For a quasi-projective variety S over a field,

$\mathrm{ICH}^r(S)$, the intersection Chow group, is defined;

properties (some of which are conjectural) are discussed.

Cf. Barthel, Brasselet, Fieseler, Gabber and Kaup: Relèvement de cycles algébriques et homomorphismses associés en homologie d'intersection, Ann. Math. 141 (1995).

S : variety / $\mathbb{C}$, $d = \dim S$

$$\begin{array}{ccccc} CHC^r(S) & \longrightarrow & ICH^r(S) & \xrightarrow{①} & CH_{d-r}(S) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2r}(S) & \longrightarrow & IH^{2r}(S) & \longrightarrow & H^{BM}_{2(d-r)}(S) \end{array}$$

$CH_*(S)$ is Chow group of S ($\otimes \mathbb{Q}$),

$CHC^r(S)$ is Chow cohomology.

① conjecturally exists and <u>surjective</u>.

<u>Thm.</u> [BBFGK]

$$\mathrm{Im}\left[CH_{d-r}(S) \to H^{BM}_{2(d-r)}(S) \right]$$

$$\subset \mathrm{Im}\left[IH^{2r}(S) \longrightarrow H^{BM}_{2(d-r)}(S) \right].$$

Let $p : X \to S$ be a projective map (with X smooth).

There is a Whitney stratification

$$S = S_0 \supset S_1 \supset \cdots \supset S_\alpha \supset \cdots$$

of S, and resolutions
$\tilde{X}_\alpha \to X_\alpha = p^{-1}S_\alpha$ such that
$\tilde{X}_\alpha \to S_\alpha$ is smooth over $S_\alpha - S_{\alpha+1}$.

$$\begin{array}{ccc} & & \tilde{X}_\alpha \\ & \overset{i_\alpha}{\swarrow} & \downarrow \\ X & \supset & X_\alpha \\ p\downarrow & & \downarrow \\ S & \supset & S_\alpha \end{array}$$

Now take $p : X \to S$ to be a resolution of singularities. One has $(d = \dim S)$

$$\mathrm{CH}_{d-r}(\tilde{X}_\alpha) \stackrel{\iota_{\alpha *}}{\to} \mathrm{CH}^r(X) \stackrel{\iota_\alpha^*}{\to} \mathrm{CH}^r(\tilde{X}_\alpha).$$

Each group has a filtration $F_S^\bullet$ (to be explained later).

Define intersection Chow group by:

$\mathrm{ICH}^r(S) :=$

$$\frac{\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \, \mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha *} F_S^{2r-d+1} \, \mathrm{CH}_{d-r}(\tilde{X}_\alpha)}$$

Theorem. $\mathrm{ICH}^r(S)$ is well-defined (indep. of choice of stratification and resolution).

There is a map
$\mathrm{ICH}^r(S) \to IH^{2r}(S)$.

Bloch, Beilinson, Murre, Shuji Saito (for case $S = \mathrm{Spec}\, k$).

Example. X smooth projective variety.

$CH^r(X)$

$\supset F^1\,\mathrm{CH}^r(X)$ homologically trivial

$\supset F^2\,\mathrm{CH}^r(X)$ Kernel of Abel-Jacobi map ?

Relative canonical filtration. Let X be smooth, and $p : X \to S$ be a projective map.

There is a filtration $F_S^\bullet$ on $\mathrm{CH}^r(X)$ satisfying:

(1) $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$.

Functorial: for $q : W \to S$ and

$\Gamma \in \mathrm{CH}_{\dim X+s}(W \times_S X)$, the induced map

$\Gamma_* : \mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)$

respects $F_S^{\bullet}$.

(2) If the induced map $[\Gamma]$: ${}^p\mathcal{H}^{2r+2s-\nu} Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu} Rp_*\mathbb{Q}_X$ is zero, then Γ_* sends F_S^{ν} to $F_S^{\nu+1}$.

(3) The filtration is the smallest with properties (1) and (2).

Proposition. Under $\boxed{\text{Conjectures}}$, $F_S^{\nu}\,\mathrm{CH}^r(X) = 0$ for $\nu >> 0$.

Theorem 1 (Under $\boxed{\text{Conjectures}}$) There is a natural surjective map $\mathrm{ICH}^r(S) \to \mathrm{CH}^r(S)$.

<u>Theorem</u> 2. (<u>Without</u> Conjectures)

$$\mathrm{Im}\,[\,\mathrm{CH}_{d-r}(S) \to H^{BM}_{2(d-r)}(S)\,]$$
$$\subset \mathrm{Im}\,[\,\mathrm{IH}^{2r}(S) \to H^{BM}_{2(d-r)}(S)\,].$$

Proof of <u>Thm 1.</u> in special case.

Assume :

$$\begin{array}{ccccc} & & \tilde{Y} & & \\ & \overset{\iota}{\swarrow} & \downarrow & \searrow^{q} & \\ X & \hookleftarrow & Y & & \\ p\downarrow & & \downarrow & & \\ S & \hookleftarrow & \Sigma & & \end{array}$$

$\tilde{Y}$: smooth over Σ

$\dim Y = d'$, $\dim q = e$

$\dim \Sigma = d' - e$.

Must show : For $\forall a \in CH_{d+r}(S)$

$\exists\, b \in CH^r(X)$ s.t.

(i) $(\iota^*)^{-1} b \in F_S^{2r-d+1} CH^r(\tilde{Y})$,

(ii) $p_*(b) = a$.

To prove, take any b satisfying (ii), and modify by $\iota_*(C)$ using the following three lemmas.

Under conjectures,

① In the sequence

$$Gr^{\nu}_{F_S} CH_{d-r}(\tilde{Y}) \xrightarrow{\iota_*} Gr^{\nu}_{F_S} CH^r(X) \xrightarrow{\iota^*} Gr^{\nu}_{F_S} CH^r(\tilde{Y})$$

ι^* is injective for $\nu \neq 2r-d$

ι_* is surjective for $\nu \neq 2r-d$.

② $CH^r(\tilde{Y}) = F_S^{2r-d'+e} CH^r(\tilde{Y})$

[perverse degree of $Rq_* \mathbb{Q}_Y$ is in $[d'-e, d'+e]$]

③ $F_S^{2r-d'-e} CH_{d-r}(\Sigma) = 0$

[perv. degree of $D_\Sigma^{\cdot}$ is $\geq -\dim \Sigma$]

Conjectures

Grothendieck's Standard conjecture ($\Rightarrow$ semi-simplicity of the category of pure homological motives).

Bloch-Beilinson-Murre: Existence of Chow-Künneth decomposition (with properties...)

($\Rightarrow$ $h(X) = \bigoplus h^i(X)$ in the category of Chow motives.)

Beilinson-Soulé: vanishing of motivic cohomology with negative degree.

Topological theory:

$D_c^b(S)$: derived category of sheaves with cohomology constructible sheaves;

For a map $f: S \to S'$, $f^*, f_*, f^!, f_!$;

Poincaré-Verdier duality formulas;

For $p: X \to S$,
$H^i(X, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_S, Rp_*\mathbb{Z}_X[i])$.

perverse t-structure. In particular, perverse cohomology functors

$$^p\mathcal{H}^\nu : D_c^b(S) \to Perv(S).$$

Motivic theory:

$\mathcal{D}(S)$: triangulated category of motives over S:

For a map f, $f^*, f_*, f^!, f_!$;

For $p : X \to S$, $H^i_{\mathcal{M}}(X, \mathbb{Z}(r)) = \mathrm{Hom}(\mathbb{Z}_S(0), Rp_*\mathbb{Z}_X(r)[i])$.

Poincaré-Verdier duality formulas;

perverse t-structure.

Realization functor

$$\rho : \mathcal{D}(S) \to D^b_c(S).$$

Theorem. (assume $\mathrm{ch} = 0$ for simplicity) There is a triangulated category $\mathcal{D}(S)$ (called the category of mixed motives over S) with properties:

(1) There is a functor h : $(\text{Quasi-Projective}\,/S)^{opp} \to \mathcal{D}(S)$

There are Tate objects $\mathbb{Z}_S(r)$.

(2) Natural isomorphism $\mathrm{Hom}(\mathbb{Z}_S(0), h(X/S) \otimes \mathbb{Z}(r)[2r-n]) = \mathrm{CH}^r(X, n)$.

In particular,

$\mathrm{Hom}(\mathbb{Z}_S(0), h(X/S) \otimes \mathbb{Z}_S(r)[2r]) = \mathrm{CH}^r(X)$.

(4) There are functors $\otimes$, f^*, f_*, $f^!$, $f_!$ among the categories $\mathcal{D}(S)$,

satisfying the correct properties (such as Verdier duality).

(5) ($k \subset \mathbb{C}$) There is the realization functor

$$\rho : \mathcal{D}(S) \to D^b_c(S(\mathbb{C}))$$

such that $\rho \circ h$ is the cohomology functor for varietites.

From now, write $\mathcal{D}(S)$ for $\mathcal{D}(S)_{\mathbb{Q}}$.

Theorem. (Under the conjectures of Grothendieck, Bloch-Beilinson-Murre, and Beilinson-Soulé)

(1) There is a Whitney stratification $\{S_\alpha\}$ of S, local systems $\mathcal{V}^i_\alpha$ on $S_\alpha - S_{\alpha+1}$, and a non-canonical direct sum decomposition

$h(X/S) = \bigoplus_{i,\alpha} h^i_\alpha(X/S)$ in $\mathcal{D}(S)$

such that $\rho(h^i_\alpha(X/S)) \cong IC_{S_\alpha}(\mathcal{V}^i_\alpha)[-i + \dim S_\alpha]$.

(Work with Corti)

(2) There is a t-structure on $\mathcal{D}(S)_{\mathbb{Q}}$ such that ρ is compatible with it and the perverse t-structure on $D^b_c(S)$. (As a consequence, there is an abelian subcategory $M\mathcal{M}(S)$, and functors

$$^p\mathcal{H}^\nu : \mathcal{D}(S) \to M\mathcal{M}(S).)$$

The category $M\mathcal{M}(S)$ is abelian, and the induced functor $M\mathcal{M}(S) \to Perv(S)$ is exact and faithful.