多変数の複素関数論の入門講義

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このノートでは, 複素領域での多変数の正則関数の基礎的な性質を解説する. 次を読者に理解させるのが目的である. 局所的に考える場合は, 「正則関数を扱うこと」と「収束べき級数を扱うこと」とは同値である.
一変数の正則関数の基礎的事項（２, ３年次の複素関数論）は既知とする.

In this note I will survey some basic results in the theory of holomorphic functions of several variables in the complex domain. The purpose of this section is to explain to the readers the following fact: treating holomorphic functions in the germ sense is almost equivalent to treating convergent power series.
It is assumed that the readers know the basic part of the theory of holomorphic functions of one variable.

Contents:

1. 形式的べき級数と収束べき級数
2. 多変数の正則関数とは？
3. テイラー展開
4. 一致の原理
5. 部分テイラー展開

1. 形式的べき級数と収束べき級数

ここでは, 形式的べき級数と収束べき級数の基本的な事柄を解説する.

In this section we use the following notations: $\boldsymbol{N} = \{0, 1, 2, \ldots\}$, $n \geq 1$, $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$, $a = (a_1, \ldots, a_n) \in \boldsymbol{C}^n$ $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $(z-a)^\alpha = (z_1-a_1)^{\alpha_1} \cdots (z_n-a_n)^{\alpha_n}$.

Definition 1.1. *By a formal power series in z centered at a* we mean a formal series of the form

$$\sum_{\alpha} c_\alpha (z-a)^\alpha \quad \left(= \sum_{\alpha_1,\dots,\alpha_n \in \boldsymbol{N}} c_{\alpha_1,\dots,\alpha_n} (z_1-a_1)^{\alpha_1} \cdots (z_n-a_n)^{\alpha_n} \right)$$

where c_α $(\alpha \in \boldsymbol{N}^n)$ are complex numbers.

Set

$$f(z) = \sum_{\alpha} c_\alpha (z-a)^\alpha.$$

From elementary calculus we know the following:

(1^0) We say that the formal power series $f(z)$ is convergent (or converges to $s \in \boldsymbol{C}$) at $z = z^0 \in \boldsymbol{C}^n$, if the series

$$\sum_{|\alpha| \le N} c_\alpha (z^0-a)^\alpha$$

is convergent (or converges to s) as $N \longrightarrow \infty$. In this case we write

$$\sum_{\alpha} c_\alpha (z^0-a)^\alpha = \lim_{N\to\infty} \sum_{|\alpha| \le N} c_\alpha (z^0-a)^\alpha = s.$$

(2^0) We say that the formal power series $f(z)$ is absolutely convergent at $z = z^0 \in \boldsymbol{C}^n$, if we have

$$\sum_{\alpha} \left| c_\alpha (z^0-a)^\alpha \right| < +\infty.$$

It is easy to see that if $f(z)$ is absolutely convergent at $z = z^0$ then $f(z)$ is convergent at $z = z^0$.

(3^0) If the formal power series $f(z)$ is convergent at $z = z^0 (= (z_1^0, \dots, z_n^0)) \in \boldsymbol{C}^n$, then $f(z)$ is absolutely and uniformly convergent on any compact subset of

$$D = \{ z \in \boldsymbol{C}^n \, ; \, |z_i - a_i| < |z_i^0 - a_i| \quad \text{for } i = 1, \dots, n \}$$

and therefore $f(z)$ defines a continuous function on D.

Lemma 1.2. *Assume that*

$$|c_\alpha| \le \frac{M}{r^\alpha} \left(= \frac{M}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}} \right) \qquad \textit{for any } \alpha \in \boldsymbol{N}^n$$

holds for some $M > 0$ and $r_1 > 0, \dots, r_n > 0$. Then we have:

(1) *The formal power series $f(z) = \sum_\alpha c_\alpha (z-a)^\alpha$ is absolutely convergent at any point in the polydisk*

$$D(a,r) = \{z \in \boldsymbol{C}^n\,;\, |z_i - a_i| < r_i \quad for\ \ i = 1, \dots, n\}$$

(with $r = (r_1, \dots, r_n)$) and therefore $f(z)$ defines a continuous function on $D(a,r)$.

(2) If we write $z_i = x_i + \sqrt{-1}y_i \in \boldsymbol{R} + \sqrt{-1}\boldsymbol{R}$ $(i = 1, \dots, n)$, $f(z)$ is of C^1 class in the $2n$ real variables (x, y) and satisfies the following formula of termwise differentiation:

$$\begin{aligned} \frac{\partial f}{\partial x_i}(z) &= \sum_\alpha c_\alpha \frac{\partial}{\partial x_i}\left((z-a)^\alpha\right), \quad for\ \ i = 1, \dots, n, \\ \frac{\partial f}{\partial y_i}(z) &= \sum_\alpha c_\alpha \frac{\partial}{\partial y_i}\left((z-a)^\alpha\right), \quad for\ \ i = 1, \dots, n. \end{aligned}$$

Proof. (1) is verified as follows. Take any $z^0 \in D(a,r)$. Then we have $|z_i^0 - a_i| < r_i$ for $i = 1, \dots, n$ and

$$\begin{aligned} &\sum_\alpha \left|c_\alpha (z^0 - a)^\alpha\right| \\ &= \sum_{\alpha_1 \geq 0, \dots, \alpha_n \geq 0} |c_\alpha| \left|z_1^0 - a_1\right|^{\alpha_1} \cdots \left|z_n^0 - a_n\right|^{\alpha_n} \\ &\leq \sum_{\alpha_1 \geq 0, \dots, \alpha_n \geq 0} \frac{M}{r_1^{\alpha_1} \cdots r_1^{\alpha_1}} \left|z_1^0 - a_1\right|^{\alpha_1} \cdots \left|z_n^0 - a_n\right|^{\alpha_n} \\ &= \sum_{\alpha_1 \geq 0, \dots, \alpha_n \geq 0} M \left(\frac{|z_1^0 - a_1|}{r_1}\right)^{\alpha_1} \times \cdots \times \left(\frac{|z_n^0 - a_n|}{r_n}\right)^{\alpha_n} \\ &= M \frac{1}{\left(1 - \dfrac{|z_1^0 - a_1|}{r_1}\right)} \times \cdots \times \frac{1}{\left(1 - \dfrac{|z_n^0 - a_n|}{r_n}\right)} \\ &< +\infty, \end{aligned}$$

which implies that $f(z)$ is absolutely convergent at $z = z^0$. Combining this with (3^0) we obtain the result (1).

Note that by differentiating $1/(1-x) = \sum_{p=0}^\infty x^p$ we have $1/(1-x)^2 = \sum_{p=1}^\infty p x^{p-1}$. By using this let us show (2). We have

$$\sum_\alpha c_\alpha \frac{\partial}{\partial x_1}\left((z-a)^\alpha\right) = \sum_\alpha c_\alpha \alpha_1 (z_1 - a_1)^{\alpha_1 - 1} \cdots (z_n - a_n)^{\alpha_n}$$

and for any $z^0 \in D(a,r)$

$$
\begin{aligned}
&\sum_\alpha \left| c_\alpha \alpha_1 (z_1^0 - a_1)^{\alpha_1 - 1} \cdots (z_n^0 - a_n)^{\alpha_n} \right| \\
&= \sum_\alpha |c_\alpha| \ \alpha_1 \ \left| z_1^0 - a_1 \right|^{\alpha_1 - 1} \left| z_2^0 - a_2 \right|^{\alpha_2} \cdots \left| z_n^0 - a_n \right|^{\alpha_n} \\
&\leq \sum_\alpha \frac{M}{r_1^{\alpha_1} \cdots r_1^{\alpha_1}} \ \alpha_1 \ \left| z_1^0 - a_1 \right|^{\alpha_1 - 1} \left| z_2^0 - a_2 \right|^{\alpha_2} \cdots \left| z_n^0 - a_n \right|^{\alpha_n} \\
&= \sum_{\alpha_1 \geq 1, \alpha_2 \geq 0, \ldots, \alpha_n \geq 0} M \, \frac{\alpha_1}{r_1} \left(\frac{|z_1^0 - a_1|}{r_1} \right)^{\alpha_1 - 1} \times \left(\frac{|z_2^0 - a_2|}{r_2} \right)^{\alpha_2} \times \\
&\qquad\qquad\qquad\qquad \times \cdots \times \left(\frac{|z_n^0 - a_n|}{r_n} \right)^{\alpha_n} \\
&= \frac{M}{r_1} \frac{1}{\left(1 - \dfrac{|z_1^0 - a_1|}{r_1} \right)^2} \frac{1}{\left(1 - \dfrac{|z_2^0 - a_2|}{r_2} \right)} \times \cdots \times \frac{1}{\left(1 - \dfrac{|z_n^0 - a_n|}{r_n} \right)} \\
&< +\infty.
\end{aligned}
$$

Since $z^0 \in D(a,r)$ is arbitrary, this implies that $\sum_\alpha c_\alpha (\partial/\partial x_1)((z-a)^\alpha)$ is convergent on any compact subset of $D(a,r)$. Hence by a theorem of term by term differentiation in elementary calculus it is easy to conclude that $f(z) = \sum_\alpha c_\alpha (z-a)^\alpha$ is differentiable in x_1 and

$$
\frac{\partial f}{\partial x_1}(z) = \sum_\alpha c_\alpha \frac{\partial}{\partial x_1} \left((z-a)^\alpha \right),
$$

and that $(\partial f/\partial x_1)(z)$ is continuous in $D(a,r)$.

By the same argument with x_1 being replaced by the other x_i or y_i we can get the result (2).

2. 多変数の正則関数とは？

ここでは, 多変数の正則関数を定義する. 簡単のため, $f(z)$ は領域で C^1 級であることを仮定する. Hartogs の定理などの難しい部分には立ち入らない.

Let $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$, $z_i = x_i + \sqrt{-1} y_i \in \boldsymbol{R} + \sqrt{-1}\boldsymbol{R}$ for $i =$

$1, \dots, n$, and

$$\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right), \qquad i = 1, \dots, n.$$

Definition 2.1. A complex valued function $f(z)$ defined on an open subset D of $\boldsymbol{C}^n$ is said to be a holomorphic function on D, if it satisfies the following conditions 1) and 2):
1) $f(z)$ is of C^1 class on D,
2) $(\partial f/\partial \overline{z}_i)(z) \equiv 0$ on D for $i = 1, \dots, n$.

First we note:

Lemma 2.2. Let $f(z)$ be a complex valued function of C^1 class on D. Then the following conditions (1) and (2) are equivalent:
(1) $f(z)$ is holomorphic on D.
(2) For each $i = 1, \dots, n$, $f(z)$ is holomorphic with respect to z_i as a function of one variable z_i.

Proof. Let $f = u + \sqrt{-1}v$. Then we see

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}_i} &= \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)(u + \sqrt{-1}v) \\ &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial y_i}\right) + \sqrt{-1}\left(\frac{\partial u}{\partial y_i} + \frac{\partial v}{\partial x_i}\right)\right] \end{aligned}$$

and therefore $\partial f/\partial \overline{z}_i \equiv 0$ is equivalent to

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i} \quad \text{and} \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}$$

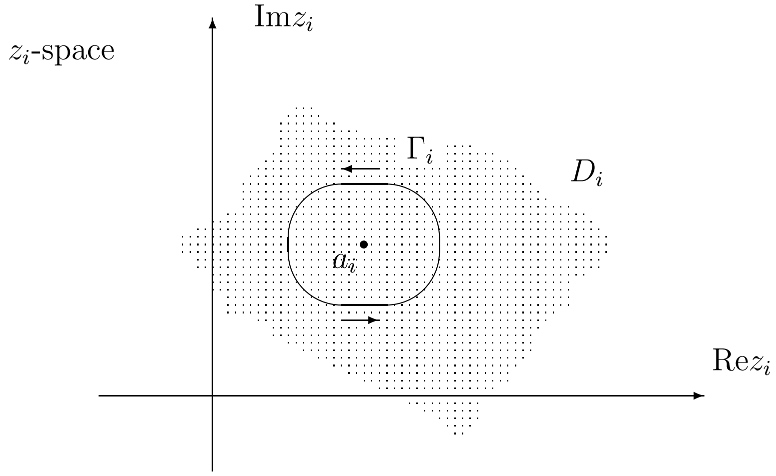
which implies that u and v satisfy the Cauchy-Riemann's relation with respect to the variable z_i. In other words, $\partial f/\partial \overline{z}_i \equiv 0$ is equivalent to that $f(z)$ is holomorphic with respect to z_i as a function of one variable z_i. This immediately leads us to Lemma 2.2.

Next let us see the Cauchy's integral formula:

Theorem 2.3. (Cauchy's integral formula). Let $D = D_1 \times \cdots \times D_n$ be a domain in $\boldsymbol{C}^n$ and let $f(z)$ $(= f(z_1, \dots, z_n))$ be a holomorphic function on D. Then for any $a = (a_1, \dots, a_n) \in D$ we have

$$f(a) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - a_1)\cdots(\zeta_n - a_n)} d\zeta_1 \cdots d\zeta_n,$$

where Γ_i is a Jordan closed curve in D_i $(i = 1, \dots, n)$ illustrated in the following picture.



Proof. For simplicity let us consider the case $n = 2$. Since $f(z_1, a_2)$ is a holomorphic function in z_1 (as a function of one variable z_1), by applying the Cauchy's integral formula in one variable case to $f(z_1, a_2)$ we have

$$f(a_1, a_2) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{f(\zeta_1, a_2)}{(\zeta_1 - a_1)} d\zeta_1.$$

Moreover, since the integrant $f(\zeta_1, z_2)$ is a holomorphic function in z_2, by applying the Cauchy's integral formula again to $f(\zeta_1, a_2)$ we have

$$f(a_1, a_2) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{1}{(\zeta_1 - a_1)} \left\{ \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_2 - a_2)} d\zeta_2 \right\} d\zeta_1.$$

This proves Theorem 2.3, because the repeated integration is expressed in the form of the double integration.

By applying $(\partial/\partial a)^\alpha$ $(= (\partial/\partial a_1)^{\alpha_1} \cdots (\partial/\partial a_n)^{\alpha_n})$ to both sides of the Cauchy's integral formula in Theorem 2.3 and by using a theorem of differentiation under integration in the elementary calculus we obtain

Corollary 2.4. Under Theorem 2.3 we have

$$f^{(\alpha)}(a)\left(=\frac{\partial^{|\alpha|}f}{\partial {z_1}^{\alpha_1}\cdots\partial {z_n}^{\alpha_n}}(a)\right)$$
$$=\frac{\alpha!}{(2\pi\sqrt{-1})^n}\int_{\Gamma_1}\cdots\int_{\Gamma_n}\frac{f(\zeta_1,\ldots,\zeta_n)}{(\zeta_1-a_1)^{\alpha_1+1}\cdots(\zeta_n-a_n)^{\alpha_n+1}}d\zeta_1\cdots d\zeta_n.$$

3. テイラー展開

ここでは,

1. 正則関数は収束べき級数に展開されること, 及び
2. 収束べき級数は正則関数を定義すること,

の２つを証明する.

In this section we will see that holomorphic functions are expanded into convergent power series.

Proposition 3.1 *(*Taylor expansions: *from holomorphic functions to convergent power series).* Let $r=(r_1,\ldots,r_n)$ with $r_1>0,\ldots,r_n>0$, and $a=(a_1,\ldots,a_n)\in \boldsymbol{C}^n$. If $f(z)$ is a holomorphic function in a neighborhood of $K=\{z\in\boldsymbol{C}^n\,;\,|z_i-a_i|\le r_i\, i=1,\ldots,n\}$, $f(z)$ is expanded into the power series

$$(*)\qquad f(z)=\sum_{\alpha\in\boldsymbol{N}^n}c_\alpha(z-a)^\alpha\quad\text{with}\ \ c_\alpha=\frac{1}{\alpha!}f^{(\alpha)}(a)$$

which is convergent at any point in $D(a,r)$. Moreover we see that the coefficients c_α ($\alpha\in\boldsymbol{N}^n$) satisfy the estimates

$$|c_\alpha|\le\frac{M}{r^\alpha}\quad\text{for any}\ \ \alpha\in\boldsymbol{N}^n$$

where M is the maximum of $|f(z)|$ on K.

The power series $(*)$ is called the Taylor expansion of $f(z)$ centered at $z=a$. When we get $(*)$ from $f(z)$, we say that we expand $f(z)$ into the Taylor series at $z=a$.

Proof of Proposition 3.1. Let $\Gamma_i = \{z_i \in \boldsymbol{C}\,;\, |z_i - a_i| = r_i\}$ $(i = 1, \dots, n)$. Take any $z = (z_1, \dots, z_n) \in D(a, r)$. Then by Cauchy's integral formula we have

$$(**) \qquad f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Since ζ_i moves in Γ_i, we have $|\zeta_i - a_i| = r_i$ and therefore $|z_i - a_i| < r_i = |\zeta_i - a_i|$ that is

$$\left| \frac{z_i - a_i}{\zeta_i - a_i} \right| < 1.$$

Hence

$$\begin{aligned} \frac{1}{\zeta_i - z_i} &= \frac{1}{(\zeta_i - a_i) - (z_i - a_i)} = \frac{1}{\zeta_i - a_i} \frac{1}{1 - \left(\dfrac{z_i - a_i}{\zeta_i - a_i} \right)} \\ &= \frac{1}{\zeta_i - a_i} \sum_{\alpha_i = 0}^{\infty} \left(\frac{z_i - a_i}{\zeta_i - a_i} \right)^{\alpha_i} = \sum_{\alpha_i \geq 0} \frac{(z_i - a_i)^{\alpha_i}}{(\zeta_i - a_i)^{\alpha_i + 1}} \end{aligned}$$

which is absolutely convergent. By sustituting this into $(**)$ and by using a theorem of termwise integration in the elementary calculus we obtain

$$\begin{aligned} f(z) &= \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\alpha_1 \geq 0, \dots, \alpha_n \geq 0} \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(\zeta_1, \dots, \zeta_n) \times \\ &\qquad \times \frac{(z_1 - a_1)^{\alpha_1}}{(\zeta_1 - a_1)^{\alpha_1 + 1}} \cdots \frac{(z_n - a_n)^{\alpha_n}}{(\zeta_n - a_n)^{\alpha_n + 1}} d\zeta_1 \cdots d\zeta_n \\ &= \sum_{\alpha_1 \geq 0, \dots, \alpha_n \geq 0} c_\alpha (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n} \end{aligned}$$

where

$$\begin{aligned} c_\alpha &= \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - a_1)^{\alpha_1 + 1} \cdots (\zeta_n - a_n)^{\alpha_n + 1}} d\zeta_1 \cdots d\zeta_n \\ &= \frac{1}{\alpha!} f^{(\alpha)}(a). \end{aligned}$$

This proves the former half of Proposition 3.1.

Next let us prove the estimates of c_α. Since $|f(\zeta)| \leq M$ for $\zeta \in \Gamma =$

$\Gamma_1 \times \cdots \times \Gamma_n$, we see

$$\begin{aligned}
|c_\alpha| &\le \frac{1}{(2\pi)^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{M}{|\zeta_1 - a_1|^{\alpha_1+1} \cdots |\zeta_n - a_n|^{\alpha_n+1}} |d\zeta_1| \cdots |d\zeta_n| \\
&= \frac{1}{(2\pi)^n} \frac{M}{r_1^{\alpha_1+1} \cdots r_n^{\alpha_n+1}} \int_{\Gamma_1} \cdots \int_{\Gamma_n} |d\zeta_1| \cdots |d\zeta_n| \\
&= \frac{1}{(2\pi)^n} \frac{M}{r_1^{\alpha_1+1} \cdots r_n^{\alpha_n+1}} (2\pi r_1) \cdots (2\pi r_n) \\
&= \frac{M}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}}.
\end{aligned}$$

Proposition 3.2 *(From convergent power series to holomorphic functions).* If

$$|c_\alpha| \le \frac{M}{r^\alpha} \left(= \frac{M}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}} \right) \quad \text{for any } \alpha \in \boldsymbol{N}^n$$

holds for some $M > 0$ and $r_1 > 0, \ldots, r_n > 0$, the power series

$$f(z) = \sum_{\alpha \in \boldsymbol{N}^n} c_\alpha (z - a)^\alpha$$

defines a holomorphic function on $D(a, r)$.

Proof of Proposition 3.2. By Lemma 1.2 we already know:
1) f defines a function of C^1 class on $D(a, r)$.
2) By term by term differentiation we have

$$\frac{\partial f}{\partial \overline{z}_i}(z) = \sum_\alpha c_\alpha \frac{\partial}{\partial \overline{z}_i} \left((z - a)^\alpha \right).$$

Since $(z_i - a_i)^{\alpha_i}$ is a polynomial in z_i, it is holomorphic in z_i and so $(\partial / \partial \overline{z}_i)(z - a)^\alpha \equiv 0$. Thus we obtain $(\partial f / \partial \overline{z}_i) \equiv 0$ for any $i = 1, \ldots, n$.

4. 一致の原理

ここでは, 正則関数の「一致の原理」について解説する. この一致の原理より「正則関数と収束べき級数とが一対一に対応している」ことが分かる.

Theorem 4.1 (Principle of analytic continuation). Let $f(z)$ be a holomorphic function on a domain (*i.e.* a connected open set) D in $\boldsymbol{C}^n$, and let $a \in D$. The following three conditions are equivalent to each other:

(1) $f \equiv 0$ on D;

(2) $f \equiv 0$ in a neighborhood of a;

(3) $f^{(\alpha)}(a) = 0$ for any $\alpha \in \boldsymbol{N}^n$.

Proof. It is clear that (1) implies (2) and that (2) implies (3). Since f is holomorphic on D, by Taylor expansion we know that f is expressed in the form

$$f(z) = \sum_{\alpha} \frac{f^{(\alpha)}(a)}{\alpha!}(z-a)^{\alpha}$$

in a neighborhood of $z = a$. Hence it is also clear that (3) implies (2).

Let us show that (2) implies (1). Assume (2). Set $E = \{z^0 \in D\,;\, f \equiv 0 \text{ in a neighborhood of } z^0\}$. If we prove (i) $E \neq \emptyset$, (ii) E is open in D, and (iii) E is closed in D, then by the connectedness of D we have $E = D$ and hence (1). Thus we have only to show (i), (ii) and (iii). Since (2) implies $a \in E$, we have (i). (ii) is clear from the definition of E. To prove (iii) it is enough to show that if a sequence $\{z_{(p)}\}$ in E converges to $z^0 \in D$ (as $p \longrightarrow \infty$) then we have $z^0 \in E$. This is verifies as follows.

Let $\{z_{(p)}\}$ be a sequence in E and assume that it converges to $z^0 \in D$ (as $p \longrightarrow \infty$). Since $z_{(p)} \in E$, we have $f^{(\alpha)}(z_{(p)}) = 0$ for any $\alpha \in \boldsymbol{N}^n$ and hence

$$f^{(\alpha)}(z^0) = \lim_{p\to\infty} f^{(\alpha)}(z_(p)) = 0 \qquad \text{for any } \alpha \in \boldsymbol{N}^n.$$

This implies that $f(z) \equiv 0$ in a neighborhood of $z = z^0$, since it is already proved that (2) and (3) are equivalent. Thus we obtain $z^0 \in D$.

5. 部分テイラー展開

偏微分方程式論では, 一部の変数のみに関するテイラー展開を使うことが暫々ある. 簡単にまとめておく.

Denote: $z = (z_1, z')$, $z' = (z_2, \dots, z_n)$, $r_1 > 0$, $r' = (r_2, \dots, r_n)$, and

$$\begin{aligned} D'(a', r') &= \{z' \in \boldsymbol{C}^{n-1}\,;\, |z_i - a_i| < r_i,\ i = 2, \dots, n\}, \\ K' &= \{z' \in \boldsymbol{C}^{n-1}\,;\, |z_i - a_i| \le r_i,\ i = 2, \dots, n\}. \end{aligned}$$

Proposition 5.1 (From holomorphic functions to convergent power series). Let $r_1 > 0, r_2 > 0, \dots, r_n > 0$ and $a = (a_1, a_2, \dots, a_n) \in \boldsymbol{C}^n$. If $f(z)$ is a holomorphic function in a neighborhood of $K = \{z \in \boldsymbol{C}^n\,;\, |z_i - a_i| \le r_i,\ i = 1, \dots, n\}$, $f(z)$ is expanded into the power series

$$f(z_1, z') = \sum_{p=0}^{\infty} \varphi_p(z')(z_1 - a_1)^p$$

which is convergent at any point in $D_1(a_1, r_1) \times K'$. Moreover we have:

1) $\varphi_p(z')$ is given by

$$\begin{aligned} \varphi_p(z') &= \frac{1}{p!}\frac{\partial^p f}{\partial z_1^p}(a_1, z') \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, z')}{(\zeta_1 - a_1)^{p+1}} d\zeta_1. \end{aligned}$$

2) $\varphi_p(z')$ is holomorphic in a neighborhood of K'.

3) If we set $M = \max_{z \in K} |f(z)|$, then we have

$$|\varphi_p(z')| \le \frac{M}{r_1^p} \quad \text{on } K' \text{ for any } p = 0, 1, 2, \dots.$$

Proof. Since $f(z_1, z')$ is a holomorphic function in z_1, by the Cauchy's integral formula we have

$$f(z_1, z') = \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, z')}{(\zeta_1 - z_1)} d\zeta_1.$$

Applying the same argument to this formula as in the proof of Proposition 3.1 we can easily obtain Proposition 5.1.

Proposition 5.2 (From convergent power series to holomorphic functions). If $\varphi_p(z')$ $(p = 0, 1, 2, \dots)$ are holomorphic functions in a neighborhood of K' and if the estimates

$$|\varphi_p(z')| \le \frac{M}{r_1^p} \quad \text{on } K' \text{ for any } p = 0, 1, 2, \dots$$

is valid for some $M > 0$ and $r_1 >$, then the series

$$f(z_1, z') = \sum_{p=0}^{\infty} \varphi_p(z')(z_1 - a_1)^p$$

defines a holomorphic function on $D(a, r)$.

Proof. By Proposition 3.1 we see that $\varphi_p(z')$ is expanded into the form

$$\varphi_p(z') = \sum_{\alpha'} c_{p,\alpha'}(z' - a')^{\alpha'}$$

for any $z' \in D'(a', r')$ and the coefficients $c_{p,\alpha'}$ satisfy

$$|c_{p,\alpha'}| \leq \frac{M}{r_1^p} \frac{1}{r_2^{\alpha_2} \cdots r_n^{\alpha_n}}.$$

Then by these estimates and Proposition 3.2 we see that

$$g(z) = \sum_{p,\alpha'} c_{p,\alpha'}(z_1 - a_1)^p (z' - a')^{\alpha'}$$

defines a holomorphic function on $D(a, r)$. Since $g(z)$ is absolutely convergent at any point $z \in D(a, r)$, by changing the order of summations we have

$$\begin{aligned} g(z) &= \sum_{p=0}^{\infty} \left(\sum_{\alpha'} c_{p,\alpha'}(z' - a')^{\alpha'} \right) (z_1 - a_1)^p \\ &= \sum_{p=0}^{\infty} \varphi_p(z')(z_1 - a_1)^p = f(z_1, z'). \end{aligned}$$

This proves Proposition 5.2.

Corollary 5.3. If $\varphi_p(z')$ $(p = 0, 1, 2, \ldots)$ are holomorphic functions in a neighborhood of K' and if the series

$$\sum_{p=0}^{\infty} \left(\max_{z' \in K'} |\varphi_p(z')| \right) (z_1 - a_1)^p$$

is convergent on $D_1(a_1, r_1)$, then the series

$$f(z_1, z') = \sum_{p=0}^{\infty} \varphi_p(z')(z_1 - a_1)^p$$

defines a holomorphic function on $D(a,r)$.

Proof. Put $M_p = \max_{z' \in K'} |\varphi_p(z')|$. By the assumption we know that the series

$$h(z_1) = \sum_{p=0}^{\infty} M_p (z_1 - a_1)^p$$

is convergent on $D_1(a_1, r_1)$.

Take any $0 < \rho < r_1$. Then $h(z_1)$ is holomorphic in a neiborhood of $\{z_1 \in \boldsymbol{C}\,;\, |z_1 - a_1| \le \rho\}$ and $|h(z_1)| \le L$ for some $L > 0$. Hence by Proposition 3.1 we have $M_p \le L/\rho^p$ for any $p = 0, 1, 2, \ldots$; this implies that

$$|\varphi_p(z')| \le \frac{L}{\rho^p} \quad \text{on } K' \text{ for any } p = 0, 1, 2, \ldots.$$

Thus by applying Proposition 5.2 we see that

$$f(z_1, z') = \sum_{p=0}^{\infty} \varphi_p(z')(z_1 - a_1)^p$$

defines a holomorphic function on $D_1(a_1, \rho) \times D'(a', r')$.

Since ρ is an arbitrary number with $0 < \rho < r_1$, this concludes that $f(z_1, z')$ defines a holomorphic function on $D(a,r)$.

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