Algebraic links with 3-bridge presentations

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Outline

1. Introduction
   - n-bridge links
   - Goal
   - Algebraic links
2. Results
   - Non-Montesinos case
   - Montesinos case
3. Proofs
   - Algebraic links vs Graph manifolds
   - 3-bridge spheres vs genus-2 Heegaard surfaces
   - Proofs of the Main Theorems

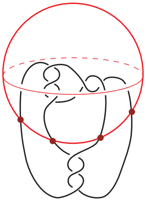
n-bridge links

n-bridge links

- $\mathbf{2}$-bridge link

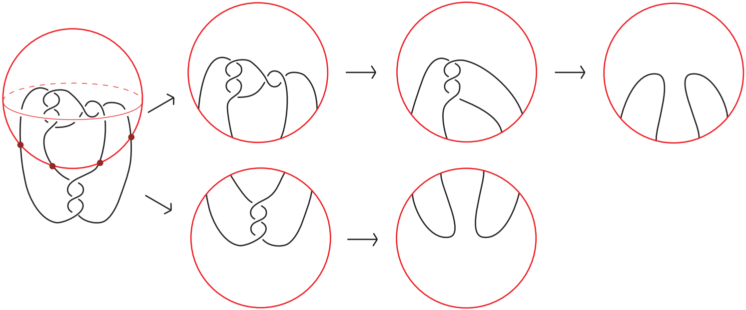
n-bridge links

- 2-bridge link



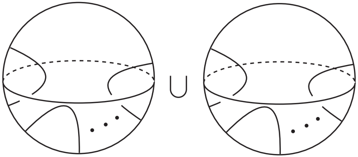
n-bridge links

- **2**-bridge link



n-bridge links

- *n*-bridge presentation



n-bridge links

Theorem (Schubert) Two-bridge links are completely classified. Moreover, each 2-bridge link admits a unique 2-bridge presentation up to isotopy.

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Question How about for 3-bridge links?

— We focus on algebraic links (in the sense of Conway).

Goal

Goal :

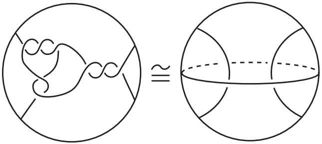
- Classification of 3-bridge algebraic links
- Classification of their 3-bridge spheres

Algebraic links

Rational tangles

B^3 : 3-ball, T : two arcs properly embedded in B^3

- rational tangle (B^3, T) :



Rational tangles

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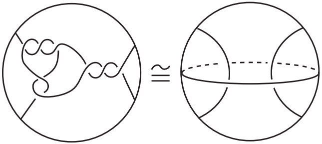
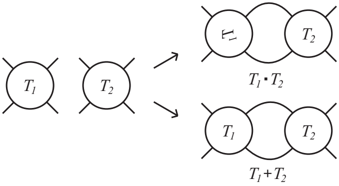


$$3 + \cfrac{1}{-2 + \cfrac{1}{3}} = \frac{12}{5}$$

Algebraic links

- algebraic tangle : a tangle obtained from rational tangles by the following operations



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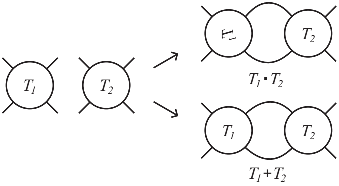
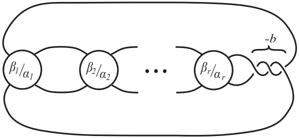


- algebraic link : a link obtained from two algebraic tangles by glueing their boundaries

Montesinos links

- Montesinos links $L(b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))$



a Montesinos link

Montesinos links

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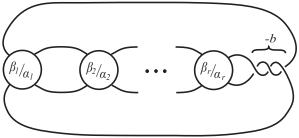
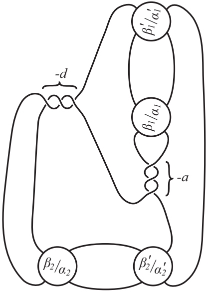


a Montesinos link

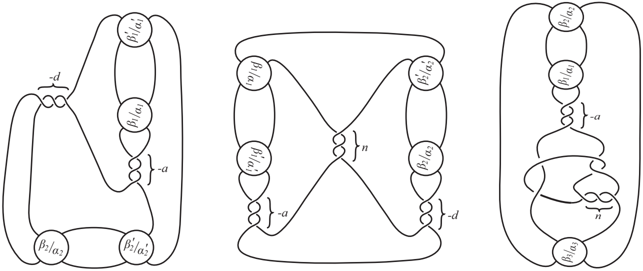


not a Montesinos link

Main Theorem (Non-Montesinos case)

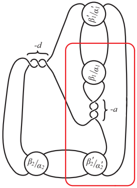
Non-Montesinos case

Theorem 1 Let L be a 3-bridge algebraic link and suppose that L is not a Montesinos link. Then L is isotopic to one of the following types of links.

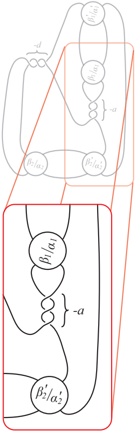


(1) (2) n>1 (3)

Non-Montesinos case



Non-Montesinos case



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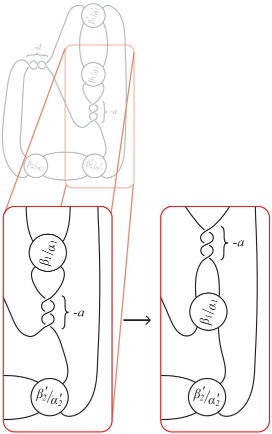

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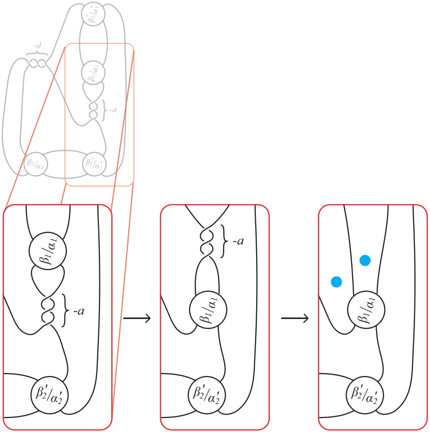

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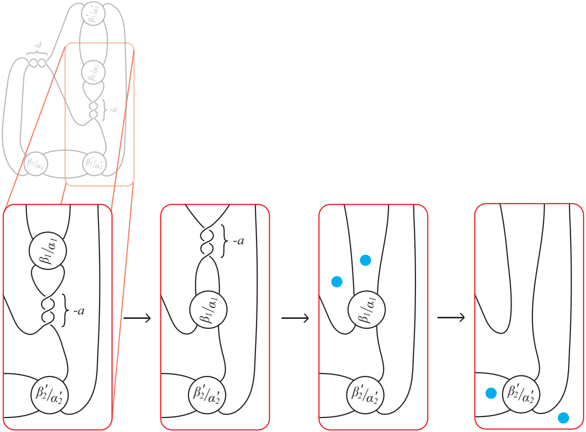

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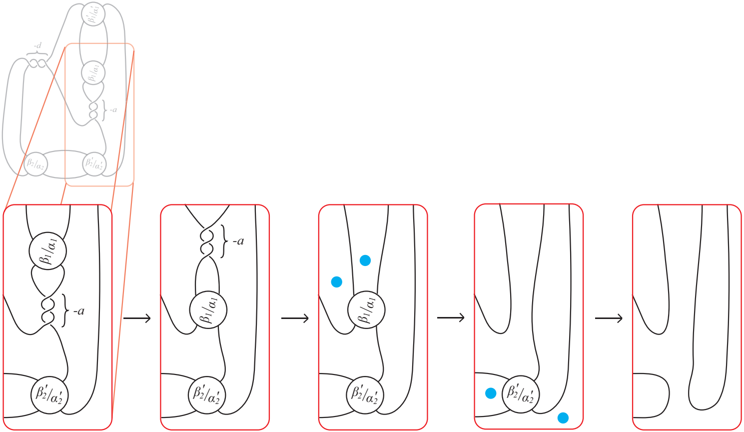
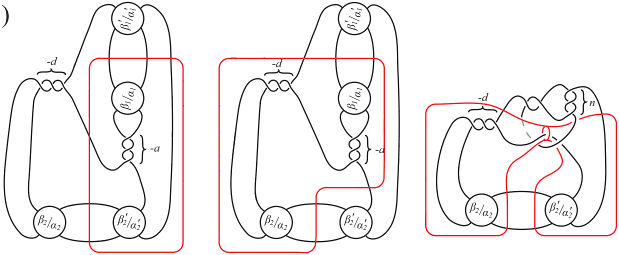


Non-Montesinos case

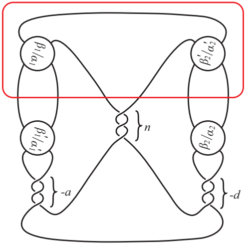
Theorem 2 Let L be a 3-bridge algebraic link in Theorem 1 and let S be a 3-bridge sphere for L. Then S is isotopic to one of the followings.

(Case 1)

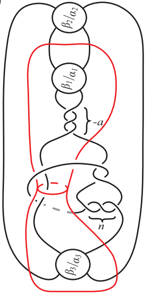


Non-Montesinos case

(Case 2) $n>1$



(Case 3)

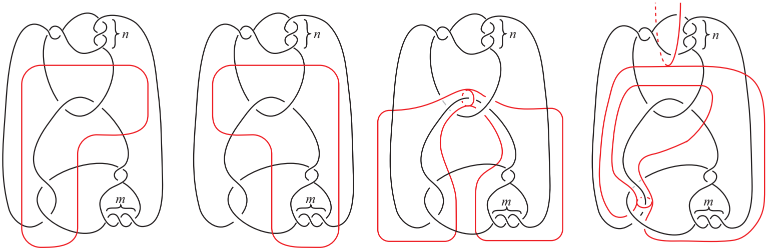


Non-Montesinos case

Theorem 3 : The isotopy classification of 3-bridge spheres in Theorem 2.

Non-Montesinos case

Corollary 1 (Answer to a question by K. Morimoto ('89)) The maximal number of isotopy classes of 3-bridge spheres for a link in Theorem 1, is 4.

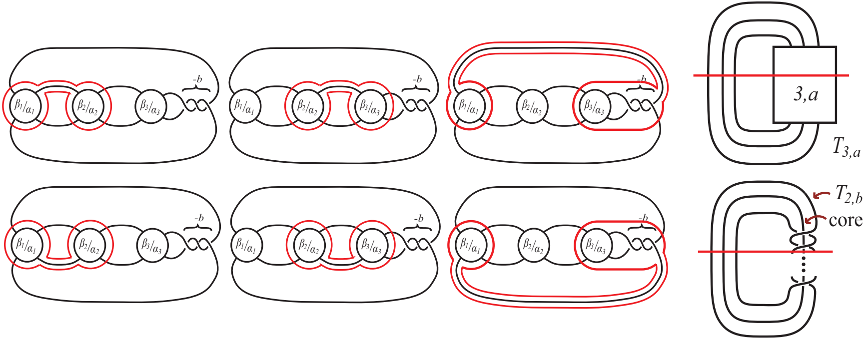


$(m, n > 1)$

Main Theorem (Montesinos case)

Montesinos case

Theorem 4 A 3-bridge Montesinos link L admits at most 6+1 3-bridge spheres up to isotopy.



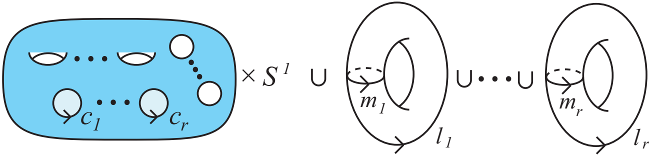
Montesinos case

Remark If L is elliptic
(i.e. $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, n(\geq 2)), (2, 3, 3), (2, 3, 4), (2, 3, 5)$),
then L admits a unique 3-bridge sphere up to isotopy.

Algebraic links
vs
Graph manifolds

Graph manifolds

- Seifert fibered space $F((\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))$ or $F(b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))$



$$\alpha_i c_i + \beta_i f \leftarrow m_i$$

Graph manifolds

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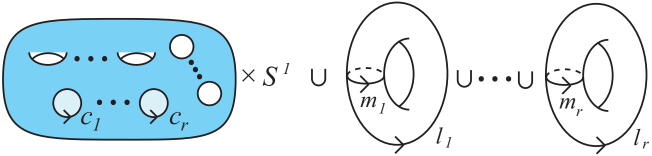
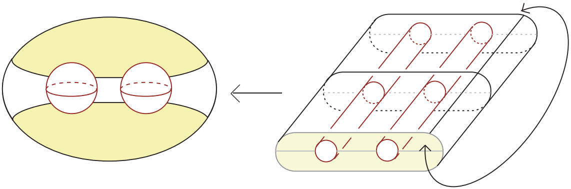


$$\alpha_i c_i + \beta_i f \leftarrow m_i$$

- graph manifold : a 3-manifold obtained from Seifert fibered spaces by glueing their boundaries

Algebraic links vs Graph manifolds

- The double branched covering of S^3 branched over a Montesinos link is a Seifert fibered space over S^2.
- The double branched covering of S^3 branched over an algebraic link is a graph manifold.



3-bridge spheres vs Genus-2 Heegaard surfaces

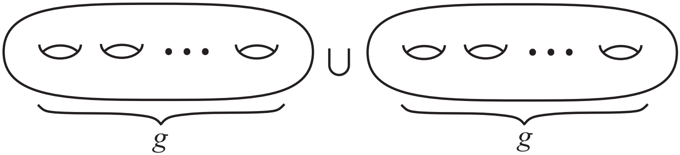
3-bridge spheres vs genus-2 Heegaard surfaces

- $(V_1, V_2; F)$: genus-g Heegaard splitting of M

  $\overset{def}{\Longleftrightarrow}$ (i) V_1, V_2 : handlebodies of genus g

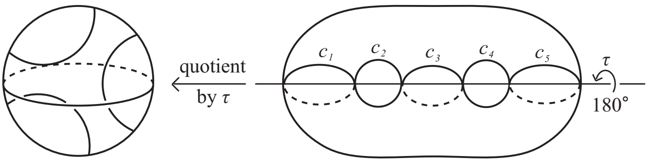
  (ii) $V_1 \cup V_2 = M$

  (iii) $V_1 \cap V_2 = \partial V_1 = \partial V_2 = F$



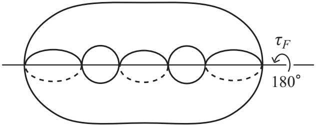
3-bridge spheres vs genus-2 Heegaard surfaces

- 3-bridge sphere of $L \longleftrightarrow$ genus-2 Heegaard surface of $M_2(L)$



Hyper-elliptic involutions

- Heegaard surface F of M
  $\rightsquigarrow$ hyper-elliptic involution $\tau_F : M \to M$



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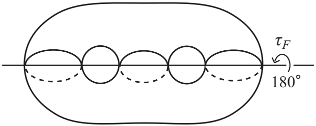

- isotopy class of genus-2 Heegaard surfaces
  $\rightsquigarrow$ $\exists 1$ strong equivalence class of hyper-elliptic involutions

  (i.e. $\exists f : M \to M$ s.t. $f(F) = F'$, $f \sim id_M$
  $\Rightarrow \exists g : M \to M$ s.t. $g\tau_F g^{-1} = \tau_{F'}$, $g \sim id_M$)

3-bridge spheres vs genus-2 Heegaard surfaces

- (Birman-Hilden)
  $\{(L, S) | L : \text{3-bridge link}, \ S : \text{3-bridge sphere for } L\}/\cong$
  $\to \{(M, F) | M : \text{3-manifold}, F : \text{g-2 Heegaard surface}\}/\cong$
  is bijective.

3-bridge spheres vs genus-2 Heegaard surfaces

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- L : 3-bridge link
  $M_2(L)$: the double branched covering of S^3
  τ_L : the covering transformation

3-bridge spheres vs genus-2 Heegaard surfaces

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  $\Phi : \{\text{3-bridge spheres for } L\}/\sim$
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3-bridge spheres vs genus-2 Heegaard surfaces

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  $\quad \to \{\text{genus-2 Heegaard surfaces } F \text{ of } M_2(L) \text{ s.t. } \tau_F = \tau_L\}/\sim$
  : well-defined, surjective

3-bridge spheres vs genus-2 Heegaard surfaces

$$\Phi : \{\text{3-bridge spheres for } L\}/\sim$$
$$\rightarrow \{\text{genus-2 Heegaard surfaces } F \text{ of } M_2(L) \text{ s.t. } \tau_F = \tau_L\}/\sim$$

3-bridge spheres vs genus-2 Heegaard surfaces

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Lemma 1 L : sufficiently complicated **&** not a Montesinos link
$\Rightarrow \boldsymbol{\Phi}$: bijective

3-bridge spheres vs genus-2 Heegaard surfaces

$\mathbf{\Phi : \{3\text{-bridge spheres for } L\}/\sim}$
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Lemma 1 L : sufficiently complicated **&** not a Montesinos link
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Lemma 2 L : elliptic Montesinos link
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3-bridge spheres vs genus-2 Heegaard surfaces

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Lemma 3 L : non-elliptic Montesinos link
$\Rightarrow \sharp\Phi^{-1}([F]) \leq 2 \quad (\forall [F])$

3-bridge spheres vs genus-2 Heegaard surfaces

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Remark L : algebraic link **&** not an elliptic Montesinos link
$\Rightarrow L$: sufficiently complicated

3-bridge spheres vs genus-2 Heegaard surfaces

Definition

- $\widetilde{M_2(L)}$: the universal cover of $M_2(L)$

  $O(L) := \langle$ all lifts of $\tau_L \rangle < \mathrm{Diff}(\widetilde{M_2(L)})$: π-orbifold group of L
  $\cong \pi_1(S^3 \setminus L)/\langle\langle m^2 \rangle\rangle$

3-bridge spheres vs genus-2 Heegaard surfaces

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- L : sufficiently complicated (s.c.)
  $\overset{def}{\Longleftrightarrow}$ L : prime, non-splittable & $O(L)$: infinite

3-bridge spheres vs genus-2 Heegaard surfaces

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Theorem (Boileau-Zimmermann)

L : sufficiently complicated link $\Rightarrow \mathrm{Sym}(S^3, L) \cong \mathrm{Out}(O(L))$

3-bridge spheres vs genus-2 Heegaard surfaces

$\mathbf{\Phi : \{\text{3-bridge spheres for } \boldsymbol{L}\}/\sim}$
$\quad \mathbf{\to \{\text{genus-2 Heegaard surfaces } \boldsymbol{F} \text{ of } \boldsymbol{M_2(L)} \text{ s.t. } \boldsymbol{\tau_F = \tau_L}\}/\sim}$

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3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 1 (Outline - Injectivity)

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 1 (Outline - Injectivity)
S, S' : 3-bridge spheres for L
$F := p^{-1}(S)$, $F' := p^{-1}(S')$
($p : M_2(L) \to S^3$: double branched covering branched over L)

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 1 (Outline - Injectivity)

S, S' : 3-bridge spheres for L

$F := p^{-1}(S)$, $F' := p^{-1}(S')$

($p : M_2(L) \to S^3$: double branched covering branched over L)

Assume that F and F' are isotopic.

i.e. $\exists f : M_2(L) \to M_2(L)$ s.t. $f(F) = F'$, $f \sim id_{M_2(L)}$

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 1 (Outline - Injectivity)

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$\rightsquigarrow \exists \widetilde{f} : \widetilde{M_2(L)} \to \widetilde{M_2(L)}$ s.t. $\widetilde{f}(\widetilde{F}) = \widetilde{F'}$, $\iota_{\widetilde{f}} = id_{\widetilde{M_2(L)}}$ in $\mathrm{Out}(O(L))$

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 1 (Outline - Injectivity)

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$\rightsquigarrow \exists \tilde{f} : \widetilde{M_2(L)} \to \widetilde{M_2(L)}$ s.t. $\tilde{f}(\tilde{F}) = \widetilde{F'}$, $\iota_{\tilde{f}} = id_{\widetilde{M_2(L)}}$ in $\mathrm{Out}(O(L))$

$\rightsquigarrow \exists g : (S^3, L) \to (S^3, L)$ s.t. $g(S) = S'$, $g \sim id_{(S^3,L)}$

i.e. **S and S' are isotopic.**

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 3 (Outline - Injectivity)

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 3 (Outline - Injectivity)

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3-bridge spheres vs genus-2 Heegaard surfaces

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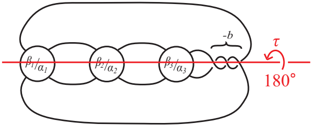
or $\iota_{\widetilde{f}} = \widetilde{\tau}$



3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 3 (Outline - Injectivity)

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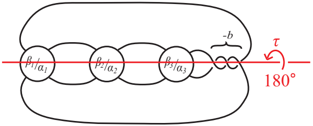
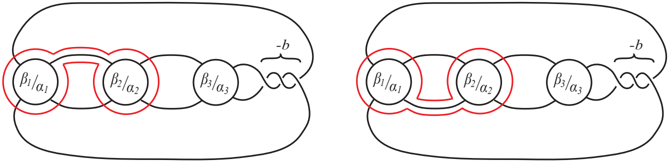
$\rightsquigarrow \exists \widetilde{f} : \widetilde{M_2(L)} \to \widetilde{M_2(L)}$ s.t. $\widetilde{f}(\widetilde{F}) = \widetilde{F'}$, $\iota_{\widetilde{f}} = id_{\widetilde{M_2(L)}}$ in $\mathbf{Out}(O(L))$

or $\iota_{\widetilde{f}} = \widetilde{\tau}$



$\therefore S'$ **is isotopic to S or $\tau(S)$.**

3-bridge spheres vs genus-2 Heegaard surfaces



Problem :

Determine if the above 3-bridge spheres are isotopic or not.

3-bridge spheres vs genus-2 Heegaard surfaces

$\mathbf{\Phi : \{3\text{-bridge spheres for } \mathit{L}\}/\sim}$
$\mathbf{\to \{genus\text{-}2\ Heegaard\ surfaces\ \mathit{F}\ of\ \mathit{M}_2(\mathit{L})\ s.t.\ \tau_F = \tau_L\}/\sim}$

Lemma 2 L : elliptic Montesinos link
$\Rightarrow \mathbf{\Phi}$: bijective

3-bridge spheres vs genus-2 Heegaard surfaces

Proof of Lemma 2 (Outline - Injectivity)

- The double branched covering of S^3 branched over an elliptic Montesinos link admits a unique genus-2 Heegaard surface up to isotopy. (known)
- We prove that an elliptic Montesinos link admits a unique 3-bridge sphere up to isotopy.

Proof of Theorem 4

Classification of 3-bridge spheres for Montesinos links :

Proof of Theorem 4

Classification of 3-bridge spheres for Montesinos links :

Lemma 2, Lemma 3
+
Classification of genus-2 Heegaard surfaces
of Seifert fibered spaces

Proof of Theorem 4

- genus-2 Heegaard surfaces of Seifert fibered spaces over S^2 : done by [Boileau-Collins-Zieschang] etc.

Proof of Theorem 4

- genus-2 Heegaard surfaces of Seifert fibered spaces over S^2 : done by [Boileau-Collins-Zieschang] etc.

To distinguish the Heegaard surfaces up to isotopy, they compared the commutator invariants.

Lemma M : closed connected orientable 3-manifold
$(V_1, V_2; F)$, $(W_1, W_2; G)$: two genus 2 Heegaard splittings of M
$\{v_i^1, v_i^2\}$, $\{w_i^1, w_i^2\}$: generating systems of $\pi_1(V_i)$, $\pi_1(W_i)(i = 1, 2)$
If $(V_1, V_2; F)$ and $(W_1, W_2; G)$ are isotopic,
then $[v_1^1, v_1^2] \sim [w_1^1, w_1^2]^{\pm 1}$ and $[v_2^1, v_2^2] \sim [w_2^1, w_2^2]^{\pm 1}$.

Proof of Theorems 1, 2

List-up of 3-bridge algebraic links and their 3-bridge spheres:

Proof of Theorems 1, 2

List-up of 3-bridge algebraic links and their 3-bridge spheres:

Lemma 1

+

(T. Kobayashi ’84) Characterization
of non-simple manifolds of genus-2

+

(T. Saito ’04) Characterization of 1-bridge knot exteriors in
lens spaces with incompressible tori

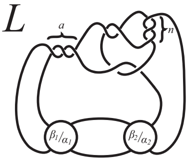
+

careful arguments

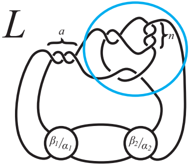
Proof of Theorem 3

Distinction of 3-bridge spheres in Theorem 2

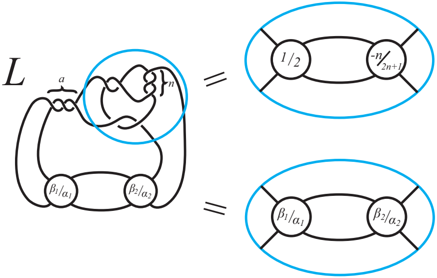
Proof of Theorem 3



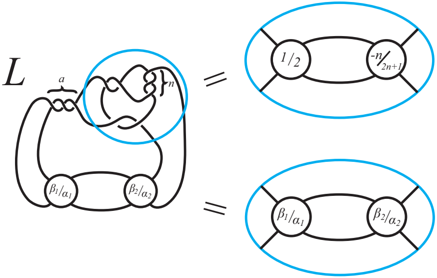
Proof of Theorem 3



Proof of Theorem 3



Proof of Theorem 3

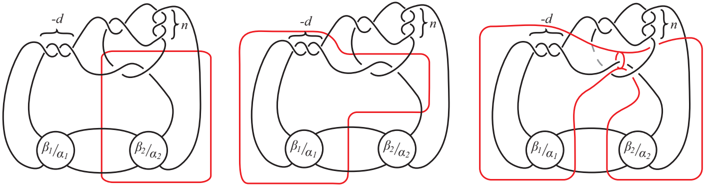


$$\longleftarrow D(1/2, -n/(2n+1)) = E(T_{2,n})$$

$$\longleftarrow D(\beta_1/\alpha_1, \beta_2/\alpha_2)$$

Proof of Theorem 3

By Theorem 2, we have the following three 3-bridge spheres for L.



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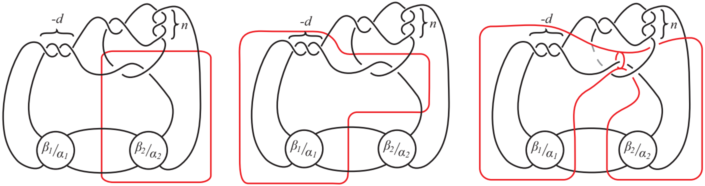

$\rightsquigarrow$ $[u_1, v_2], [u_2, v_1]$ $[u_1, v_1], [u_2, v_2]$ $[u_1, \tau_2 u_2' \tau_2^{-1}], [\tau_2\tau_1, h]$

Here, $\{u_1, u_2\}$: exceptional fibers of $D(\beta_1/\alpha_1, \beta_2/\alpha_2)$,
$\{v_1, v_2\}$: exceptional fibers of $D(1/2, -n/(2n+1))$, ...

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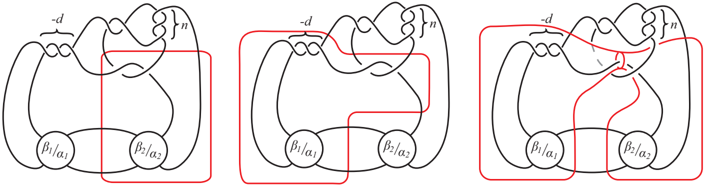

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Proof of Theorem 3

Key point of the proof :

$$M_2(L) = D(\beta_1/\alpha_1, \beta_2/\alpha_2) \cup_T D(1/2, -n/(2n+1))$$

$\rightsquigarrow$

$$\pi_1(M_2(L)) = \pi_1(D(\beta_1/\alpha_1, \beta_2/\alpha_2)) *_{\pi_1(T)} \pi_1(D(1/2, -n/(2n+1)))$$

Proof of Theorem 3

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$$= \langle c_1, c_2, h \mid c_i^{\alpha_i} h^{\beta_i}, [c_i, h] \rangle$$

$$*_{\pi_1(T)} \langle c_1', c_2', h' \mid c_1'^2 h', c_2'^{2n+1} h'^{-n}, [c_i', h'] \rangle$$

Proof of Theorem 3

Key point of the proof :

$$M_2(L) = D(\beta_1/\alpha_1, \beta_2/\alpha_2) \cup_T D(1/2, -n/(2n+1))$$

$\rightsquigarrow$

$$\begin{aligned}
\pi_1(M_2(L)) &= \pi_1(D(\beta_1/\alpha_1, \beta_2/\alpha_2)) *_{\pi_1(T)} \pi_1(D(1/2, -n/(2n+1))) \\
&= \langle c_1, c_2, h \mid c_i^{\alpha_i} h^{\beta_i}, [c_i, h] \rangle \\
&\qquad *_{\pi_1(T)} \langle c_1', c_2', h' \mid c_1'^2 h', c_2'^{2n+1} h'^{-n}, [c_i', h'] \rangle \\
&= \langle c_1, c_2, c_1', c_2', h, h' \mid \\
&\qquad c_i^{\alpha_i} h^{\beta_i}, [c_i, h], c_1'^2 h', c_2'^{2n+1} h'^{-n}, [c_i', h'], \\
&\qquad c_1' c_2' = h, h' = (c_1 c_2) h^d \rangle
\end{aligned}$$

Proof of Theorem 3

(i) $[u_1, v_2]$ vs $[u_1, v_1]$

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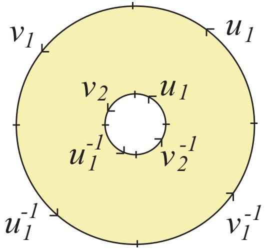
$\leadsto \quad \exists \Psi : S^1 \times I \to M$ s.t.



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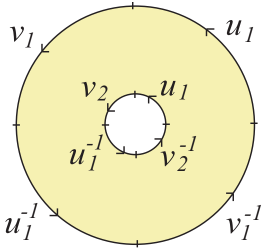
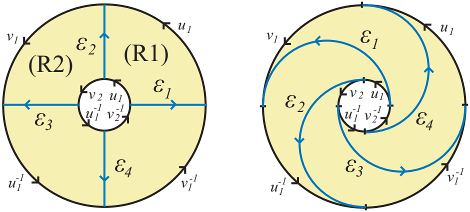


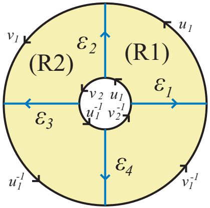
By studying the intersection of the annulus and the attaching torus, we have one of the following :

Proof of Theorem 3

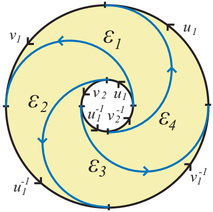


(Case1) (Case2)

Proof of Theorem 3



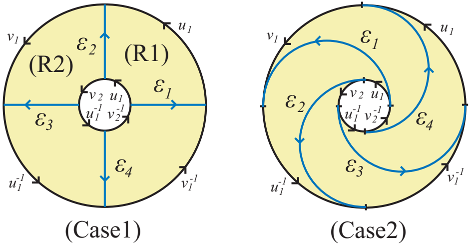
(Case1)



(Case2)

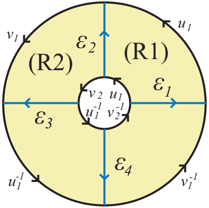
$$\pi_1(D(\beta_1/\alpha_1, \beta_2/\alpha_2)) = \langle c_1, c_2, h \mid c_i^{\alpha_i} h^{\beta_i}, [c_i, h] \rangle$$

Proof of Theorem 3



$$\pi_1(D(\beta_1/\alpha_1, \beta_2/\alpha_2)) = \langle c_1, c_2, h \mid c_i^{\alpha_i} h^{\beta_i}, [c_i, h] \rangle$$
$$\rightsquigarrow G := \pi_1(D(\beta_1/\alpha_1, \beta_2/\alpha_2))/\langle h \rangle = \langle c_1, c_2 \mid c_i^{\alpha_i} \rangle \cong \mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2}$$

Proof of Theorem 3



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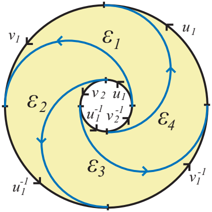

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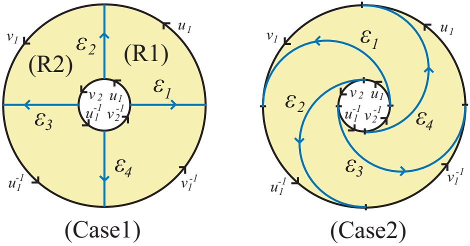
(R1) $\rightsquigarrow \varepsilon_1^{-1} u_1 \varepsilon_2 = u_1$

$$\rightsquigarrow ((c_1 c_2)^a h^b)(c_1^{\gamma_1} h^{\delta_1})((c_1 c_2)^d h^e) = c_1^{\gamma_1} h^{\delta_1}$$

Proof of Theorem 3



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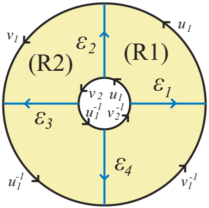
Proof of Theorem 3



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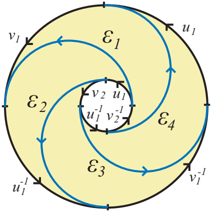

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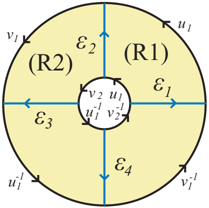
$\rightsquigarrow a = d = 0$ and $e = -b$

Proof of Theorem 3



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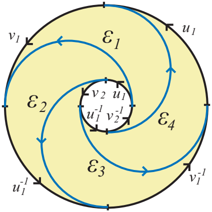
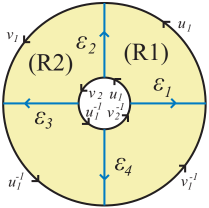


(Case2)

$\therefore \varepsilon_1 = \varepsilon_2 = h^e$.

Proof of Theorem 3



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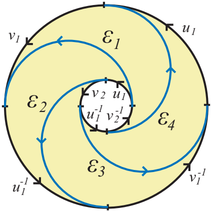

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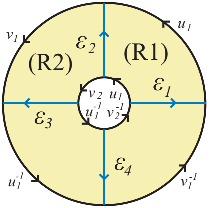
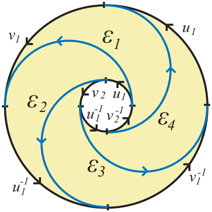
Similarly, we have $\varepsilon_3 = \varepsilon_4 = h^f$.

Proof of Theorem 3



(Case1)



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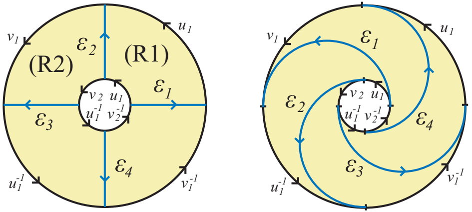
(R2) $\rightsquigarrow \cdots \rightsquigarrow n = 1$

Proof of Theorem 3



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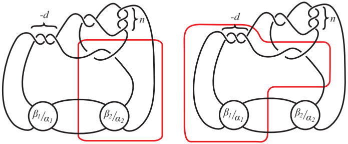
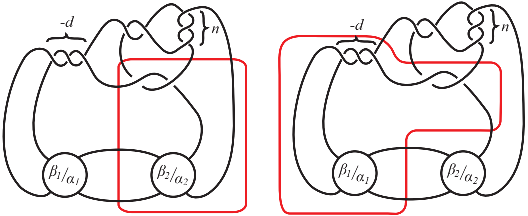


Proof of Theorem 3



By studying all cases, we prove that

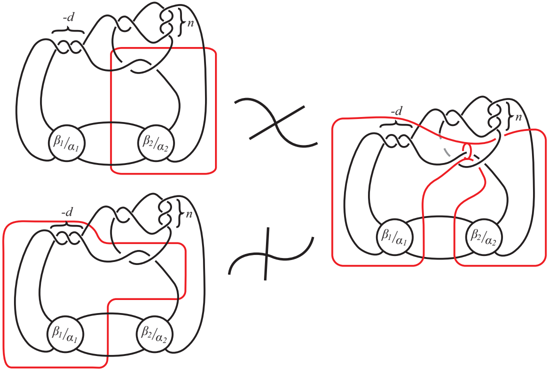
the above 3-bridge spheres are isotopic

$\Longleftrightarrow$ (i) $n = 1$, or

(ii) $\beta_1 = \pm 1 + k_1\alpha_1$, $\beta_2 = \pm 1 + k_2\alpha_2$ and $d = k_1 + k_2$.

Proof of Theorem 3

We also prove that



Thank you!

¡Gracias!