim \mathfrak{g}$, then
  $\{\text{left-invariant Riem. metrics on } G\} \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$.
- This is too large...
  (e.g., $G := \mathrm{SL}(3, \mathbb{R})$; $\dim \mathrm{GL}(8, \mathbb{R})/\mathrm{O}(8) = 63 - 28 = 35$)

Note

- $\{\text{left-inv. metrics on } G \text{ with signature } (p, q)\}$
  can be identified with $\mathrm{GL}(p + q, \mathbb{R})/\mathrm{O}(p, q)$, similarly.

Review on the Riemannian case (1/6)

Slogan

- Left-invariant Riem. metrics can be studied by isometric actions on some noncompact Riem. symmetric spaces.

Character

- some noncompact Riem. symmetric spaces:

$$\begin{aligned}\widetilde{\mathfrak{M}}_G &:= \{\text{left-inv. Riem. metrics on } G\}\\ &\cong \{\text{positive definite inner products on } \mathfrak{g}\}\\ &\cong \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n).\end{aligned}$$

- isometric actions:

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}_G.$$

Review on the Riemannian case (2/6)

- We are interested in $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}_G$.

Note

- $\mathbb{R}^\times \curvearrowright \widetilde{\mathfrak{M}}_G$ gives "scaling".
- $\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}_G$ gives "isometry".

Def. (cf. Kodama-Takahara-T. (2011)):

- $\mathfrak{PM}_G := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_G$ (the orbit space) is called the **moduli space** of left-invariant Riem. metrics on G.

Review on the Riemannian case (3/6)

- We are interested in $\mathfrak{PM}_G := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_G$.

Note

- Milnor (1976) constructed so-called the "Milnor frames" for 3-dim. unimodular Lie groups.
- Milnor frames essentially express $\mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_G$.
- Therefore, $\mathfrak{PM}_G$ is a kind of generalization (higher-dimensional extension) of Milnor frames.

Review on the Riemannian case (4/6)

- We are interested in $\mathfrak{PM}_G := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_G$.

Note

- describe $\mathfrak{PM}_G \longleftrightarrow$ classify left-invariant metrics

Note

- In order to solve existence/nonexistence problems of nice left-invariant metrics, it is enough to study $\mathfrak{PM}_G$.
- Therefore, if $\mathfrak{PM}_G$ is "small", then such problems could be handled.

Review on the Riemannian case (5/6)

Thm. (Lauret (2003), Kodama-Takahara-T. (2011))

- $\mathfrak{PM}_G = \{\mathrm{pt}\}$ iff $\mathfrak{g}$ is one of the following:
  - (1) $\mathbb{R}^n$ (abelian);
  - (2) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ (the Lie algebra of $\mathbb{R}\mathrm{H}^n$);
  - (3) $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ (3-dim. Heisenberg plus abelian).

Note

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is a parabolic subgroup for the above $\mathfrak{g}$.

Review on the Riemannian case (6/6)

Note

- $\exists$ many examples of G with $\mathfrak{PM}_G$ is one-dimensional, e.g.,
  (1) all 3-dim. solvable Lie algebras (unless $\mathfrak{PM}_G = \{\mathrm{pt}\}$);
  (2) some 4-dim. solvable Lie algebras;
  (3) several n-dim. solvable Lie algebras.
- A classification of such Lie algebras is still open.

Note

- $\mathfrak{PM}(G)$ is 1-dim. iff $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}(G)$ is cohomogeneity one.
- One can apply results of cohomogeneity one actions on noncompact symmetric spaces (e.g., studied by Berndt-T.).

Our Framework (1/3)

Slogan

- Left-invariant **pseudo-Riem**. metrics can be studied by isometric actions on some **pseudo-Riem**. symmetric spaces.

Character

- some pseudo-Riem. symmetric spaces:

$$\widetilde{\mathfrak{M}}_{(p,q)}(G) := \{\text{left-inv. metrics on } G \text{ of signature } (p,q)\}$$
$$\cong \mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q).$$

- isometric actions:

$$\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}_{(p,q)}(G).$$

Our Framework (2/3)

Def. (cf. Kubo-Onda-Taketomi-T. (2016)):

- $\mathfrak{PM}_{(p,q)}(G) := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{(p,q)}(G)$ is called the **moduli space** of left-invariant metrics on G of signature (p, q).

Note

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}_{(p,q)}(G) \cong \mathrm{GL}(p+q, \mathbb{R}) / \mathrm{O}(p, q)$ is an isometric action on a pseudo-Riem. symmetric space.

Ex.

- For $\mathfrak{g} := \mathbb{R}^{p+q}$ (abelian), one has $\mathfrak{PM}_{(p,q)}(G) = \{\mathrm{pt}\}$.

Our Framework (3/3)

Problem

- Describe $\mathfrak{PM}_{(p,q)}(G)$ for some other (easy) G.

Note

- $\mathfrak{PM}_{(n,0)}(G) = \{\mathrm{pt}\}$ iff
  $\mathfrak{g} \cong \mathbb{R}^n$, $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, or $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.
- Therefore, for studying $\mathfrak{PM}_{(p,q)}(G)$, we start with
  $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}, \ \mathfrak{h}^3 \oplus \mathbb{R}^{p+q-3}$.

Results (1/7)

- We describe $\mathfrak{PM}_{(p,q)}(G)$ for some G (and some (p,q)).

Thm 1 (Kubo-Onda-Taketomi-T. (2016))

- $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) = 3$ for every $p, q \in \mathbb{Z}_{\geq 1}$.

Thm 2 (Kondo-T. (in progress))

- $\# \mathfrak{PM}_{(3,1)}(\mathfrak{h}^3 \oplus \mathbb{R}) = 6$.

Results (2/7)

Technique of Proof

- In one word, “matrices calculations”...
- Recall that
$$\mathfrak{PM}_{(p,q)}(G) := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}(G).$$
- This can be identified with the double coset space:
$$\mathfrak{PM}_{(p,q)}(G) \cong \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q).$$

Fact

- $\# \, \mathfrak{PM}_{(n-1,1)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = 3$ (Nomizu 1979).
- $\# \, \mathfrak{PM}_{(2,1)}(\mathfrak{h}^3) = 3$ (Rahmani 1992).

(In both cases, techniques are different from ours.)

Results (3/7)

Summary (list of $\#\mathfrak{PM}_{(p,q)}(\mathfrak{g})$)

$\mathfrak{g}$	$\#\mathfrak{PM}_{(p,q)}(\mathfrak{g})$	condition	
$\mathbb{R}^n$	1	for $\forall(p,q)$	
$\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$	1	Riem.	
	3	Lorentz	Nomizu 1979
	3	generic	**KOTT 2016**
$\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$	1	Riem.	
	3	sig. $(2,1)$	Rahmani 1992
	6	sig. $(3,1)$	**Kondo-T.**
	???	others	

Results (4/7)

- We try to explain intuitive reasons why 3 or 6 points...

Intuitive Explanation (1/6)

- One can also identify $\mathfrak{PM}_{(p,q)}(G)$ with the orbit space of
  $$\mathrm{O}(p,q) \curvearrowright \mathrm{GL}(p+q,\mathbb{R})/(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})).$$
- For our cases,
  $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are parabolic, and hence
  $\mathrm{GL}(p+q,\mathbb{R})/\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are real frags, or R-spaces.

Results (5/7)

Intuitive Explanation (2/6)

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = \left\{ \left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{array} \right) \right\}$: type $(1, n-1)$.
- $\mathrm{GL}(p+q, \mathbb{R})/(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})) = G_{p+q-1}(\mathbb{R}^{p+q}) \cong \mathbb{R}\mathrm{P}^{p+q-1}$.

Intuitive Explanation (3/6)

- $\mathrm{O}(p,q) \curvearrowright \mathbb{R}\mathrm{P}^{p+q-1}$ has exactly 3 orbits:

$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{spacelike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{lightlike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{timelike}\}.$$

Results (6/7)

Intuitive Explanation (4/6)

- $H := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{h}^3 \oplus \mathbb{R})$ is parabolic of type $(1,1,2)$:

$$H \cong \left\{ \left(\begin{array}{c|c|cc} * & 0 & 0 & 0 \\ \hline * & * & 0 & 0 \\ \hline * & * & * & * \\ * & * & * & * \end{array} \right) \in \mathrm{GL}(4,\mathbb{R}) \right\}.$$

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3})$ is similar; parabolic of type $(1,1,n-2)$.

Results (7/7)

Intuitive Explanation (5/6)

- $\mathrm{GL}(4,\mathbb{R})/H \cong F_{1,2}(\mathbb{R}^4)$: flag, where
- $F_{1,2}(\mathbb{R}^4) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset \mathbb{R}^4 \,(\text{subsp}),\ \dim V_k = k\}$.

Intuitive Explanation (6/6)

- $\mathrm{O}(3,1) \curvearrowright F_{1,2}(\mathbb{R}^4)$ has 6 orbits:

$$V_2 : (2,0) \Rightarrow V_1 : \text{positive},$$
$$V_2 : \text{degenerate} \Rightarrow V_1 : \text{positive/degenerate},$$
$$V_2 : (1,1) \Rightarrow V_1 : \text{positive/degenerate/negative}.$$

Summary (1/3)

Summary

- Left-invariant Riem. metrics can be studied by isometric actions on some noncompact Riem. symmetric spaces.

- Left-invariant **pseudo-Riem**. metrics can be studied by isometric actions on some **pseudo-Riem**. symmetric spaces.

- We classify (determine the moduli space of) left-invariant pseudo-Riemannian metrics on some particular Lie groups.

Summary (2/3)

Summary (list of $\#\mathfrak{PM}_{(p,q)}(\mathfrak{g})$)

$\mathfrak{g}$	$\#\mathfrak{PM}_{(p,q)}(\mathfrak{g})$	condition
$\mathbb{R}^n$	1	for $\forall(p,q)$
$\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$	1	Riem.
	3	non-Riem.
$\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$	1	Riem.
	3	dim. 3, sig. $(2,1)$
	6	dim. 4, sig. $(3,1)$
	???	others

Summary (3/3)

Problem (in progress)

- Determine $\# \mathfrak{PM}_{(p,q)}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3})$. Is it finite?

Problem (near future...?)

- If $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}) < +\infty$, then $\mathfrak{g}$ is one of above?
- Describe $\mathfrak{PM}_{(p,q)}(\mathfrak{g})$ for some $\mathfrak{g}$ (e.g., three/four-dim., ...)
- Study isometric actions on pseudo-Riem. symmetric spaces.

Thank you very much!