Cohomogeneity one actions on non-compact symmetric spaces

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Abstract. The classification of homogeneous codimension one foliations on irreducible Riemannian symmetric spaces of non-compact type will be mentioned. This classification is obtained by the joint work with Dr. Jürgen Berndt ([5]). In this article, we will describe the rough story of the classification and give some explicit examples.

0. Introduction

In this article we consider isometric actions of Lie groups G on connected complete Riemannian manifolds M. The *cohomogeneity* of an action of G on M is defined by the codimension of the regular orbit. An orbit is called *regular* if the dimension is maximal. A transitive action has cohomogeneity zero, since the regular orbit is M itself. Note that the regular orbits of cohomogeneity one actions are homogeneous hypersurfaces in M.

Two isometric actions on a Riemannian manifold M are said to be *orbit equivalent* if there exists an isometry of M mapping the orbits of one of these actions onto the orbits of the other. For a given Riemannian manifold, it is a natural and classical problem to determine the moduli space of all isometric cohomogeneity one actions on M modulo orbit equivalence.

In this article we study isometric cohomogeneity one actions on an irreducible symmetric space M of non-compact type. There are the following two cases (see e.g., [2]).

(F) Every orbit is regular. In this case the set of orbits induces the foliation on M.

(S) The action has a singular orbit. In this case there exists exactly one singular orbit.

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The main purpose of this article is to mention the classification of the actions which satisfy the property (F).

Cohomogeneity one actions on some manifolds have been classified. It was shown by Hsiang and Lawson ([9]) that the moduli space of cohomogeneity one actions on the sphere S^n is isomorphic to the set of $(n+1)$-dimensional symmetric spaces of non-compact type and of rank two. The bijective correspondence is given as follows. Take the isotropy representation of such a symmetric space, which is a representation of a compact Lie group K on $\mathbb{R}^{n+1}$. Note that the rank of a symmetric space coincides with the cohomogeneity of the isotropy representation. Therefore K acts on $\mathbb{R}^{n+1}$ with cohomogeneity two, and on the unit sphere S^n in $\mathbb{R}^{n+1}$ with cohomogeneity one.

Cohomogeneity one actions on simply connected irreducible symmetric spaces of compact type have been classified by Kollross ([13]). The essential tool for his result is the classification of maximal subgroups of the isometry groups. Note that the isometry groups of irreducible symmetric spaces of compact type are compact semi-simple Lie groups. Consequently a compact symmetric space can admit just finitely many cohomogeneity one actions.

We are interested in the cohomogeneity one actions on symmetric spaces of non-compact type M. The method of the above works can not be applied for our case. The group which acts on M with cohomogeneity one is not compact in general, and there are infinitely many maximal subgroups in the isometry groups of M. In fact there are infinitely many cohomogeneity one actions if rank > 1.

E. Cartan ([6]) classified the cohomogeneity one actions on the real hyperbolic spaces $\mathbb{R}\mathrm{H}^n$, which is the symmetric space of non-compact type and of rank one. There are $n+1$ cohomogeneity one actions on $\mathbb{R}\mathrm{H}^n$, which will be mentioned in Section 1. The essential tool for his classification is the Gauss-Codazzi equation for a submanifold, which is too complicated to apply for our case.

Our new strategy to study the cohomogeneity one actions is to use the theory of solvable Lie groups. Let G be the identity component of the full isometry group of M, and $G = KAN$ denote the Iwasawa decomposition (see Appendix A for Iwasawa decompositions). It is well known that the subgroup AN is solvable and M is isometric to AN equipped with certain left-invariant metric. Therefore it seems to be natural to use the solvable Lie groups for the studies on symmetric spaces M of non-compact type. If we take a codimension one subgroup H in AN, then the action of H on $M \cong AN$ has obviously cohomogeneity one and satisfies (F). This is the way to construct examples.

In section 1 we mention the classification of cohomogeneity one actions on the real hyperbolic spaces $\mathbb{R}\mathrm{H}^n$. Two of them induce codimension one foliations. For general irreducible symmetric spaces of non-compact type M, we can construct two types of examples by taking codimension one subgroups in AN. In fact every homogeneous codimension one foliations on M can be constructed in this way up to isometric congruence. The classification is completed by checking which of them are isometrically congruent. We need to know the geometry of foliations to check the congruence, which is studied in Sections 3 and 4.

1. Cohomogeneity one actions on $\mathbb{R}\mathrm{H}^n$

In this section we mention the classification of cohomogeneity one actions on the real hyperbolic spaces, obtained by E. Cartan ([6]). It seems to be good to know this result, since the situation of general cases is quite similar.

Let $\mathbb{R}\mathrm{H}^n = SO^0(n,1)/SO(n)$ be the real hyperbolic space, where $SO^0(n,1)$ is the identity component of the Lorentz group $SO(n,1)$. Take the Iwasawa decomposition $SO^0(n,1) = KAN$. Note that $K = SO(n)$ is a maximal compact subgroup, A is abelian and N is nilpotent. One knows that $S^{(n)} := AN$ is a solvable Lie group of dimension n and we call it the *solvable part*. It is remarkable that $S^{(n)}$ acts simply transitively on $\mathbb{R}\mathrm{H}^n$.

Theorem 1.1 ([6]). *A cohomogeneity one action on $\mathbb{R}\mathrm{H}^n$ is orbit equivalent to one of the followings.*

(1) *The action of N on $\mathbb{R}\mathrm{H}^n$, which satisfies (F).*

(2) *The action of $SO^0(n-1,1)$ on $\mathbb{R}\mathrm{H}^n$, which satisfies (F).*

(3) *The action of $SO^0(n-k,1) \times SO(k)$ on $\mathbb{R}\mathrm{H}^n$, which satisfies (S) for $k = 2, \dots, n$.*

There are $n+1$ cohomogeneity one actions on $\mathbb{R}\mathrm{H}^n$ and two of them satisfy (F). On the action (1) note that the group N is a codimension one subgroup of AN. The orbits of this action are the horospheres in $\mathbb{R}\mathrm{H}^n$, and this action induces the so-called horosphere foliation. On the action (2) the orbit through the origin is a totally geodesic $\mathbb{R}\mathrm{H}^{n-1}$, and the other orbits are the equidistant hypersurfaces. Let us take the solvable part $S^{(n-1)}$ of $SO^0(n-1,1)$. The action (2) is orbit equivalent to the $S^{(n-1)}$-action and $S^{(n-1)}$ is a codimension one subgroup of AN.

Regarding the above results, one can observe the following. To classify cohomogeneity one actions on $\mathbb{R}\mathrm{H}^n$ satisfying (F), it is enough to consider codimension one subgroups of AN. In fact this is true for our general setting. Solvable groups are quite essential for our classification of (F)-actions.

Here we note on the action (3). This action is orbit equivalent to the $S^{(n-k)} \times SO(k)$-action, where $S^{(n-k)}$ denotes the solvable part of $SO^0(n-k,1)$. The singular orbit is a totally geodesic $\mathbb{R}\mathrm{H}^{n-k}$ on which $S^{(n-k)}$ acts simply transitively. The regular orbits are $\mathbb{R}\mathrm{H}^{n-k} \times S^{k-1}$, the tubes around the singular orbit. In this case $S^{(n-k)} \times SO(k)$ is the direct product of a solvable group and a compact one. Therefore solvable groups might play an important role for studying (S)-actions. In fact Berndt and Brück ([4]) obtained many examples of (S)-actions on hyperbolic spaces (i.e., rank one symmetric spaces of non-compact type) in terms of solvable groups.

2. Codimension one subalgebras

Let $M = G/K$ be an irreducible symmetric space of non-compact type, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa decomposition. In this section we classify codimension one subalgebras in $\mathfrak{a} + \mathfrak{n}$.

Let $\xi \in \mathfrak{a} + \mathfrak{n}$ be a non-zero vector and denote by $\mathfrak{s}_\xi$ the orthogonal complement of $\mathbb{R}\xi$ in $\mathfrak{a} + \mathfrak{n}$. We decide the condition for $\mathfrak{s}_\xi$ to be a subalgebra. We need the root system Δ, the root spaces $\mathfrak{g}_\alpha$, the root vectors H_α, the set of simple roots Λ, and the set of positive roots Δ^+. It is known that $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}$ is generated by $\sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha$. See Appendix A for definitions.

Proposition 2.1. *$\mathfrak{s}_\xi$ is a subalgebra if and only if*

(I) $\xi \in \mathfrak{a}$, *or*

(II) $\xi \in \mathbb{R}H_\alpha + \mathfrak{g}_\alpha$ *for some* $\alpha \in \Lambda$.

This proposition can be proved by direct Lie algebraic calculations.

The connected subgroup S_ξ in AN with Lie algebra $\mathfrak{s}_\xi$ acts on M with cohomogeneity one. For the cohomogeneity one actions on real hyperbolic space $\mathbb{R}\mathrm{H}^n$, the group N is of type (I), and the group $S^{(n-1)}$ is of type (II).

These subgroups S_ξ provide infinitely many cohomogeneity one actions. For the classification we have to do the followings :

- determine which of them are orbit equivalent, and
- prove that every cohomogeneity one action is orbit equivalent to one of the above.

In the next two sections we will discuss the geometry of foliations associated with S_ξ-actions, which is useful to study the orbit equivalence. The following is useful to study the geometry.

Lemma 2.2. *The Levi-Civita connection of* AN *is given by*

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle$$

for $X, Y, Z \in \mathfrak{a}+\mathfrak{n}$. *The shape operator* A_ξ *of* $S_\xi \cdot o$ *at* o *with respect to* ξ *is given by*

$$A_\xi : \mathfrak{s}_\xi \to \mathfrak{s}_\xi : X \mapsto \frac{1}{2}[\xi - \theta(\xi), X]_{\mathfrak{s}_\xi},$$

where θ *denotes the Cartan involution and the subscript* $\mathfrak{s}_\xi$ *means the* $\mathfrak{s}_\xi$*-component.*

Therefore the principal curvatures and the mean curvatures can be calculated in terms of Lie algebras.

3. The foliations of type (I)

We study the geometry of the foliations associated with S_ξ for $\xi \in \mathfrak{a}$. This case contains the horosphere foliations on real hyperbolic spaces.

Let $\xi \in \mathfrak{a}$ be a unit vector and put $\ell := \mathbb{R}\xi$. Denote by S_ℓ the simply-connected Lie group whose Lie algebra is $\mathfrak{s}_\ell := (\mathfrak{a} \ominus \ell) + \mathfrak{n}$.

PROPOSITION 3.1. *Every S_ℓ-orbit is conjugate under an isometry.*

Proof. Let o be the origin and $p \in M$ be an arbitrary point. We show that $S_\ell \cdot o$ is conjugate to $S_\ell \cdot p$. Consider the geodesic starting from o with tangent vector ξ, which is $\exp(t\xi) \cdot o$. This geodesic meets $S_\ell \cdot p$ at some point, say $g \cdot o$ where $g \in A$. We can assume $p = g \cdot o$ without loss of generality. One can see that

$$g^{-1} S_\ell \cdot p \;=\; g^{-1} S_\ell g \cdot g^{-1} p \;=\; I_{g^{-1}}(S_\ell) \cdot o \;=\; S_\ell \cdot o.$$

The last equality follows from the fact that $\xi \in \mathfrak{a}$, $g \in A$ and A is abelian. $\square$

Each leaf of the foliation associated with S_ℓ-action is conjugate. Therefore it is enough to calculate the curvatures of $S_\ell \cdot o$.

PROPOSITION 3.2. *The principal curvatures of $S_\ell \cdot o$ are $\{0\} \cup \{\alpha(\xi) \mid \alpha \in \Delta^+\}$. The mean curvature is given by*

$$\mu_\ell = \frac{1}{n-1} \sum_{\alpha \in \Delta^+} (\dim \mathfrak{g}_\alpha) \alpha(\xi),$$

where n is the dimension of M.

Proof. Since $\xi \in \mathfrak{a}$, Lemma 2.2 leads that the shape operator of $S_\ell \cdot o$ is given by $A_\xi(X) = [\xi, X]$. Thus the claim on principal curvatures is completed by the definition of the root systems. The formula for the mean curvature can be obtained by just summing up the principal curvatures. $\square$

By looking at the formula of the mean curvature, which is a polynomial of degree one, one can see that $\mu_\ell = 0$ has a solution if the rank is high.

COROLLARY 3.3. *If $r := \mathrm{rank}(M) \geq 2$ then there exists an $(r-2)$-dimensional family of homogeneous minimal foliations on M.*

A foliation is called *minimal* if every leaf is minimal. A minimal foliation is also called a harmonic foliation, since the canonical projection from M onto the space of leaves is harmonic. We refer [20] for minimal foliations. In fact this corollary is mentioned in [20].

Here we consider the case $M = SL(4, \mathbb{R})/SO(4)$ as an example. See Appendix B for notations.

EXAMPLES 3.4. *Let* $\xi := \mathrm{diag}(e_1, e_2, e_3, e_4) \in \mathfrak{a}$ *be a normal vector and* $\ell := \mathbb{R}\xi$. *The principal curvatures of* $S_\ell \cdot o$ *are* $\{0\} \cup \{e_i - e_j \mid 1 \le i < j \le 4\}$. *The mean curvature is given by* $\mu_\ell = \frac{1}{8}(3e_1 + e_2 - e_3 - 3e_4)$.

Therefore, if $\ell = \mathbb{R} \cdot \mathrm{diag}(1, -1, -1, 1)$ then the associated foliation is minimal. If $\ell = \mathbb{R} \cdot \mathrm{diag}(3, -1, -1, -1)$ then the type number is 3, and if ℓ is generic then the type number is 6.

4. The foliations of type (II)

We study the geometry of the foliations associated with S_ξ for $\xi \in \mathbb{R}H_\alpha \oplus \mathfrak{g}_\alpha$ with $\alpha \in \Lambda$. We always assume that the $\mathfrak{g}_\alpha$-component is nonzero. This case contains the foliations on real hyperbolic spaces $\mathbb{R}\mathrm{H}^n$ whose leaves consist of the totally geodesic $\mathbb{R}\mathrm{H}^{n-1}$ and its equidistant hypersurfaces.

LEMMA 4.1. *Let* $\xi \in \mathfrak{g}_\alpha$ *be a unit vector with* $\alpha \in \Lambda$ *and put*

$$\xi_t := \frac{1}{\cosh(|\alpha|t)}\xi - \frac{1}{|\alpha|}\tanh(|\alpha|t)H_\alpha$$

for $t \in \mathbb{R}$. *Then the* S_ξ*-orbit with oriented distance* t *in direction of* ξ *from* o *is isometrically congruent to the orbit* $S_{\xi_t} \cdot o$.

Proof. The subspace $\mathbb{R}\xi_{\mathfrak{a}} \oplus \mathbb{R}\xi_{\mathfrak{g}_\alpha}$ forms a subalgebra isomorphic to the solvable part of $sl(2, \mathbb{R})$. Therefore it is enough to prove the lemma only on $SL(2, \mathbb{R})$-case. One can show it directly. $\square$

This lemma means that the S_ξ-action is orbit equivalent to the S_{ξ_t}-action. Furthermore, we can identify the set of S_ξ-orbits, $\{S_\xi \cdot p \mid p \in M\}$, with the set of S_{ξ_t}-orbits through o, $\{S_{\xi_t} \cdot o \mid t \in \mathbb{R}\}$. This is useful for studying the geometry of S_ξ-orbits. The following is a direct consequence of Lemma 2.2.

PROPOSITION 4.2. *The shape operator* A_{ξ_t} *of* $S_{\xi_t} \cdot o$ *at* o *with respect to* ξ_t *is given by*

$$A_{\xi_t}(X) = \left[\frac{1}{2\cosh(|\alpha|t)}(\xi - \theta\xi) - \frac{1}{|\alpha|}\tanh(|\alpha|t)H_\alpha,\ X\right]_{\mathfrak{s}_{\xi_t}}$$

for every $X \in \mathfrak{s}_{\xi_t}$.

Therefore the principal curvatures and the mean curvatures can be calculated in terms of Lie algebras. Such calculations lead that

PROPOSITION 4.3. *Let $\xi \in \mathfrak{g}_\alpha$ with $\alpha \in \Lambda$. Then $S_\xi \cdot o$ is the only minimal leaf among the S_ξ-orbits.*

In general the calculations are complicated. We just mention an example.

EXAMPLES 4.4. *Let us consider the symmetric space*

$$M := SL(4, \mathbb{R})/SO(4).$$

Let $\xi \in \mathfrak{g}_{\alpha_1}$ be a unit vector. Then the principal curvatures of $S_\xi \cdot o$ are 0 with multiplicity 4 and $\pm\frac{1}{2}$ with multiplicities 2. The orbit $S_\xi \cdot o$ is minimal.

Proof. In this case we have

$$\xi := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_\xi(X) = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X\right]_{\mathfrak{s}_\xi}.$$

Therefore one can calculate the principal curvatures directly. In fact, one can do it as follows. One has $\xi - \theta(\xi) \in \mathfrak{g}_{\alpha_1} + \mathfrak{g}_{-\alpha_1}$. Therefore if $\alpha \pm \alpha_1$ are not roots then $A_\xi = 0$ on $\mathfrak{g}_\alpha$. Thus $\mathfrak{a} + \mathfrak{g}_{\alpha_3}$ is the principal curvature space of principal curvature 0. Furthermore, A_ξ preserves $\sum \mathfrak{g}_{\alpha + k\alpha_1}$. In this case A_ξ preserves $\mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2}$ and $\mathfrak{g}_{\alpha_2+\alpha_3} + \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$. These observations make the calculations easier. □

In general the principal curvature spaces are closely related to the root strings. For the case $\eta \in \mathfrak{g}_{\alpha_2}$, the above observation leads that the shape operator A_η of $S_\eta \cdot o$ satisfies

$$\begin{aligned} &A_\eta = 0 \quad \text{on } \mathfrak{a} + \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}, \text{ and} \\ &A_\eta \text{ preserves } \mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_1+\alpha_2} \text{ and } \mathfrak{g}_{\alpha_3} + \mathfrak{g}_{\alpha_2+\alpha_3}. \end{aligned}$$

One can check that the principal curvatures of $S_\eta \cdot o$ and $S_\xi \cdot o$ coincide, counted with multiplicities. As we see in Appendix C, these two actions are not orbit equivalent.

The following holds in general.

Proposition 4.5. *Let $\xi \in \mathfrak{g}_\alpha$ and $\eta \in \mathfrak{g}_\beta$ be normal vectors. If $|\alpha| = |\beta|$, then for every $p \in M$, the principal curvatures of $S_\xi \cdot p$ and $S_\eta \cdot p$ coincide with each other, counted with multiplicities.*

5. The sketch of the proof for classification

In this section we mention the sketch of the proof of our main theorem.

Theorem 5.1. *Let M be an irreducible symmetric space of non-compact type. The moduli space $\mathcal{M}$ of isometric actions on M satisfying (F) is*

$$\mathcal{M} \;\cong\; \mathbb{R}\mathrm{P}^{r-1} \cup \{1, \dots, r\} \;/\; \mathrm{Aut}(DD),$$

where $r := \mathrm{rank}(M)$ and $\mathrm{Aut}(DD)$ denotes the automorphism group of the Dynkin diagram.

Note that $\mathrm{Aut}(DD)$ acts on $\mathbb{R}\mathrm{P}^{r-1} \cup \{1, \dots, r\}$ as follows. An element $f \in \mathrm{Aut}(DD)$ is a permutation of the set of simple roots Λ. Therefore f can act on $\{1, \dots, r\}$ identified with Λ. Furthermore, f can be extended to the linear map $\mathfrak{a}^* \to \mathfrak{a}^*$. By taking the metric dual, f can act on $\mathfrak{a}$. We identify $\mathbb{R}\mathrm{P}^{r-1}$ with $\mathrm{P}(\mathfrak{a})$, the projective space of $\mathfrak{a}$. One can induce the natural action of f on $\mathrm{P}(\mathfrak{a})$.

Let us define the map $\mathcal{F}$ from $\mathbb{R}\mathrm{P}^{r-1} \cup \{1, \dots, r\}$ to $\mathcal{M}$. For $\ell \in \mathbb{R}\mathrm{P}^{r-1}$, define $\mathcal{F}(\ell)$ by the orbit equivalence class of the action of S_ℓ, which we constructed in Section 3. Note that we identify $\mathbb{R}\mathrm{P}^{r-1}$ with $\mathrm{P}(\mathfrak{a})$. For $i \in \{1, \dots, r\}$, take $\mathfrak{g}_{\alpha_i} \ni \xi \neq 0$ and define $\mathcal{F}(i)$ by the orbit equivalence class of the action of S_ξ, which we constructed in Section 4. We have to show that $\mathcal{F}(i)$ is well-defined.

Proposition 5.2. *Let $\xi, \eta \in \mathfrak{g}_{\alpha_i}$ be non-zero vectors. Then the actions of S_ξ and S_η are orbit equivalent.*

Proof. We can assume that $\dim \mathfrak{g}_{\alpha_i} > 1$. The centralizer of $\mathfrak{a}$ in K, denoted by K_0, preserves $\mathfrak{a} + \mathfrak{n}$ and acts transitively on the unit sphere in $\mathfrak{g}_{\alpha_i}$ (see e.g., [21]). Therefore, there exists $g \in K_0$ such that $\mathrm{Ad}(g)(\mathfrak{s}_\xi) = \mathfrak{s}_\eta$. Then S_ξ and S_η are congruent. □

Let $f \in \mathrm{Aut}(DD)$. For $\ell \in \mathrm{P}(\mathfrak{a})$, one can see that S_ℓ-action and $S_{f(\ell)}$-action are orbit equivalent by the same arguments as above.

For $\xi \in \mathfrak{g}_{\alpha_i}$ and $\eta \in \mathfrak{g}_{f(\alpha_i)}$, one can show that S_ξ-action and S_η-action are orbit equivalent. Therefore, $\mathcal{F}$ can induce the map from $\mathbb{R}\mathrm{P}^{r-1} \cup \{1, \dots, r\}$ / $\mathrm{Aut}(DD)$ to $\mathcal{M}$.

Now we have only to show that $\mathcal{F}$ is bijective.

Proposition 5.3. *The map $\mathcal{F}$ is surjective.*

Proof. Let S be a Lie group acting on M with (F).

Claim 1: We can assume that S is solvable and acts on M freely. Take Levi-decomposition $S = L \cdot R$, where L is semi-simple and R is solvable. Moreover, let $L = L_K \cdot L_{AN}$ be an Iwasawa decomposition. Then the action of the solvable group $L_{AN} \cdot R$ is orbit equivalent to the S-action. We may decompose $L_{AN} \cdot R = T \cdot B$, where T is compact and B is k-solvable (see e.g. [15]). The group B satisfies the condition of the claim, and orbit equivalent to the S-action.

Claim 2: We can assume that S is contained in AN. Let $\mathfrak{s}$ be the Lie algebra of S. The maximal solvable subalgebras of real semi-simple Lie algebras have been classified by Mostow ([16]). From the classification list and the fact that S acts on M freely, one can see that $\mathfrak{s} \subset \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{t}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let $\mathfrak{s}_{\mathfrak{a}+\mathfrak{n}}$ be the image of the orthogonal projection of $\mathfrak{s}$ onto $\mathfrak{a}+\mathfrak{n}$. One can show that $\mathfrak{s}_{\mathfrak{a}+\mathfrak{n}}$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$. Furthermore, the action of the corresponding sugroup $S_{\mathfrak{a}+\mathfrak{n}}$ of AN is orbit equivalent to S-action.

Therefore we can assume $\mathfrak{s}$ is a codimension one subalgebra. The action of S is orbit equivalent to one of the actions constructed before. $\square$

Proposition 5.4. *The map $\mathcal{F}$ is injective.*

Proof. We will show that, if the two actions constructed by elements of $\mathbb{R}\mathrm{P}^{r-1} \cup \{1, \dots, r\}$ / $\mathrm{Aut}(DD)$ are orbit equivalent, then these groups are related by an element of $\mathrm{Aut}(DD)$.

Claim 1: S_ℓ-action and S_ξ-action can not be orbit equivalent. All of the orbits of S_ℓ are congruent (Proposition 3.1). Among the S_ξ-orbits, if ξ is a vector in a simple root space, $S_\xi \cdot o$ is the only minimal one (Proposition 4.3). Therefore $S_\xi \cdot o$ can not be congruent to any other orbits.

Claim 2: If the actions of S and S' are orbit equivalent, then their Lie algebras are isomorphic. The groups involved in $\mathbb{R}\mathrm{P}^{r-1} \cup$

$\{1, \ldots, r\}$ are complete solvable, that is, the adjoint representation can be represented by upper triangular matrices. Therefore the claim follows from the theorem of Alekseevskii ([1]).

Therefore it is enough to show that

- if $\mathfrak{s}_\ell$ is isomorphic to $\mathfrak{s}_{\ell'}$ then ℓ can be mapped to ℓ' by an element of $\mathrm{Aut}(DD)$,

- if $\mathfrak{s}_\xi$ is isomorphic to $\mathfrak{s}_\eta$, where $\xi \in \mathfrak{g}_{\alpha_i}$ and $\eta \in \mathfrak{g}_{\alpha_j}$, then α_i can be mapped to α_j by an element of $\mathrm{Aut}(DD)$.

For proving these we need long and complicated arguments on Lie algebras. One has to study the structures of solvable Lie algebras $\mathfrak{s}_\ell$ and $\mathfrak{s}_\xi$. See Appendix C in which we will see some examples. □

Appendix

Appendix A. The Iwasawa decomposition

In this section we describe the Iwasawa decompositions of semi-simple Lie algebras. We will see that the root systems are useful for studying the structures of the solvable parts of the Iwasawa decompositions.

Let M be an irreducible symmetric space of non-compact type, and G be the connected component of the isometry group. Fix a point $o \in M$, called the *origin*. Let K be the isotropy subgroup at o, that is, $K := \{g \in G \mid g \cdot o = o\}$. One can express $M = G/K$.

The Cartan involutions. Let s_o be the symmetry at o, which is an involutive isometry of M. The differential of $I_{s_o} : G \to G : g \mapsto s_o \circ g \circ s_o^{-1}$, denoted by θ, is called the *Cartan involution* of $\mathfrak{g}$. The eigenspace decomposition of $\mathfrak{g}$ with respect to θ gives a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, called the *Cartan decomposition.* Note that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$, since M is irreducible.

The natural inner product on $\mathfrak{g}$. Let B be the Kiling form of $\mathfrak{g}$, which is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{k}$. Define the inner product by $\langle X, Y \rangle := -B(X, \theta(Y))$. One can see that $\mathrm{Ad}\,|_{\mathfrak{k}}$ is skew-symmetric and $\mathrm{Ad}\,|_{\mathfrak{m}}$ is symmetric with respect to this inner product.

The root space decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{m}$, which is unique up to conjugation. For $\alpha \in \mathfrak{a}^*$, define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a},\ [H, X] = \alpha(H)X\}.$$

A non-zero α is called a *root* if $\mathfrak{g}_\alpha \neq 0$. Since $\mathrm{Ad}\,|_{\mathfrak{a}}$ is symmetric, one get $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum \mathfrak{g}_\alpha$, which is called the *root space decomposition*. Denote by Δ the set of roots. Note that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

Simple roots. A subset $\Lambda := \{\alpha_1, \ldots, \alpha_r\} \subset \Delta$ is called a *set of simple roots* if (i) Λ is a basis of the dual space of $\mathfrak{a}$, and (ii) every $\alpha \in \Delta$ can be expressed as $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r$, where every c_i is non-negative integer or every c_i is non-positive integer. Note that there exists the unique set of simple roots up to the automorphisms of Δ. Therefore Δ can be decomposed into the positive roots Δ^+ and the negative roots Δ^-. A root α is said to be *highest* if $\alpha + \alpha_i \notin \Delta$ for all $i = 1, \ldots, r$. The highest root is unique.

The natural gradation. Put $\mathfrak{g}_k := \displaystyle\sum_{c_1+\cdots+c_r=k} \mathfrak{g}_{c_1\alpha_1+\cdots+c_r\alpha_r}$. Thus one has the gradation $\mathfrak{g} = \sum_k \mathfrak{g}_k$. Note that the index k runs through from $-\nu$ to ν, where ν is the sum of the coefficients of the highest root. One can easily see that $\theta(\mathfrak{g}_k) = \mathfrak{g}_{-k}$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j. It is known that $\mathfrak{g}_1$ generates $\sum_{i>0} \mathfrak{g}_i$, that is, $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for every $i > 0$ ([12]).

Iwasawa decompositions. Put $\mathfrak{n} := \sum_{i>0} \mathfrak{g}_i$, which is obviously ν-step nilpotent. Thus we have the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, called the *Iwasawa decomposition.* Note that the decomposition is not orthogonal. In fact, $\mathfrak{k} = (\mathfrak{g}_0 \ominus \mathfrak{a}) \oplus \{X + \theta(X) \mid X \in \mathfrak{n}\}$ holds. Take a connected subgroup AN of G with Lie algebra $\mathfrak{a} + \mathfrak{n}$, which acts simply-transitively on M. The symmetric space M is isomorphic to AN endowed with the left-invariant metric induced from $\langle\ ,\ \rangle$.

Appendix B. $SL(4, \mathbb{R})$

In this section we consider the symmetric space $SL(4, \mathbb{R})/SO(4)$ and describe the Iwasawa decomposition of $sl(4, \mathbb{R})$ explicitly.

The Cartan involution of $\mathfrak{g} := sl(4, I\!R)$ is given by $\theta(X) := -{}^tX$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the eigenspace decomposition, where

$$\begin{aligned} \theta = 1 \ \text{ on } \ \mathfrak{k} &= \{X \mid X = -{}^tX\} = so(4), \\ \theta = -1 \ \text{ on } \ \mathfrak{p} &= \{X \mid \mathrm{tr}X = 0,\ X = {}^tX\}. \end{aligned}$$

The inner product on $\mathfrak{g}$ is given by $\langle X, Y\rangle := \mathrm{tr}(X \cdot {}^tY)$.

Next we decide the root system. Let $\mathfrak{a}$ be the subspace of diagonal matrices in $\mathfrak{p}$, which is maximal abelian in $\mathfrak{p}$. Define $\alpha_i \in \mathfrak{a}^*$ by

$$\alpha_i \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & e_4 \end{pmatrix} := e_i - e_{i+1}, \qquad \text{for} \quad i = 1, 2, 3.$$

Direct calculations show that the root system Δ is

$$\Delta = \pm\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3\}.$$

For instance,

$$\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_{\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{\alpha_3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

The subset $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ forms a set of simple roots. Therefore this root system is of A_3-type.

Now one can describe the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. We have already seen $\mathfrak{k}$ and $\mathfrak{a}$. The nilpotent part $\mathfrak{n}$ is given by

$$\mathfrak{n} := \sum_{\alpha > 0} \mathfrak{g}_\alpha = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is generated by $\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_3}$ and is 3-step nilpotent.

Appendix C. The root systems of solvable groups

We will give the definition of the root systems of solvable Lie algebras of Iwasawa type. This is the essential tool to determine the conjugacy of the orbits in our study.

A solvable Lie algebra $\mathfrak{s}$ endowed with an inner product is said to be of *Iwasawa-type* if it satisfies

(i) the orthogonal complement of $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$, denoted by $\mathfrak{a}$, is abelian,

(ii) the operator $\mathrm{ad}(H)$ is symmetric for all $H \in \mathfrak{a}$, and

(iii) $\mathrm{ad}(H_0)$ has positive eigenvalues for some $H_0 \in \mathfrak{a}$.

The typical examples are the solvable parts of the Iwasawa decompositions of semi-simple Lie algebras $\mathfrak{g}$. Note that the inner product is given by $\langle X, Y \rangle := -B(X, \theta(Y))$, where B is the Killing form of $\mathfrak{g}$ and θ is the Cartan involution. The subalgebras $\mathfrak{a} + (\mathfrak{n} \ominus \mathfrak{g}_\alpha)$, where α is simple, are the other examples of solvable Lie algebras of Iwasawa-type.

Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be a solvable Lie algebra of Iwasawa-type. We can define the root system of $\mathfrak{s}$ with respect to $\mathfrak{a}$ in the same way as the case of symmetric spaces. We call $\alpha \in \mathfrak{a}^*$ a *root* if $\mathfrak{n}_\alpha \neq 0$, where

$$\mathfrak{n}_\alpha := \{X \in \mathfrak{n}, \mid \forall H \in \mathfrak{a},\ [H, X] = \alpha(H)X\}.$$

The conditions (i), (ii) in the definition lead that $\mathfrak{n}$ can be decomposed into the sum of the root spaces. We say that a root α is *simple* if it can not be decomposed into the sum of two roots.

Let $\mathfrak{g}$ be a semi-simple Lie algebra. One has the attached symmetric space generated by $\mathfrak{g}$ and the Cartan involution of $\mathfrak{g}$. It is easy to see that the root system of the solvable part of $\mathfrak{g}$ coincides with the set of positive roots of the root system of the attached symmetric space.

Let us define the "Dynkin diagrams" in the following manner. Each simple root represents a vertex. Then connect two vertices α and β each other, where the numbers of the lines depend on the string relations. For instance, if α and β span the A_2-type root system then connect them by one line, if α and β span the B_2-type root system then connect them by two lines and put the arrow, and so on. Note that the Dynkin diagram is an invariant of the isomorphism class of solvable Lie algebras of Iwasawa-type.

Let $\mathfrak{s}$ be the solvable part of the Iwasawa decomposition of $sl(4, \mathbb{R})$. We use the same notations as in Appendix B. Let Δ^+ be the set of positive roots of the symmetric space $SL(4, \mathbb{R})/SO(4)$. Denote by Δ^i the root system of $\mathfrak{s} \ominus \mathfrak{g}_{\alpha_i}$ and by Λ^i the set of its simple roots. One can easily see that

$$\Delta^1 = \Delta^+ - \{\alpha_1\}, \quad \Lambda^1 = \{\alpha_2,\ \alpha_3,\ \alpha_1 + \alpha_2\},$$
$$\Delta^2 = \Delta^+ - \{\alpha_2\}, \quad \Lambda^2 = \{\alpha_1,\ \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3\}.$$

The Dynkin diagram of Λ^1 is of A_3-type, and that of Λ^2 is of $(A_2 + A_2)$-type. Therefore we conclude that $\mathfrak{s} \ominus \mathfrak{g}_{\alpha_1}$ and $\mathfrak{s} \ominus \mathfrak{g}_{\alpha_2}$ can not be isomorphic.

References

1. Alekseevskii, D.V.: Conjugacy of polar factorizations of Lie groups. *Math. USSR Sbornik* **13** (1971), 12-24.
2. Bérard Bergery, L.: Sur de nouvelles variétés riemanniennes d'Einstein. *Publ. Inst. E. Cartan* **4** (1982), 1-60.
3. Berndt, J.: Homogeneous hypersurfaces in hyperbolic spaces. *Math. Z.* **229** (1998), 589-600.
4. Berndt, J., Brück, M.: Cohomogeneity one actions on hyperbolic spaces. To appear in *J. Reine Angew. Math.*
5. Berndt, J., Tamaru, H.: Homogeneous foliations of codimension one on Riemannian symmetric spaces of non-compact type. Preprint.
6. Cartan, E.: Familles de surfaces isoparamétriques dans les espaces à courbure constante. *Ann. Mat. pura appl. IV. s.* **17** (1938), 177-191.
7. Ferus, D., Karcher, H., Münzner, H.-F.: Cliffordalgebren und neue isoparametrische Hyperflächen. *Math. Z.* **177** (1981), 479-502.
8. Helgason, S.: *Differential geometry, Lie groups, and symmetric spaces.* Academic Press, San Diego, 1978.
9. Hsiang, W.-Y., Lawson Jr., H.B.: Minimal submanifolds of low cohomogeneity. *J. Differential Geom.* **5** (1971), 1-38.
10. Iwata, K.: Classification of compact transformation groups on cohomology quaternion projective spaces with codimension one orbits. *Osaka J. Math.* **15** (1978), 475-508.
11. Iwata, K.: Compact transformation groups on rational cohomology Cayley projective plane. *Tôhoku Math. J. II. Ser.* **33** (1981), 429-442.
12. Kaneyuki, S., Asano, H.: Graded Lie algebras and generalized Jordan triple systems. *Nagoya Math. J.* **112** (1988), 81-115.

13. Kollross, A.: A classification of hyperpolar and cohomogeneity one actions. To appear in *Trans. Amer. Math. Soc.*
14. Levi-Civita, T.: Famiglie di superficie isoparametriche nell'ordinario spazio euclideo. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (6)* **26** (1937), 355-362.
15. Malcev, A.: On the theory of the Lie groups in the large. *Mat. Sb. n. Ser.* **16** (1945), 163-190. Correction: ibid. **19** (1946), 523-524.
16. Mostow, G.D.: On maximal subgroups of real Lie groups. *Ann. Math.* **74** (1961), 503-517.
17. Segre, B.: Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (6)* **27** (1938), 203-207.
18. Sugiura, M.: Conjugate classes of Cartan subalgebras in real semisimple Lie algebras. *J. Math. Soc. Japan* **11** (1959), 374-434. Correction: ibid. **23** (1971), 379-383.
19. Takagi, R.: On homogeneous real hypersurfaces in a complex projective space. *Osaka J. Math.* **10** (1973), 495-506.
20. Takagi, R., Yorozu, S.: Minimal foliations on Lie groups. *Tôhoku Math. J.* **36** (1984), 541-554.
21. Tamaru, H.: The local orbit types of symmetric spaces under the actions of the isotropy subgroups. *Diff. Geom. Appl.* **11** (1999), 29-38.
22. Uchida, F.: Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits. *Japan. J. Math.* **3** (1977), 141-189.

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