Branched coverings and three manifolds
Second lecture

José María Montesinos-Amilibia

Universidad Complutense

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Second Lecture. Universal branching sets

Universal branching sets in surfaces

- Take an arbitrary unbounded, surface Σ with an orientation O and a triangulation K of it.

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- Take an arbitrary unbounded, surface Σ with an orientation O and a triangulation K of it.
- Subdivide barycentrically K to obtain another triangulation K'. The vertexes of K' fall naturally in three classes: barycenters of vertexes (resp. edges, faces) of K called, respectively, **bary-vertexes, bary-edges or bary-faces.**

- Any face of K' has a natural orientation o, namely the one given by the following ordering of its vertexes: bary-vertex, bary-edge, bary-face.

- Any face of K' has a natural orientation o, namely the one given by the following ordering of its vertexes: bary-vertex, bary-edge, bary-face.
- Color this face **white** iff $o = O$. Otherwise color it **black**. Then we have obtained a check-board coloration of the faces of K' because two different faces sharing an edge get different colors.

- Think of the sphere S^2 as the result of pasting linearly together two triangles (one white; the other black) along their edges. Call the resulting vertexes 0, 1, 2.

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- Map linearly white (black) triangles of the surface Σ to the white (black) triangle of S^2 in such a way that bary-vertexes (resp. bary-edges, bary-faces) go to 0 (resp. 1, 2).

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- Map linearly white (black) triangles of the surface Σ to the white (black) triangle of S^2 in such a way that bary-vertexes (resp. bary-edges, bary-faces) go to 0 (resp. 1, 2).
- This map is a branched cover:

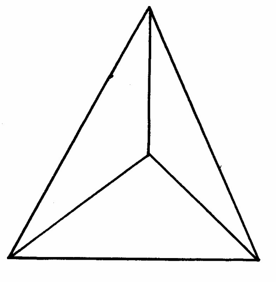
Theorem (Ramírez)

Every unbounded, orientable surface Σ is a covering of the sphere S^2 branched over three points

The argument of Ramírez works in fact for every triangulated unbounded, oriented n-manifold. For $n = 3$, therefore, we have proved the following Theorem.

Theorem (Ramírez)

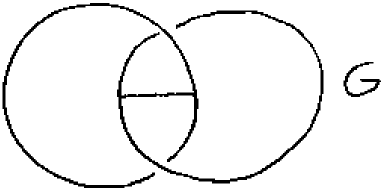
Every unbounded, orientable 3-manifold is a (combinatorial) branched covering of the sphere with branching set the set G of edges of a tetrahedron embedded in the sphere:



The graph G

Corollary (M)

Every unbounded, orientable 3-manifold is a (combinatorial) branched covering of the sphere with branching set the set :



whose exterior has fundamental group of rang two.

Universal branching set

The graph G is a UNIVERSAL BRANCHING SET in the sense that every 3-manifold branches over it. But note that, while in the case of surfaces, the branching set is a manifold, this is not the case if the dimension of the manifold is ≥ 3

Problem

Is there a universal branching set which is a manifold for every dimension?

History:
González-Acuña asked this question. Open for $n > 3$.

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2. Thurston also asked if some familiar knots and links, (like the figure eight knot, Whitehead link or the Borromean rings) were in fact universal branching sets.

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2. It was also clear at the time that some knots and links could not be universal branching sets. (like the trefoil knot).

Theorem (Hilden-Lozano-Montesinos)

The figure-eight knot and the Whitehead and Borromean links are universal branching sets for all closed, orientable 3-manifolds.

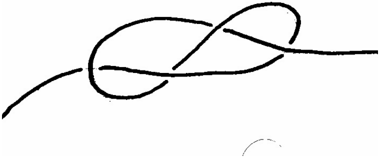
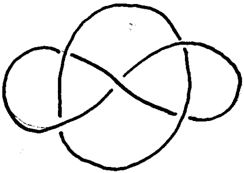
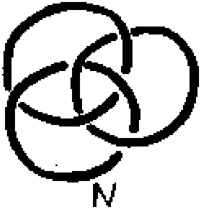


Figure eight knot



Whitehead link



Borromean ring

The proof

- Start with a closed, orientable 3-manifold M^3.

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- Start with a closed, orientable 3-manifold M^3.
- Let $p : M^3 \to S^3$ be a simple 3–fold covering branched over the colored link L.

- Assume L is a closed braid.

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- Applying Montesinos moves to L we can assume every crossing of L has 3 colors.

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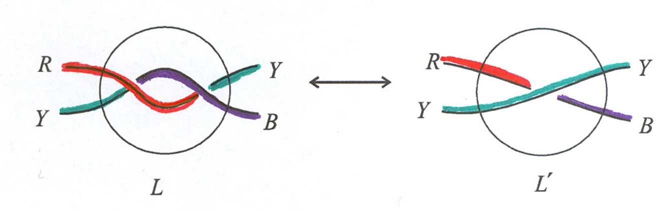


Figure: Montesinos move

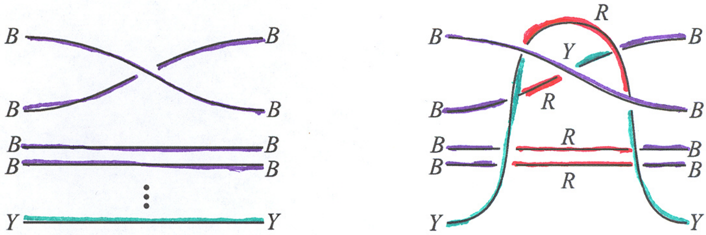


Figure: Every crossing of L has 3 colors.

Using Montesinos moves we can assume all crossings are “positive":

In fact:

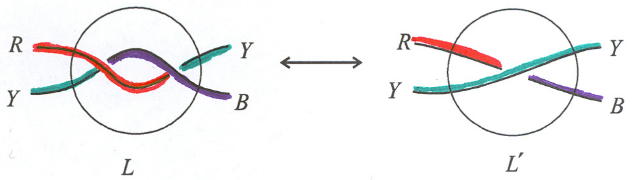
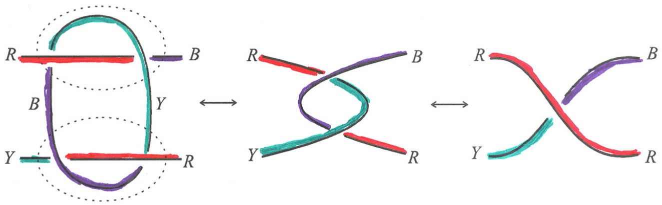
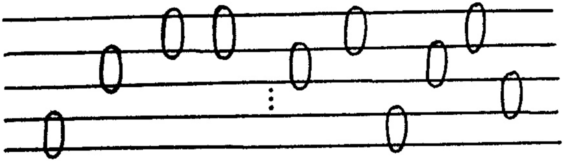


Figure: Montesinos move

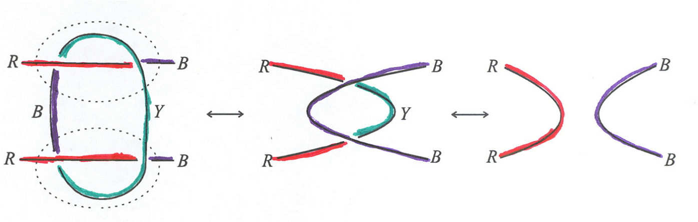
Replace each crossing with a new small circle component:



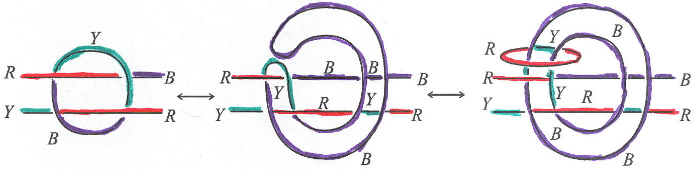
After doing this our link L has two types of components; "braid" or "horizontal" components and "small circle" components:



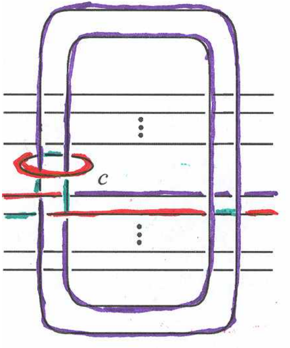
Use the following the Montesinos transformation:



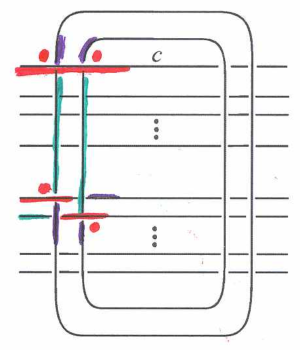
to replace each small circle component by three components as in the right hand side of:



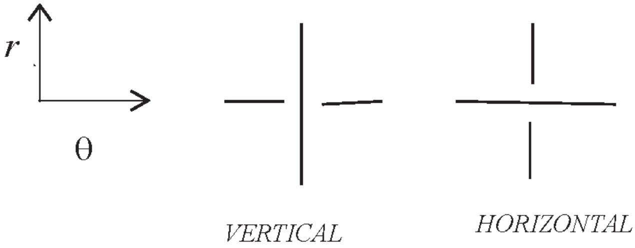
Up to isotopy, each big circle component of L extends over the top and bottom of all the horizontal components:



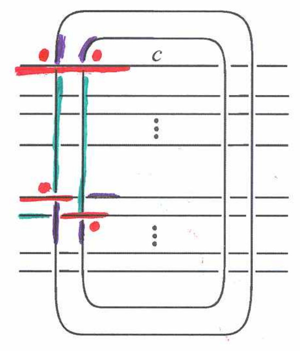
Now isotope the small circle components, **one at a time**, so that they become braid or horizontal components. As we do this to a particular small circle component "c" it becomes the topmost braid component:



Now our link L has two types of components, horizontal components and vertical components. There are also two types of crossings:

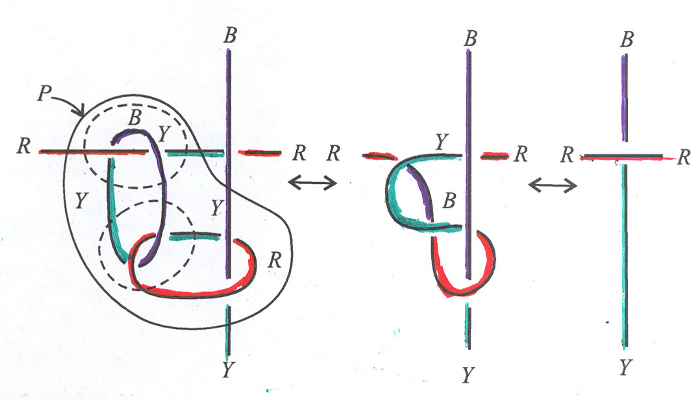


Crucial observation: Every horizontal crossing is 3-colored:

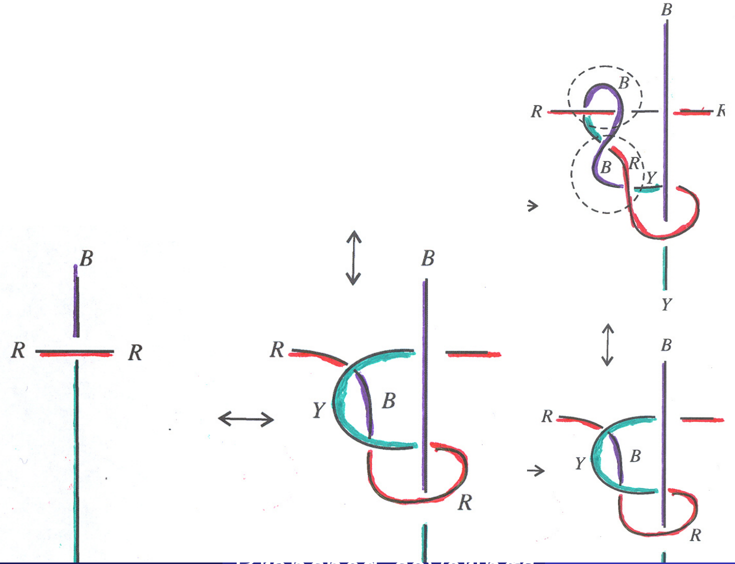


We use the two following Montesinos transformations illustrated in what follows (both are useful) to **replace each horizontal crossing by a vertical crossing.**

First transformation (from right to left):



Second transformation (first and second step):



Second transformation (third step,from right to left):

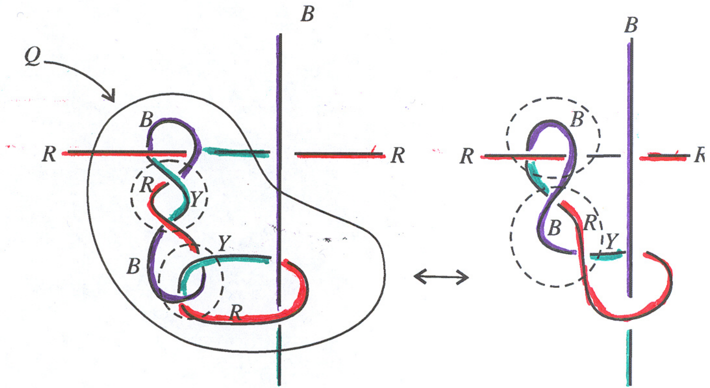
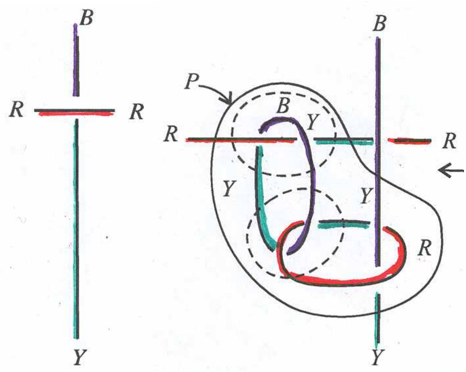
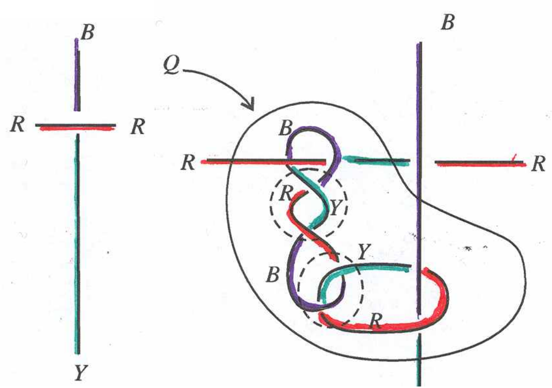


Figure: Final step

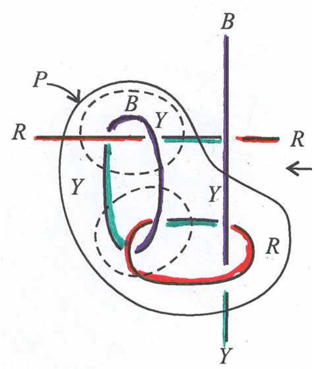
We use these two Montesinos transformations to replace each horizontal crossing by a vertical crossing.
In the course of doing this, new components, contained in the "peanut shaped" balls indicated by a "P" or "Q" are introduced:
First transformation:



Second transformation:

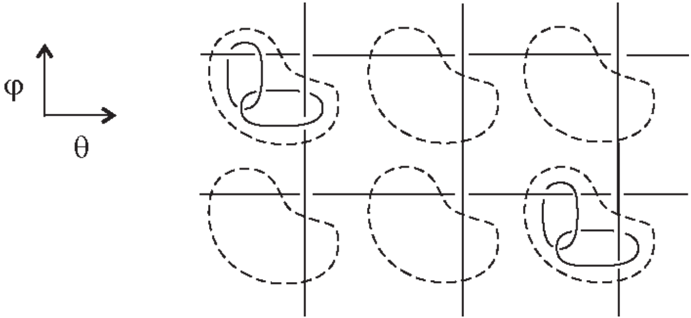


We will use for simplicity the P-peanut shape:



(We can use the P-peanut shape or the Q-peanut shape but never both in the same proof.)

Finally, after a slight isotopy our link L has three types of components; horizontal, vertical and "special". Each special component is contained in a "peanut shaped" topological ball:



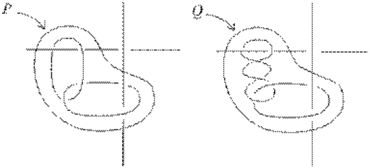
Theorem

Let M^3 be a closed oriented 3–manifold. Then there is a 3–fold simple branched covering $p : M^3 \longrightarrow S^3$ branched over a link L.
The link L has three types of components.

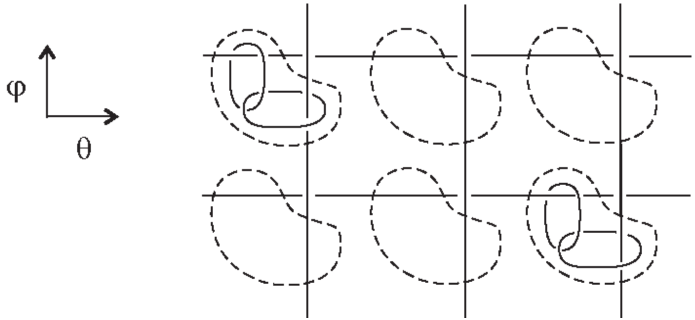
a. **Horizontal.**

b. **Vertical.**

c. **Special.** *These have local projectionsas in either the left or right hand side of the next figure:*



A portion of the image of the link L appears as follows:



Some of the "peanut shaped" balls contain two component links and arcs from a vertical and horizontal component, others contain only arcs from a vertical and horizontal component.

- From this point (Hilden-Lozano-M: Collectanea Mathematica, 34(1):19-28 (1983) the proof diverges in two different result.

- We will work out both together.

- Define two rotations T_1 and T_2 of $S^3 = E^3 \cup \{\infty\}$. The rotation T_1 is simply the m–fold rotation about the z–axis; the rotation T_1 leaves invariant the set of horizontal and the set of vertical components of the link L.

- The rotation T_2 has as its axis a circle. It leaves the set of horizontal and the set of vertical components of L invariant. It cyclically permutes the horizontal components and it sends each vertical component to itself. Its restriction to a vertical component is just the usual n–fold rotation of a circle.

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- We can and shall assume that both rotations T_1 and T_2 leave the peanuts of the link L invariant.

- Consider the map $f : S^3 \to S^3/T_1 = S^3$ which is an m–fold cyclic branched covering $S^3 \to S^3$ with branch set the trivial knot or z–axis, induced by rotation T_1.

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- (**Alternative**: *a branched cover* $f_1 : S^3 \longrightarrow S^3$ *coinciding with* f *out of a tubular nbd of the branch set which is the doble of the trivial knot trivial knot, and* with branch indexes 1 and 2).

- The branch set for the composite map $f \circ p : M^3 \to S^3$ consists of the branch set for f (the z-axis) plus the image under f of the branch set of p.

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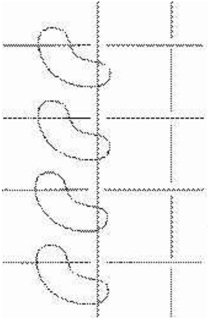
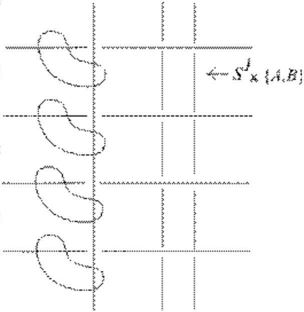
- (**Alternative**: *using* f_1 *the branch set for the composite map* $f_1 \circ p : M^3 \longrightarrow S^3$ *consists of the branch set for* f_1 *(the double of the* z*-axis) plus the image under* f *of the branch set of* p*.)*

The part $f(L)$ of the branch set of $f \circ p : M^3 \to S^3$ has one vertical component and n–horizontal components and n–"peanut" components (we depict also the image of the z-axis of rotation):



branch set of
$f \circ p$



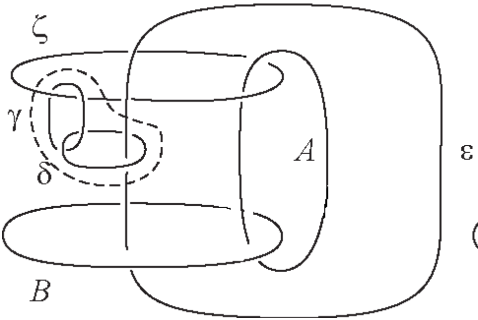
branch set of $f_1 \circ p$

- Consider the map $g : S^3 \longrightarrow S^3/T_2 = S^3$ which is an n–fold cyclic branched covering $S^3 \longrightarrow S^3$ with branch set the trivial knot, induced by rotation T_2.

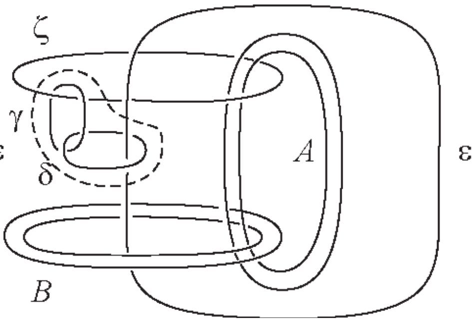
- The branch set for the composite map $g \circ f \circ p : M^3 \to S^3$ consists of the branch set for g (a circle) plus the image under g of the branch set of $f \circ p$.

- The branch set for the composite map $g \circ f \circ p : M^3 \longrightarrow S^3$ consists of the branch set for g (a circle) plus the image under g of the branch set of $f \circ p$.
- (**Alternative**: *using* g_1 *the branch set for the composite map* $g_1 \circ f \circ p : M^3 \longrightarrow S^3$ *consists of the branch set for* g_1 *(the double of a circle) plus the image under* g *of the branch set of* $f_1 \circ p$*.)*

The branching set of $g \circ f \circ p : M^3 \to S^3$ (or of $g_1 \circ f_1 \circ p$) :

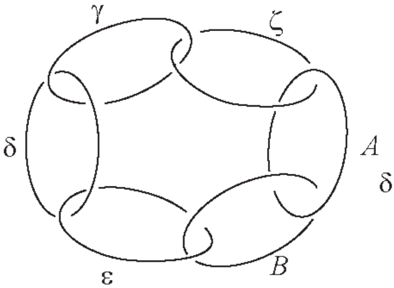


Branch set of $g \circ f \circ p : M^3 \to S^3$

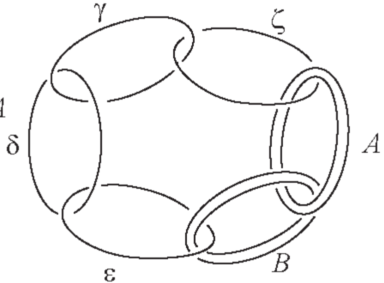


Branch set of $g_1 \circ f_1 \circ p$

Can be isotoped to the link:



Branch set of
$g \circ f \circ p : M^3 \to S^3$



Branch set of $g_1 \circ f_1 \circ p$

- The first link has a 3–fold symmetry. Let T_3 be this 3–fold rotation and let $h : S^3 \to S^3 = S^3 / T_3$ be the resulting branched covering.

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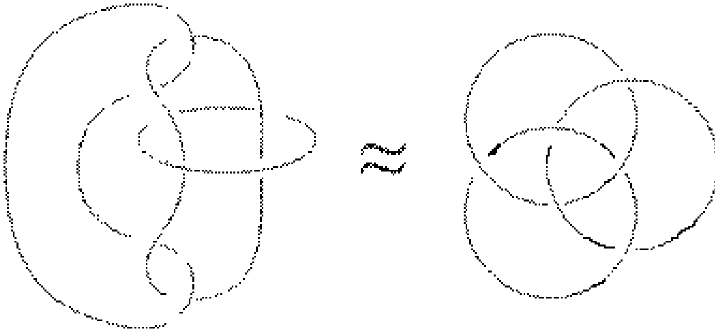
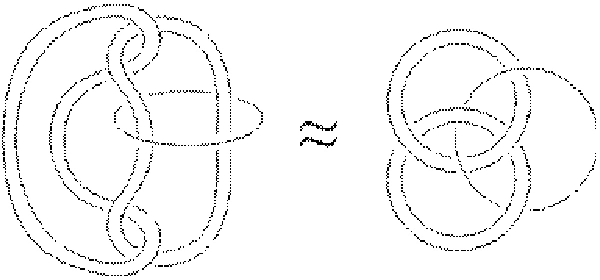
- (*Let* $h_1 : S^3 \to S^3$ *be equal to* h *except in solid torus nbd of the axis of rotation of* h *which it is replaced with two parallel axes and the branch indexes of* h_1 *are* 1 *and* 2*).*

- The map $h \circ g \circ f \circ p$ is a $9mn$ to 1 branched covering of S^3 by M^3 with branch set h (branch set $g \circ f \circ p$) together with the image of the rotational axis:





We summarize this result in the form of a theorem:

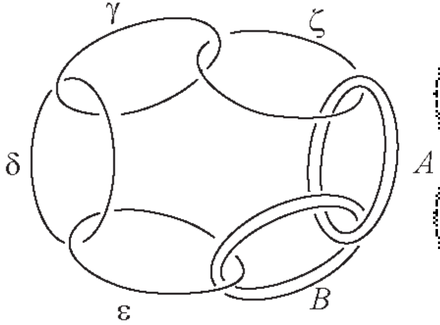
Theorem (Hilden-Lozano-M)

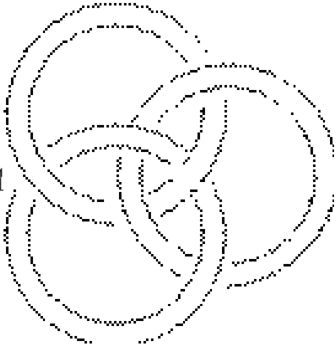
Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the Borromean rings. That is, the Borromean rings is a universal branching set.

Using the second link and the map h_1 instead of h we have proved:

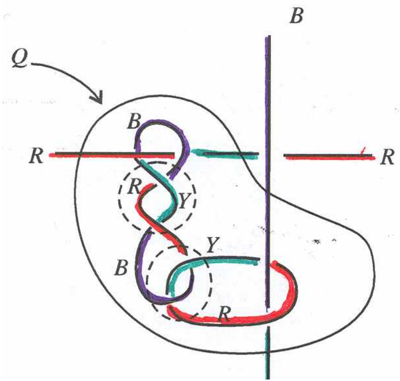
Theorem (Lozano-M)

Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the double Borromean rings and the branching indexes of the covering are 1 and 2.





If in the above proofs we use the peanut Q



instead of P we get

Theorem (Hilden-Lozano-M)

Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the Whitehead link. That is, the Whitehead link is a universal branching set.

Theorem (Lozano-M)

Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the double Whitehead link and the branching indexes of the covering are 1 and 2.

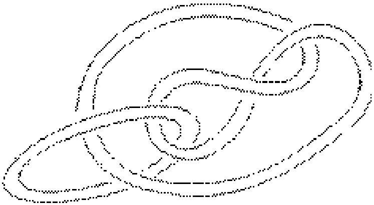


Figure: The double of the Whitehead link