Left-invariant pseudo-Riemannian metrics on some solvable Lie groups

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Abstract

Our Problem

- How many inner products $\langle , \rangle$ (with signature (p, q)) on a given Lie algebra $\mathfrak{g}$ up to automorphism and scaling?

Our Results

- We (want to) study the above question for
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$: the Lie algebra of real hyperbolic spaces,
  - $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$: the 3-dim. Heisenberg Lie algebra plus abelian.

Introduction (1/5)

General Problem

- Study/Find/Classify nice left-invariant metrics on Lie groups.

Def.

- $\langle , \rangle_1$ and $\langle , \rangle_2$ on $\mathfrak{g}$ are **equivalent up to automorphism and scaling** if $\exists c \in \mathbb{R}^\times$, $\exists \varphi \in \mathrm{Aut}(\mathfrak{g})$: $\langle , \rangle_2 = (c\varphi).\langle , \rangle_1$.

Note

- $g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$ for $g \in \mathrm{GL}(\mathfrak{g})$.
- $\langle , \rangle_1 \sim \langle , \rangle_2 \Rightarrow$ the corresponding left-invariant metrics are isometric up to scaling.

Introduction (2/5)

Def.

- $\mathfrak{PM}_{(p,q)}(\mathfrak{g}) := \{\langle , \rangle \text{ on } \mathfrak{g} \text{ with sig. } (p,q)\}/\text{"equiv."}$ is called the **moduli space**.

Our Problem

- Describe $\mathfrak{PM}_{(p,q)}(\mathfrak{g})$ for a given $\mathfrak{g}$.

Ex.

- $\mathfrak{g}_{\mathbb{R}H^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j = 2, \ldots, n)$.
  ($G_{\mathbb{R}H^n}$ (simply-connected) acts simply-transitively on $\mathbb{R}H^n$)
- $\mathfrak{h}^3 := \mathrm{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_3$. (Heisenberg)

Introduction (3/5)

Summary (the list of $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g})$)

$\mathfrak{g}$	positive def.	indefinite	comments
$\mathbb{R}^n$	1	1	
$\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$	1		
$\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$	1		

Fact (Lauret 2003, cf. Kodama-Takahara-T. 2011)

- $\#\mathfrak{PM}_{(n,0)}(\mathfrak{g}) = 1$ (in fact, finite)
  $\Longleftrightarrow \mathfrak{g} \cong \mathbb{R}^n$, $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, or $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Introduction (4/5)

Summary (the list of $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g})$)

$\mathfrak{g}$	positive def.	indefinite	comments
$\mathbb{R}^n$	1	1	
$\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$	1	3	$(p,q) = (n-1,1)$
		?	(p,q) : generic
$\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$	1	3	n=3
		?	$n \geq 4$

Fact

- $\# \mathfrak{PM}_{(n-1,1)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = 3$ (Nomizu 1979).
- $\# \mathfrak{PM}_{(2,1)}(\mathfrak{h}^3) = 3$ (Rahmani 1992).

Introduction (5/5)

Prop. (well-known)

- $\{\langle , \rangle$: with signature (p, q) on $\mathfrak{g}\} \cong \mathrm{GL}(p+q, \mathbb{R})/\mathrm{O}(p, q)$.

Prop. (observation)

- $\mathfrak{PM}_{(p,q)}(\mathfrak{g})$
  $\cong$ the orbit space of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \mathrm{GL}(p+q, \mathbb{R})/\mathrm{O}(p, q)$
  $\cong \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}(p+q, \mathbb{R})/\mathrm{O}(p, q)$
  $\cong$ the orbit space of $\mathrm{O}(p.q) \curvearrowright \mathrm{GL}(p+q, \mathbb{R})/\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$

Result 1: $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ (1/5)

- We study $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$.

Thm. (Kubo-Onda-Taketomi-T. 2016)

- $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) = 3$ if $p, q \in \mathbb{Z}_{\geq 1}$,
- these 3 metrics have positive/zero/negative constant sectional curvatures.

Result 1: $\mathfrak{g}_{\mathbb{R}H^n}$ (2/5)

Idea of Proof

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}H^n}) = \left\{ \left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{array} \right) \right\}$: type $(1, n-1)$.
- By direct matrices calculations,

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}H^{p+q}}) \backslash \mathrm{GL}(p+q, \mathbb{R}) / \mathrm{O}(p, q) \cong \{3 \text{ points}\}.$$

- We can give an intuitive explanation why "3 points".

Result 1: $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ (3/5)

Recall

- $\mathfrak{PM}_{(p,q)}(\mathfrak{g}) \cong \mathrm{O}(p,q)\backslash(\mathrm{GL}(p+q,\mathbb{R})/\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))$.

Intuitive Explanation (1)

- Recall that $H := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}})$ is “parabolic”.
- One knows

$$\mathrm{GL}(p+q,\mathbb{R})/H \cong \mathbb{R}\mathrm{P}^{p+q-1}.$$

- $\mathrm{O}(p,q) \curvearrowright \mathbb{R}\mathrm{P}^{p+q-1}$ has exactly 3 orbits:

$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{spacelike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{lightlike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{timelike}\}.$$

Result 1: $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ (4/5)

Cor. (a generalization of Milnor frames)

- $\forall\langle,\rangle$ with signature (p,q) on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}$, $\exists k>0$, $\exists\lambda\in\{0,1,2\}$, $\exists\{x_1,\ldots,x_{p+q}\}$: p-o.n.b. wrt $k\langle,\rangle$,

$$[x_1,x_i]=x_i \quad (i\in\{2,\ldots,p+q-1\}),$$
$$[x_1,x_{p+q}]=-\lambda x_1+x_{p+q},$$
$$[x_i,x_{p+q}]=-\lambda x_i \quad (i\in\{2,\ldots,p+q-1\}).$$

Intuitive Explanation (2)

- In general, $[\mathfrak{g},\mathfrak{g}]$ is preserved by $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$.
- $[\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}},\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}]$ is spanned by $\{x_2,\ldots,x_{p+q-1},-\lambda x_1+x_{p+q}\}$.
- Hence, $\langle,\rangle|_{[\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}},\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}]}$ is $(p-1,q)$, degenerate, $(p,q-1)$.

Result 1: $\mathfrak{g}_{\mathbb{RH}^n}$ (5/5)

Note

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{h}^3)$ is quite "similar" to $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_{\mathbb{RH}^3})$.
- Hence, our method can be applied to $\mathfrak{h}^3$, which gives an alternative proof for the following.

Fact (Rahmani 1992)

- $\# \, \mathfrak{PM}_{(2,1)}(\mathfrak{h}^3) = 3$.

Result 2: $\mathfrak{h}^3 \oplus \mathbb{R}$ (1/4)

- In this section, we study $\mathfrak{g} := \mathfrak{h}^3 \oplus \mathbb{R}$ (Heisenberg + abelian).

Thm. (Kondo-T., in progress)

- $\# \mathfrak{PM}_{3,1}(\mathfrak{h}^3 \oplus \mathbb{R}) = 6$.

Idea of Proof

- Direct matrices calculations! (but more complicated)

Result 2: $\mathfrak{h}^3 \oplus \mathbb{R}$ (2/4)

Note

- $H := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{h}^3 \oplus \mathbb{R})$ is parabolic of type $(1,1,2)$:

$$H \cong \left\{ \left(\begin{array}{c|c|cc} * & 0 & 0 & 0 \\ \hline * & * & 0 & 0 \\ \hline * & * & * & * \\ * & * & * & * \end{array} \right) \in \mathrm{GL}(4,\mathbb{R}) \right\}.$$

Note

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3})$ is similar; parabolic of type $(1,1,n-2)$.

Result 2: $\mathfrak{h}^3 \oplus \mathbb{R}$ (3/4)

Note

- $\mathrm{GL}(4, \mathbb{R})/H \cong F_{1,2}(\mathbb{R}^4)$: flag,
- $F_{1,2}(\mathbb{R}^4) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset \mathbb{R}^4 \text{ (subsp)},\ \dim V_k = k\}$.

Intuitive Explanation (1)

- $\mathrm{O}(3,1) \curvearrowright F_{1,2}(\mathbb{R}^4)$ has 6 orbits:

$$V_2 : (2,0) \Rightarrow V_1 : \text{positive},$$
$$V_2 : \text{degenerate} \Rightarrow V_1 : \text{positive/degenerate},$$
$$V_2 : (1,1) \Rightarrow V_1 : \text{positive/degenerate/negative}.$$

Result 2: $\mathfrak{h}^3 \oplus \mathbb{R}$ (4/4)

Note

- $\mathfrak{g} := \mathfrak{h}^3 \oplus \mathbb{R} = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_3$.
- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ normalizes

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{e_3\}, \quad Z(\mathfrak{g}) = \mathrm{Span}\{e_3, e_4\}.$$

Intuitive Explanation (2)

- $\exists$ 6 $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{h}^3 \oplus \mathbb{R}$ such that

$$\langle , \rangle|_{Z(\mathfrak{g})} : (2,0) \Rightarrow \langle , \rangle|_{[\mathfrak{g},\mathfrak{g}]} : \text{positive},$$
$$\langle , \rangle|_{Z(\mathfrak{g})} : \text{degenerate} \Rightarrow \langle , \rangle|_{[\mathfrak{g},\mathfrak{g}]} : \text{positive/degenerate},$$
$$\langle , \rangle|_{Z(\mathfrak{g})} : (1,1) \Rightarrow \langle , \rangle|_{[\mathfrak{g},\mathfrak{g}]} : \text{positive/degenerate/negative}.$$

Summary (1/2)

Summary (list of $\#\mathfrak{PM}_{(p,q)}(\mathfrak{g})$)

$\mathfrak{g}$	positive def.	indefinite	comments
$\mathbb{R}^n$	1	1	
$\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$	1	3	for $\forall(p,q)$
$\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$	1	3	n=3
		6	$n = 4$ with sig. $(3,1)$

Summary (2/2)

Problem

- $\# \, \mathfrak{PM}_{(p,q)}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}) = \, ?$ Is it finite?
- If $\# \, \mathfrak{PM}_{(p,q)}(\mathfrak{g}) < +\infty$, then $\mathfrak{g}$ is one of above?
- Describe $\mathfrak{PM}_{(p,q)}(\mathfrak{g})$ for some $\mathfrak{g}$ (e.g., three/four-dim., ...)

Thank you very much!