Branched coverings and three manifolds
An exposition. I

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Some history

- The origins of Branched Covering Theory are closely related to the history of the “uniformization” of functions.

Given a multivalued function

$$f : X \to \mathbb{C},$$

Riemann constructed:

- A new domain Σ (the Riemann surface):

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- A map (the branched cover):

$$p : \Sigma \to X,$$

and

- A new univalued function g:

$$\begin{array}{ccc} \Sigma & & \\ p\downarrow & \searrow g & \\ X & \xrightarrow{f} & \mathbb{C} \end{array}$$

such that $g = p^{-1} \circ f$.

Abstraction

At the beginning, the three objects were inseparable. But soon they gave rise to two theories.

- The concept of (abstract) Riemann surface Σ was defined and isolated from the functions f, p and g: the definition by Weyl of abstract Riemann surface soon gave rise to a complete theory of topological manifolds.

$$\begin{array}{ccc} \Sigma & & \\ p\downarrow & \searrow g & \\ X & \xrightarrow{f} & \mathbb{C} \end{array}$$

- The functions p and g were the prototype of branched covering: the pair (Σ, p) being constructed by "cuts" performed in the original domain and the function g by lifting f.

Holomorphic map between Riemann surfaces

- If we take a non-constant holomorphic map

$$f : S \rightarrow T$$

between two Riemann surfaces.

- Then

$$f : S \to T$$

is always **open** and the only singularities are **coning-points** (around isolated points where the derivative of f is zero).

- If the image of the set of cone-points is a discrete subset of T then

$$f : S \to T$$

is a "branched covering". (For instance, if S is compact.)

- But this is not always the case. Tesselate H^2 by regular pentagons, 10 around each vertex. Holomorph one pentagon onto an euclidean regular pentagon (angle $3(\frac{2}{10}\pi)$). Extend by Schwarz principle. Each vertex of the hyperbolic tessellation has branch index 3. But their images are dense in E^2.

Generalization

- It was discovered that the singularities of algebraic curves were essentially cones over classical tame knots.

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- This forced the generalization of branched coverings to other dimensions.

- Soon, branched coverings became an important instrument in the study of knots and manifolds (Reidemeister book, Tietze article).

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- "Folded coverings" were investigated by Tucker and they were put to good use much later to place the concept of orbifold, discovered by Satake and rediscovered by Thurston, on solid ground.

Foundation

However the theory of branched coverings lacked a firm topological basis.

- At those early times it was unknown if all topological (metrizable) manifolds were polyhedrons. The Haupvermutung was the key to establish the topological nature of certain knot invariants that were defined starting from a polyhedron (combinatorial invariants).

- The definition process of these invariants consisted in constructing **branched coverings** (überlagerungen, revêtements) **over** these **knots**, defined in a canonical way, and then obtaining the ordinary invariants of these covering manifolds: homology groups, linking form, etc.

- The difficulty of knowing if these invariants were topological resided in that the construction of the branched coverings depended essentially on the polyhedral structure of the base space.

Solution by Fox

- How to complete the part of the total space that covers what is outside of the singular set?

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- How to build topologically and in a unique way the part of the total space lying over the singular set?

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- How to build topologically and in a unique way the part of the total space lying over the singular set?
- How to remove all additional structure that is not topological?

- These were formidable challenges. Indeed, the singular set can now be a wild knot or a Cantor set.

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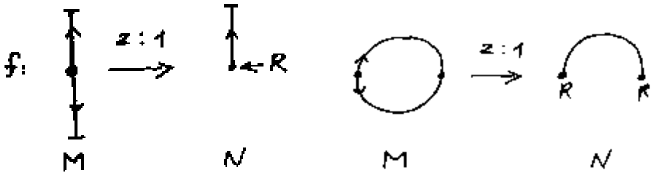
- How to define the cover over the knot if it doesn't have a tubular neighborhood?

Fox solved all these problems by generalizing the concept (until then, non topological) of branched covering.
The main Theorem of Fox is in:

- R.H.Fox, Covering spaces with singularities. In Algebraic Geometry and Topology. A Symposium in honor of S. Lefschetz. Editors: Fox et al. Princeton Univ. Press (1957) pp.243-257

Motivating Examples:

- Foldings



- **Dimension 2: regular**

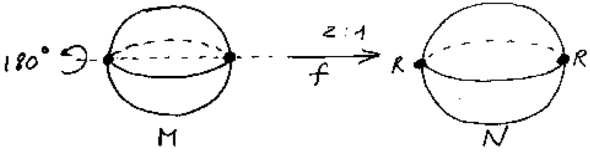


Figure: 2-fold cyclic

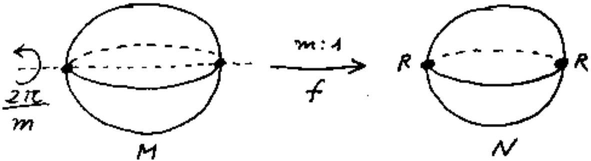


Figure: *m*-fold cyclic

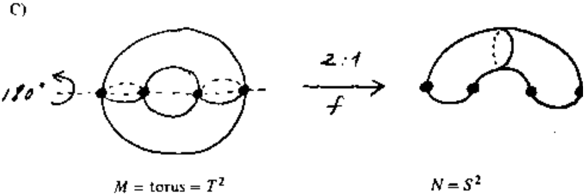


Figure: 2-fold cyclic

Theorem

Every closed, orientable surface is a 2-fold branched covering of the 2-sphere S^2.

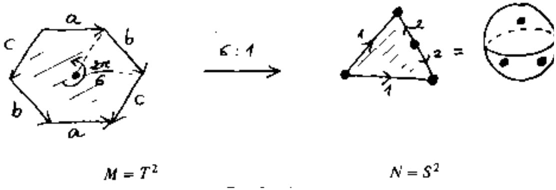


Figure: 6-fold cyclic

Dimension 2:

- **Irregular covering**

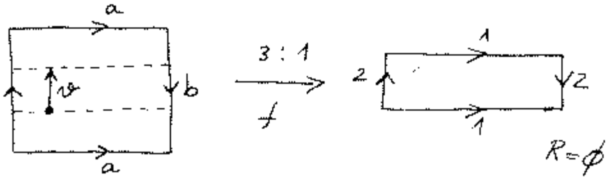


Figure: 3-fold irregular unbranched

Dimension 2:

- **Irregular foldings. Folding letter trick**

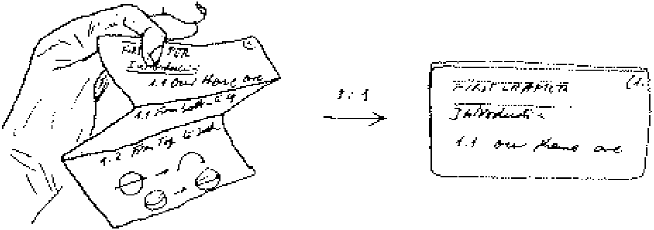


Figure: Charpenter rule folding

Passing to branched coverings

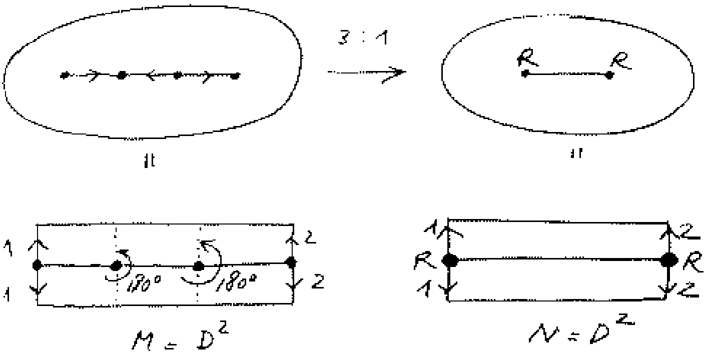
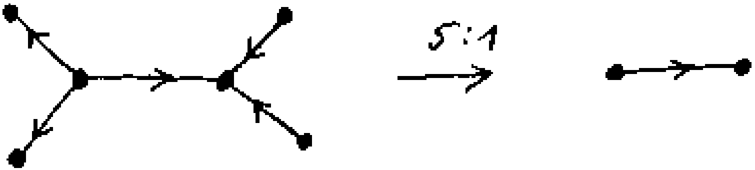
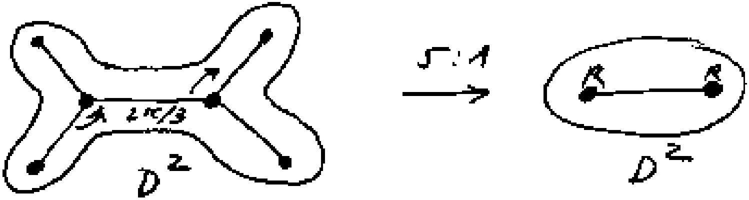


Figure: Charpenter rule branched covering

Again:



Passes to



Simplicial maps

- Take a simplicial map
$$f : S \longrightarrow T$$
between two triangulated surfaces.

- Assume S and T have no boundary and

$$f : S \to T$$

is open. Then we can have conings around vertexes (around a vertex v of S the map f behaves like $z \mapsto z^n$ for some natural number n). That is, f is a BRANCHED COVERING and n is the branching index of the CONE-POINT v.The image of the set of cone-points is the BRANCHING SET. It is a set of isolated points

- If T has non empty boundary and

$$f : S \to T$$

is open then we can also have foldings along edges. The map is a BRANCHED FOLDING. There are now CONE-POINTS, CORNER-POINTS and FOLDING POINTS.

Combinatorial definition of branched covering

- The definition of branched covering is relatively easy if the spaces involved are triangulated manifolds.

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- The definition of branched covering is relatively easy if the spaces involved are triangulated manifolds.
- Thus, if M and N are connected, unbounded, triangulable n-manifolds, then a function

$$f : M \longrightarrow N$$

is a (COMBINATORIAL) BRANCHED COVERING OF N if there are triangulations of M and N such that f is an open simplicial map.

- Assume $n = 3$ for simplicity.

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- Then $f : M \to N$ is an ordinary covering when restricted to the complement of the 1-skeletton of M.

- Therefore there is a minimal subset B of the 1-skeletton of N such that the restriction

$$f|_{(M\backslash f^{-1}(B))} : M\backslash f^{-1}(B) \to N\backslash B$$

is an ordinary covering (the ASSOCIATED COVERING).

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- We call B the BRANCHING SET of $f : M \to N$.

- A point $x \in f^{-1}(B)$ has BRANCH INDEX $b(x)$ if

$$(\textit{star of } x \textit{ minus } f^{-1}(B))$$

is mapped b to 1 onto $(\textit{star of } f(x) \textit{ minus } B)$.

- The part of $f^{-1}(B)$ with branch index $b > 1$ is called the BRANCHING COVER.

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- The part of $f^{-1}(B)$ with branch index $b = 1$ is called the PSEUDO-BRANCHING COVER.

- The ASSOCIATED UNBRANCHED COVERING

$$g : M \setminus f^{-1}(B) \to N$$

is the composition of the covering $f|_{M \setminus f^{-1}(B)}$ with the natural inclusion $N \setminus B \subset N$.; and

- The MONODROMY of the associated covering is a representation ω of $\pi_1(N \backslash B, p)$ into the group of bijections of $f^{-1}(p)$ obtained by lifting p-based loops. It is **transitive**, because M is connected.

- A pair (B, ω) where B is a subpolyhedron of the 1-skeletton of N and ω is a transitive representation into the group of bijections of a numerable or finite set F is called a REPRESENTED GRAPH of N.

- The information (B, ω) defines (or "gives rise" or "it is enough to construct") a unique (up to PL-isomorphism) covering $f : M \to N$ branched over B with monodromy ω.

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- For M to be a manifold it is sufficient (but not necessary) that B is a link.

The construction (due to Neuwirth in "Knot groups")

- Think of N as the result of pasting together tetrahedrons T_i (i belongs to a countable set) along closed faces.

- Find a 2-subpolyhedron K of N such that $\omega\Big|_{\pi_1(N\setminus K)}$ is trivial (K is a SPLITTING COMPLEX)

- Paste the tetraedrons T_i together along the faces not belonging to K to construct a 3-polyhedron (the TRIVIAL SHEET S).

- Then N is the result of pasting together pairs of free 2-faces of S. Each ordered such a pair (G_1, G_2) defines an element γ of $\pi_1(N)$.

- Paste together copies of the trivial sheet using the information given by ω: face G_1^i in sheet S^i is pasted with face G_2^j in sheet S^j iff $\omega(\gamma)(i) = j$.

- Paste together copies of the trivial sheet using the information given by ω: face G_1^i in sheet S^i is pasted with face G_2^j in sheet S^j iff $\omega(\gamma)(i) = j$.
- Fox theorem grants that the branched covering so constructed is unique.

Simple coverings

- A branched covering is SIMPLE if all the points in the fiber $f^{-1}(y)$, $y \in B$, have branch index 1, except for one point that has branch index 2.

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- A branched covering is SIMPLE if all the points in the fiber $f^{-1}(y)$, $y \in B$, have branch index 1, except for one point that has branch index 2.
- A simple branched covering is irregular except if it is 2-fold.

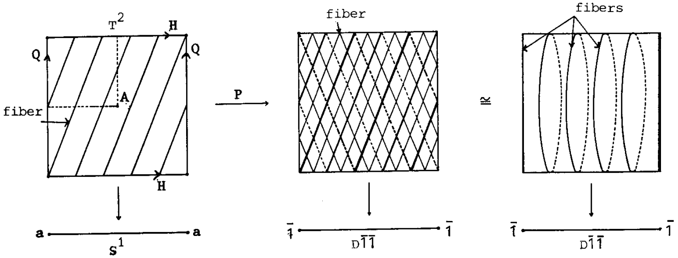
- A REPRESENTED KNOT OR LINK (B, ω) of S^3 gives rise to a simple branched covering iff ω sends meridians of B to transpositions of F. We say that such a (B, ω) is a SIMPLE REPRESENTATION.

End of the first part

Second part: Particular problems

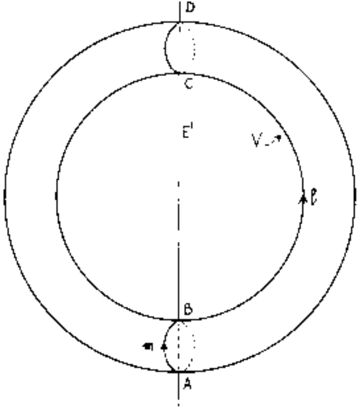
2-fold branched coverings.

- The fibration of the torus with base S^1 is the 2-fold branched covering of the "singular" fibration of the 2-sphere:

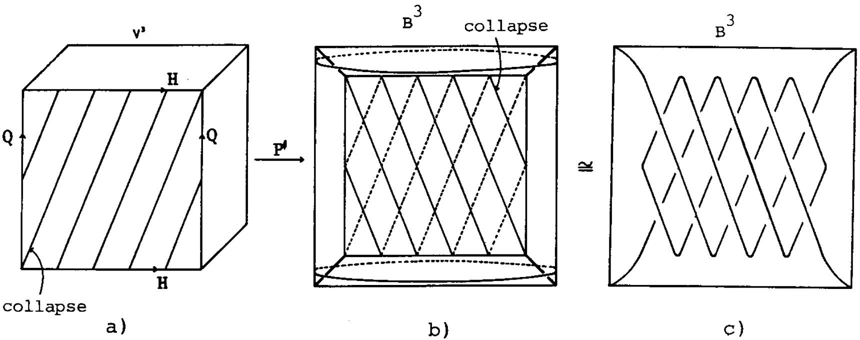


There is an involution of the torus defining the branched covering.

- This involution extends to the solid torus:



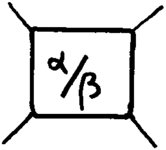
- The solid torus of the left (fibers collapsed) is the 2-fold branched covering of the 3-cell of the right.

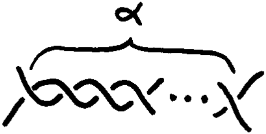


The branching set is the rational tangle 5/2, in this example.

Notation:

- the rational tangle α/β (left).





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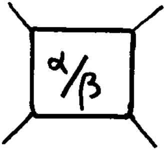
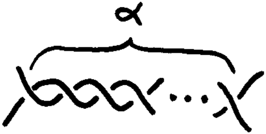
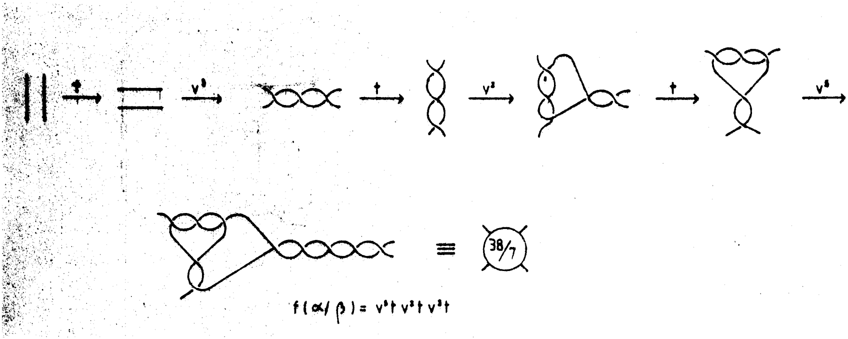




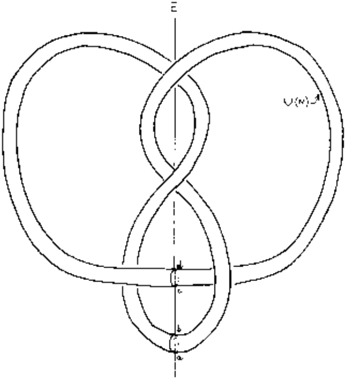
- The rational tangle $\alpha/1$ (right).

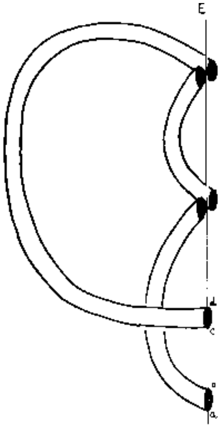
Rational tangles from continuous fraction expansion (J. Conway):



The tangle 38/7.

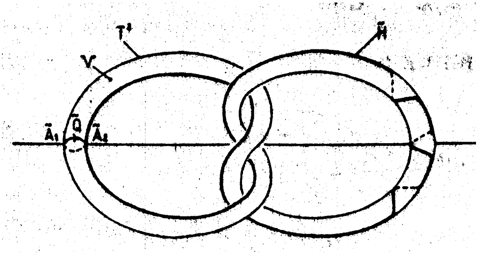
Strongly invertible knot:



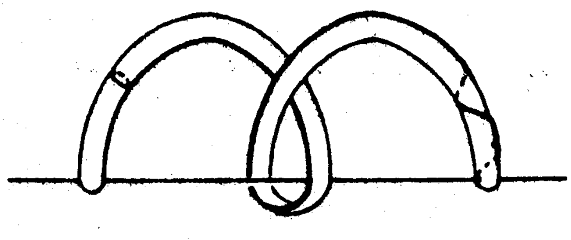
- 
E

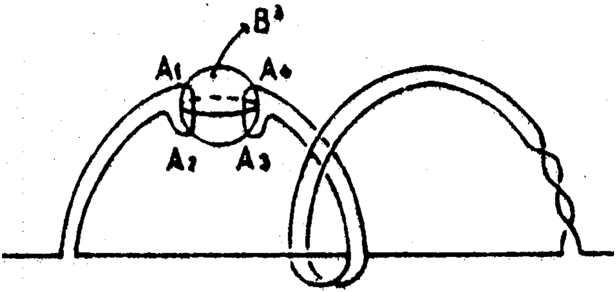
Theorem (M)

Surgery on strongly-invertible link is 2-fold branched covering of S^3.



Proof

- 


B³
A₁
A₄
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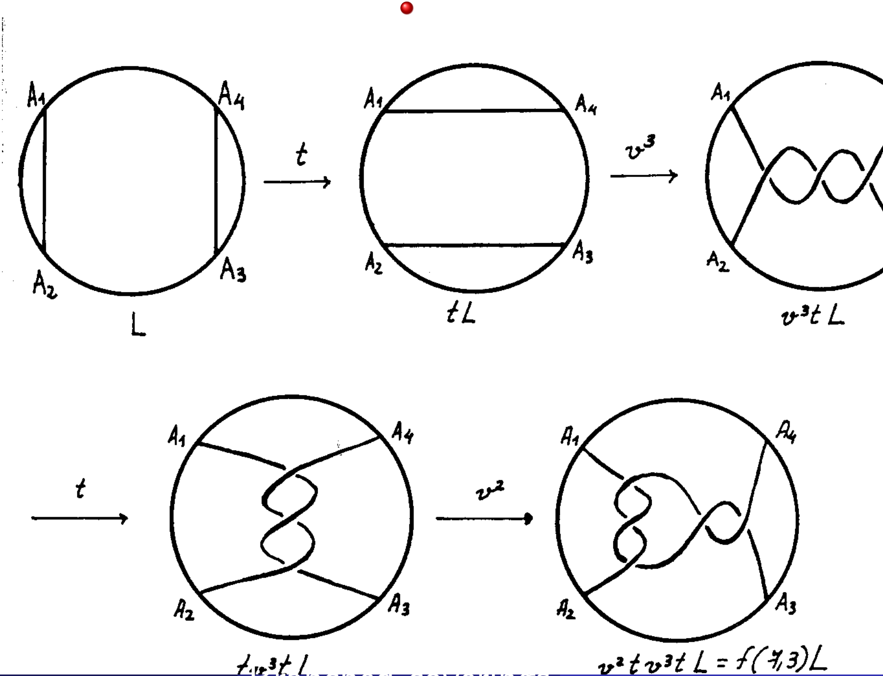
A₃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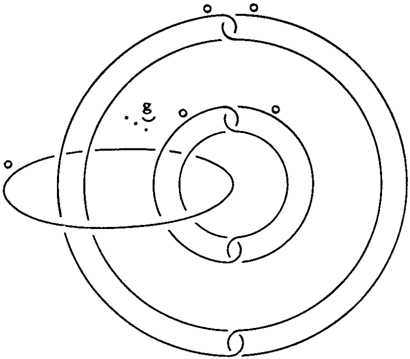
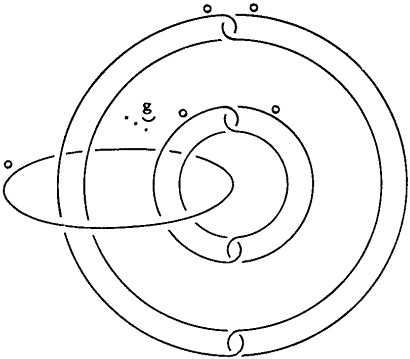
Applications

- **Seifert Manifolds**

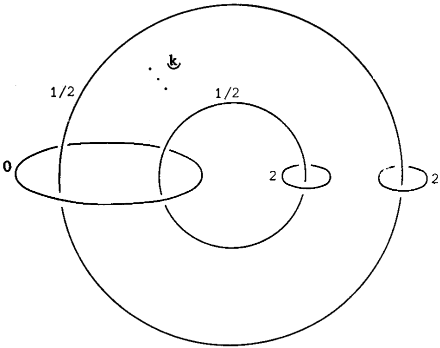


Applications

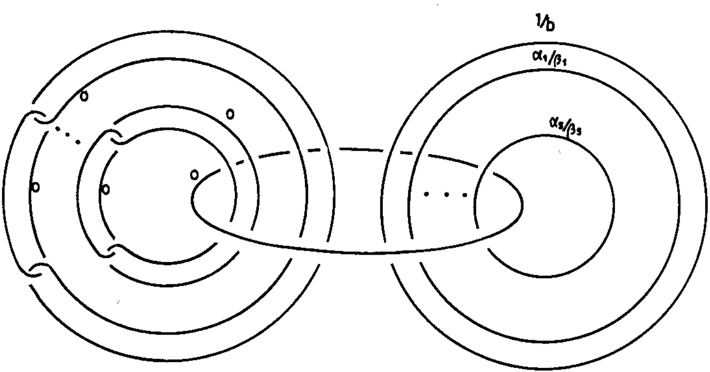
- **Seifert Manifolds**



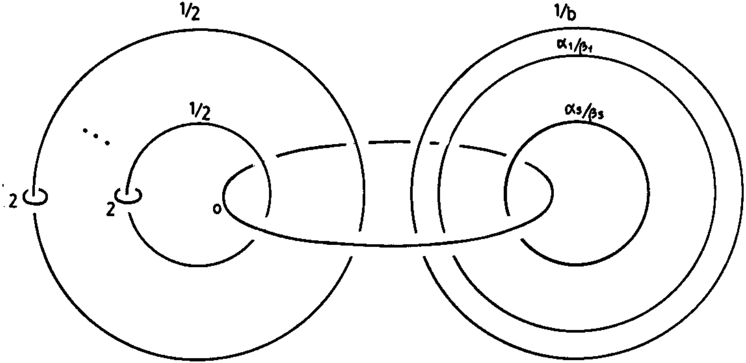
- Seifert Manifold $(O, o, g \mid 0) = S^1 \times F_g$ =Orientable fiber bundle over orientable surface of genus k (trivial).



- Seifert Manifold $(O, n, k \mid 0)$ =Orientable fiber bundle over non-orientable surface of genus k (trivial).

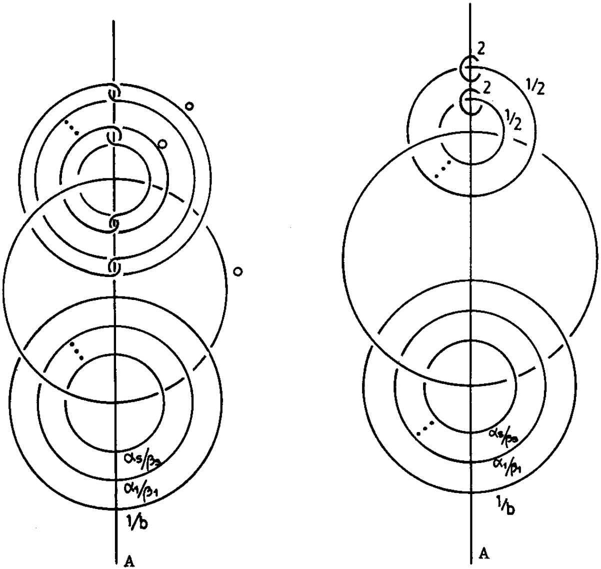


Seifert Manifold $(O, o, g \mid b; \alpha_1/\beta_1, ..., \alpha_k/\beta_k)$

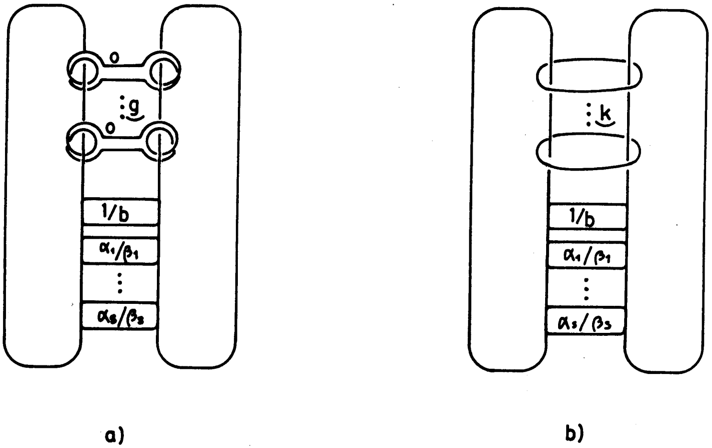


Seifert Manifold $(O, n, k \mid b; \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k)$

Strongly symmetric:



Montesinos knots:

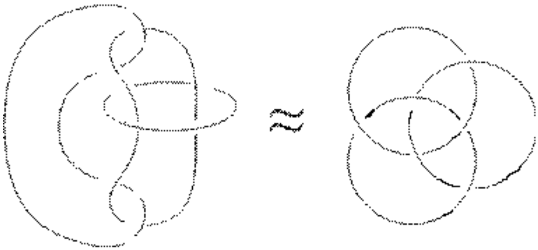


For $g = 0$ or for any $k \geq 1$ these knots (links) are in S^3.

Example

The Borromean rings is a Montesinos link. Therefore its 2-fold branched covering is the Seifert manifold

$$(O, n, 1 \mid -1; (2,1), (2,1))$$



Problem

Is every closed, orientable 3-manifold a branched covering of S^3?

Solution

Every closed, orientable 3-manifold is a covering of S^3 branched over a graph (J. W. Alexander) and over a link (indicated by Alexander; proved in detail by M).

Problem

Is every closed, orientable 3-manifold a 2-fold covering of S^3 branched over a link?

Theorem (R. H. Fox)

The 3-torus is not a 2-fold branched covering of S^3.

This was shown much later by Edmonds in a different but very important way.

Therefore:

- **The minimal number of sheets is not two.**

Theorem (Tollefson)

There are closed, orientable 3-manifolds without periodic (PL-) homeomorphisms.

- **If such a manifold branch-covers S^3 the covering cannot be regular**

Therefore it is natural to investigate irregular 3-fold branched coverings of S^3.

- We will consider simple represented knots or links (L, ω) in $X = S^3$ (resp. simple represented *string or string-link* in $X = \mathbb{R}^3$). Thus ω is a transitive homomorphism

$$\omega : \pi_1(X \backslash L) \rightarrow S_n$$

onto the symmetric group S_n of the indices $\{1, 2, 3, ..., n\}$ sending meridians to transpositions.

- When $n = 3$ we will say that the simple representation (L, ω) in $X = S^3$ or $X = \mathbb{R}^3$ is a COLORED KNOT, LINK or STRING, as the case may be, because

- $$\omega : \pi_1(X \setminus L) \to S_3$$

  sends meridians of L to transpositions $(1,2)$, $(1,3)$, or $(2,3)$ of S_3.

- and following a beautiful idea of Fox, we can represent $(1, 2)$ by color Red (R), $(1, 3)$ by color Green (G) and $(2, 3)$ by color Blue (B).

1. and we can endow each overpass of a normal projection of L with one of the three colors R, G, B in such a way that the colors meeting in a crossing are all equal or all are different.

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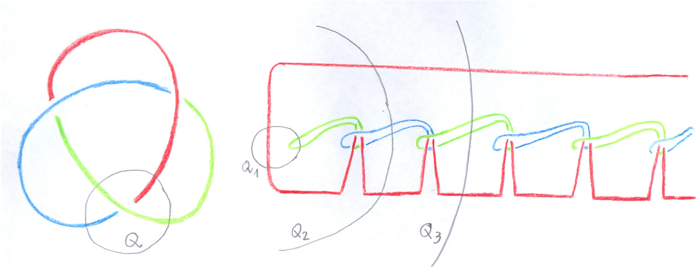
2. and at least two colors ought to be used.

- A colored knot or link (resp. string or string-link) (L, ω) in $X = S^3$ (resp. $X = \mathbb{R}^3$)


Q
Q1
Q2
Q3

- gives rise to a simple, 3-fold irregular covering $f : M(L, \omega) \longrightarrow X$ branched over L.

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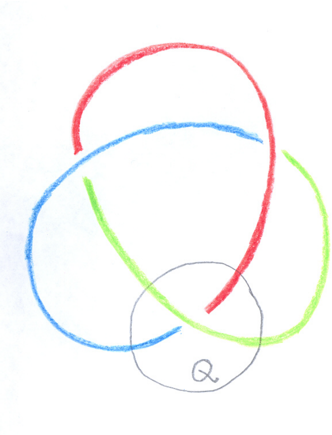
- The space $M(L, \omega)$ is a closed (resp. open), orientable 3-manifold.

Example

Consider the following colored knot (L, ω). Fox asked if $M(L, \omega)$ is S^3.



Solution (Montesinos PhD Thesis)

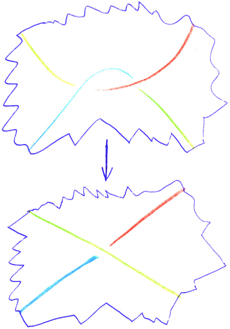
Yes: on top of the 3*-cell* Q *(resp.* $S^3 \setminus IntQ$*) lies a* 3*-cell because*

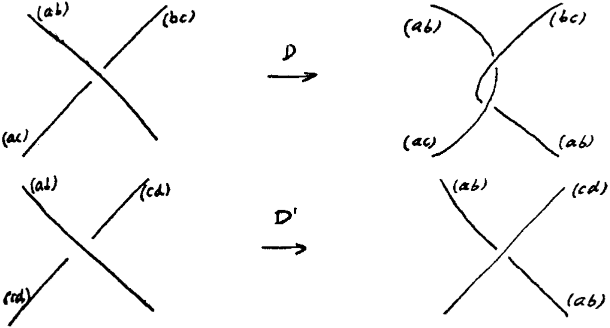
$$p|_{p^{-1}Q} : p^{-1}Q \longrightarrow Q$$

is a 3-fold simple covering of the 3-cell Q_i*, branched over two properly embedded arcs (apply the folding letter trick).*

Corollary (Montesinos PhD Thesis)

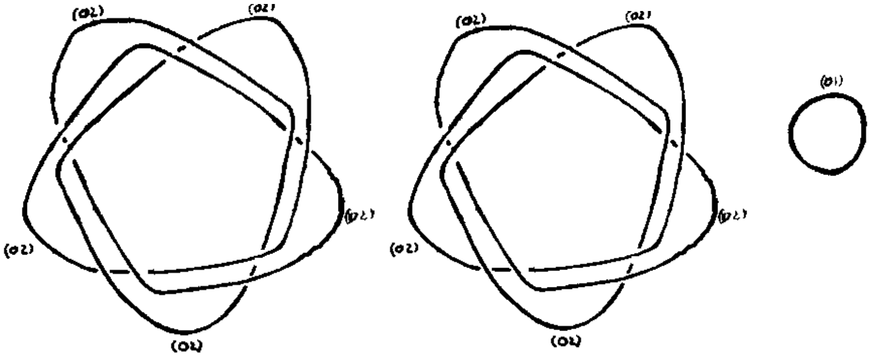
The following moves D, D' and D'' have the following property. If these moves are applied to a portion of a simply represented knot or link (resp. string or string-link), we obtain a new simply represented knot or link (resp. string or string-link) whose corresponding branched covering spaces are homeomorphic.



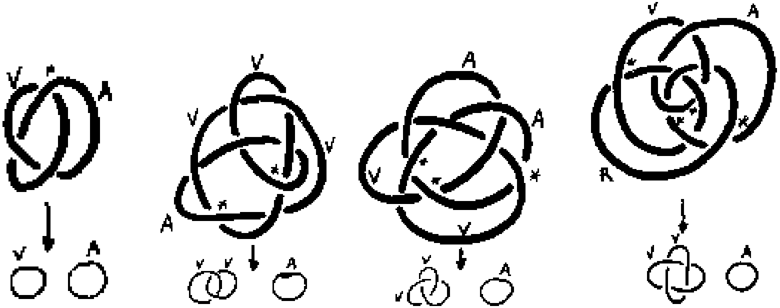


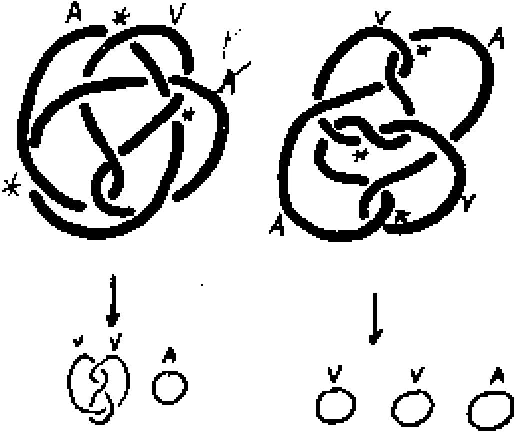
Moves D and D'

- The move D'': Adding a trivial sheet

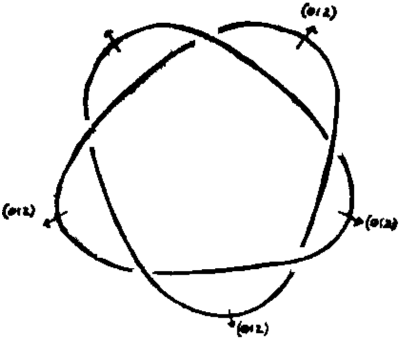


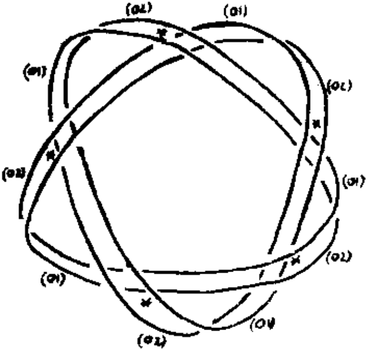
- **Applications**



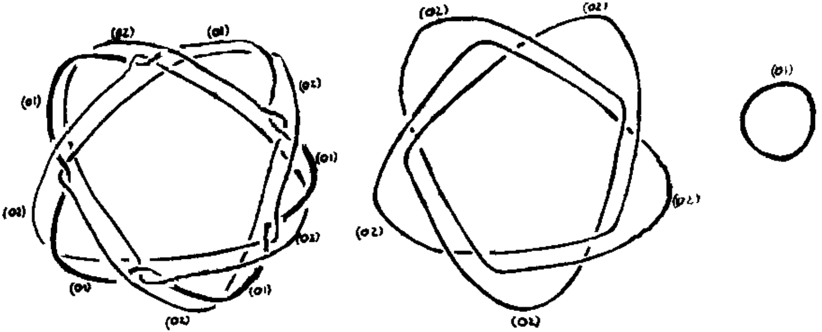


At the left we have a 3-fold cyclic covering. It is modified in a tubular nbd of the knot to a simple 3-fold covering. Note that out of that tubular nbd the covering behaves as cyclic. However the 3-fold branched covering is irregular.





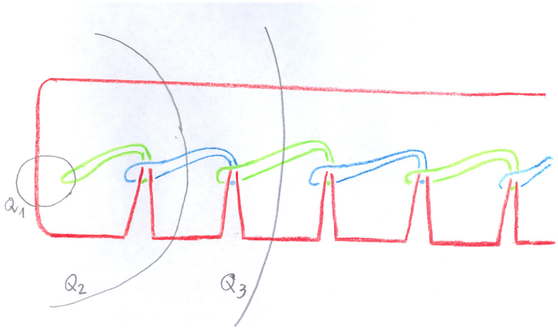
Apply moves:



This shows that the 3-fold branched cover of the toroidal knot $\{2,5\}$ (Brieskorn manifold $B(2,3,5)$) is the 2-fold branched cover of the toroidal knot $\{3,5\}$ (a result shown formerly by Seifert).

Example

Consider the next colored string (L,ω) in $\mathbb{R}^3$. This colored string was first considered by R. H. Fox in *"A remarkable simple closed curve"*. I will call L *Fox string*. I will prove that the space $M(L,\omega)$ is homeomorphic to $\mathbb{R}^3$. Thus *there exist a 3-fold simple covering* $\widehat{p}:\mathbb{R}^3\to\mathbb{R}^3$ *branched over the Fox string L*.


Q1
Q2
Q3

Let

$$p : M(L, \omega) \to \mathbb{R}^3$$

be the simple branched covering given by the representation ω.

- Select a sequence of 3-cells $\{Q_i\}_{i=1}^{\infty}$ such that $Q_i \subset Int(Q_{i+1})$ and

$$\cup_{i=1}^{\infty} Q_i = \mathbb{R}^3 = S^3 \setminus \{\infty\},$$

- Then, for $i \geq 1$, $p^{-1}(Q_i)$ is a 3-cell Q'_i. In fact,

$$p|p^{-1}(Q_i) : p^{-1}(Q_i) \to Q_i$$

is a 3-fold simple covering of the 3-cell Q_i, branched over two properly embedded arcs; these arcs are embedded exactly as in case $i = 1$.

- By the previous Example, $p^{-1}(Q_i)$ is a 3-cell Q'_i.

- By the previous Example, $p^{-1}(Q_i)$ is a 3-cell Q_i'.
- Then

$$M(L,\omega) = \cup_{i=1}^{\infty} Q_i'.$$

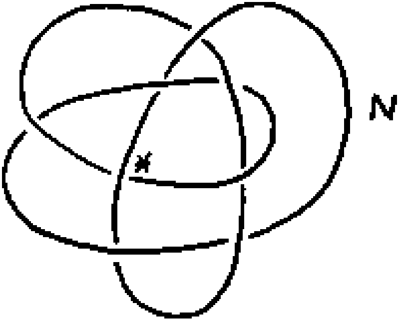
- And from this follows that $M(L,\omega)$ is homeomorphic to $\mathbb{R}^3$ (Brown Theorem). (There is an alternative proof using moves.)

Theorem (Hilden and Montesinos, independently)

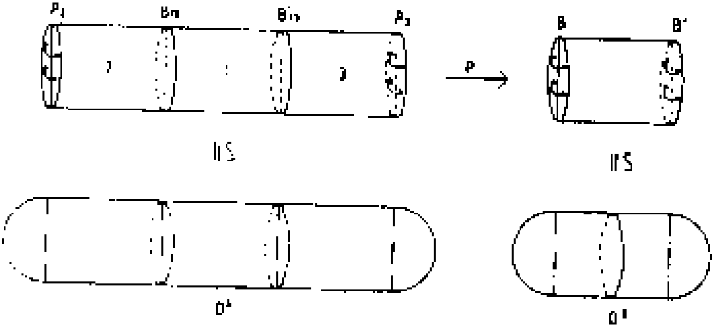
Every closed orientable 3-manifold is of the form $M(L, \omega)$ for countably many mutually inequivalent knots L. In other words, every closed orientable 3-manifold is a simple 3 fold covering of S^3 branched over a knot in many different ways.

- Starting point: *Every closed orientable* 3*-manifold is obtained by surgery on a link in* S^3 (Lickorish and Wallace independently).

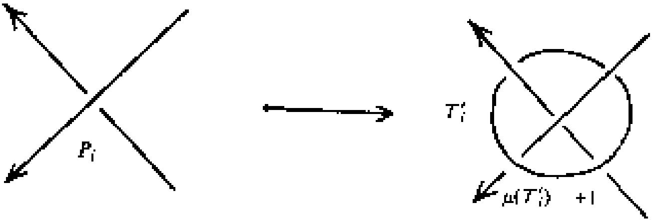
- If the link is not strongly invertible:



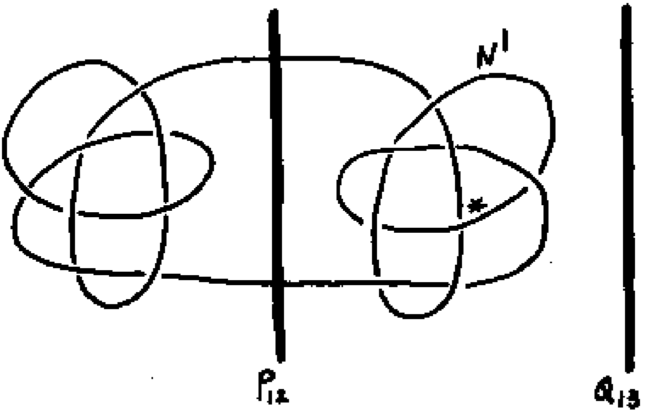
- We need the folding letter trick covering:



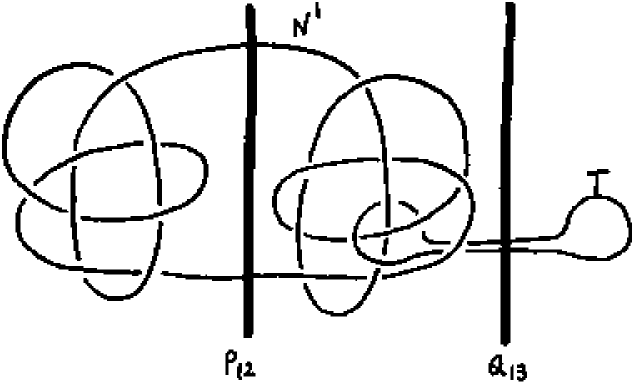
- Together with Hempel´s trick:



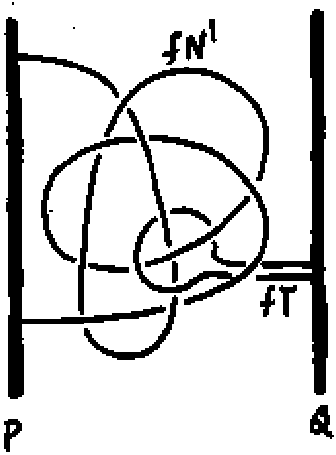
- To almost symmetrize the link



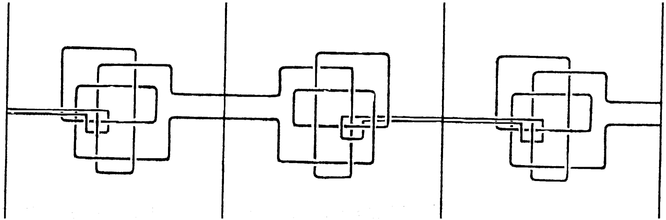
- Hempel´s trick:



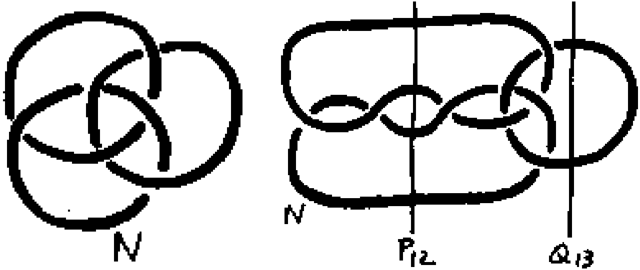
- Downstairs:

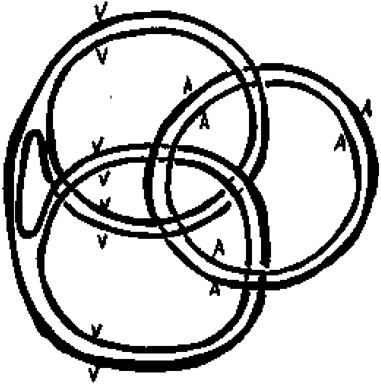


- Upstairs:

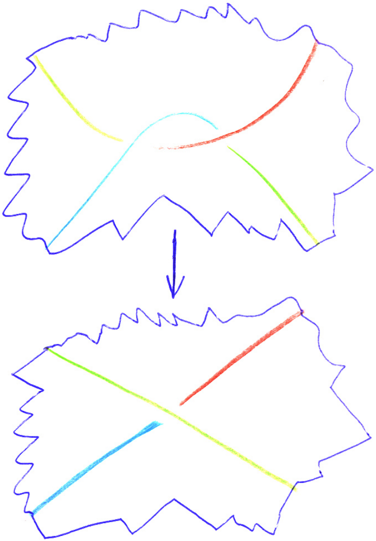


- Example: the 3-torus:




V
V
A
A
A
A
V
V
V
V
A
A
V
V

- Finally: use the move D to connect components of the branching set:



Theorem (Piergallini)

Given two colored knots or links with the same associated covering spaces it is possible to pass from one another by a finite application of moves D*,* D' *and* D''*.*

Sharpenigs:

Theorem (Hilden, M and Thickstun)

Every closed orientable 3-manifold is a simple 3 fold covering of S^3 branched over a knot in such a way that the branched cover bounds a 2-cell.

This gives an elementary proof of this theorem of Hirsch-Smale:

Corollary

Every closed orientable 3-manifold is parallelizable.

Theorem (Hilden-Lozano-M)

Every closed orientable 3-manifold has a simple 3 fold covering p of S^3, which is pullback of the simple 3 fold covering $q : S^3 \rightarrow S^3$ branched over a pair of unknotted unlinked circles.

The proof of this theorem can be used to sharpen a theorem of Hilden:

Corollary

Every closed orientable 3-manifold M has an embedding in $S^3 \times D^2$ such that $M \longrightarrow S^3 \times D^2 \longrightarrow S^3$ is a simple 3-fold branched covering.

This implies the following result of Morris Hirsch:

Corollary

Every closed orientable 3-manifold M has an embedding in S^5.

Generalization:

Theorem (M)

Every orientable, open 3-manifold M admits a (combinatorial) simple 3 fold covering $p: M \rightarrow S^3 \setminus T$ branched over a 1-manifold, where T is a subset of a tamely embedded Cantor set homeomorphic to the space of ends of M.

Corollary

Every open, 3-manifold with one end is a (combinatorial) simple 3 fold covering $p : M \longrightarrow \mathbb{R}^3$ *branched over a string link.*

Whitehead manifold is an example of this.