Branched coverings and three manifolds
Third lecture

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Third Lecture. Control on branch indexes.

Surfaces: Control on branch indexes.

- Remember Ramirez Theorem: *every unbounded, orientable surface* Σ *there is a covering* $f : \Sigma \to S$ *of the sphere* S *branched over three points* v_0, v_1, v_2, *marked, respectively,* 0, 1, 2.

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- Similarly, the branch index of $w \in f^{-1}(v_1)$ is 2.
- But there is absolutely no control on the branch index of points belonging to the fiber of v_0.

Problem

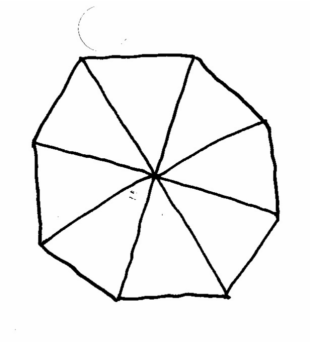
Is it possible to find, for any Σ, a covering $f : \Sigma \longrightarrow S$ of the sphere S, branched over three points v_0, v_1, v_2 with **extrict control on the branching indexes***?*

Solution

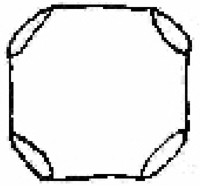
Every closed, orientable surface is a covering of S^2 branched over three points A, B and C. The branching indexes on top of A (resp. B; C) are all 2 (resp. all 3; all 4 or 8).

Proof

- Take a regular octogon Ω and from its center draw segments to its vertices. This gives a triangulation of Ω by 8 triangles.

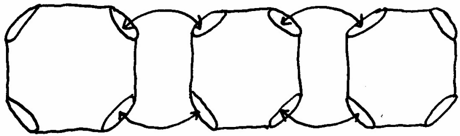


- Paste together alternating sides of two of these triangulated octogons to obtain a sphere *Soooo* with 4 holes.



- Pasting the holes in pairs we get the orientable surface F_2 of genus 2. Note that F_2 is triangulated by 16 triangles so that each vertex belongs to 8 of them (has valence 8).

- Paste n copies of *Soooo* together we can obtain a surface $F_{n-1}oooo$ of genus $n-1$ with four holes. Pasting now the holes in pairs we get the orientable surface F_{n+1} of genus $n+1$.



We have proved.

Theorem

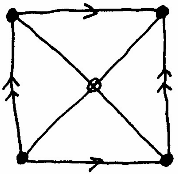
Every orientable, closed surface F_g of genus $g \geq 2$ is triangulated by $16(g-1)$ triangles with vertexes of valence 8.

The case of genus 1

- The torus F_1 is a square with opposite sides identified.

- We can divide it in 4 triangles by connecting its center to its vertexes.

- Thus the torus can be divided in triangles with one vertex of valence 4 and one vertex of valence 8.



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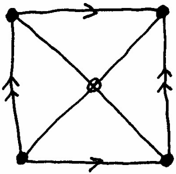


- Apply Ramírez construction to these triangulations K and the octahedral triangulation of S^2 to obtain the following Theorem.

Theorem

Every closed, orientable surface is a covering of S^2 branched over three points A, B and C. The branching indexes on top of A (resp. B; C) are all 2 *(resp. all* 3*; all* 4 *or* 8*).*

Reformulation of this theorem in orbifold terms.

Definition (Kato)

A (combinatorial) orientable 2-ORBIFOLD is (N, v), where (THE UNDERLYING SPACE) N is an unbounded, triangulable 2-manifold; and the ISOTROPY function

$$v : V \to \mathbb{N}$$

is a function from the (SINGULAR) set V of vertices of some triangulation of N into the set of natural numbers such that $v(x) = 0$ for all but a discrete subset of N.

Example

$S238$ denote the 2-orbifold with underlying space the 2-sphere S and singular points A, B, C with isotropies 2, 3, 8 respectively.

Definition (Kato)

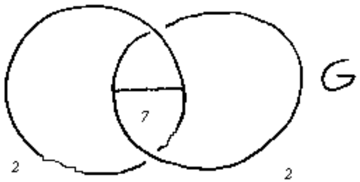
A (combinatorial) orientable 3-ORBIFOLD is (N, B, v), where (THE UNDERLYING SPACE) N is an unbounded, triangulable 3-manifold; the (SINGULAR) set B is a polyhedral graph in N ; and v is an (ISOTROPY) function that associates an integer > 1 to each component of $B \setminus V_B$, where

$$V_B = \{x \in B : valence(x) > 2\},$$

and the integer 1 to $N \setminus B$.We assume that B has no isolated point.

Example

(S^3, G, v):



Definition

Let (M, B', v') and (N, B, v) be two orientable orbifolds. An orbifold covering $f : (M, B', v') \to (N, B, v)$ is a covering $f : M \to N$ branched over B such that, $B' \subset f^{-1}(B)$ and

$$v'(x)b(x) = v(y)$$

for every $x \in f^{-1}(y)$, $y \in B \backslash V_B$.

Therefore if $f : M \longrightarrow N$ is a covering branched over B and (N, B, v) is an orientable orbifold, then M is the underlying space of an orbifold such that f is an orbifold covering iff $b(x) \mid v(f(x))$, for all $x \in M$.

Definition (Kato)

An orbifold is uniformizable if it admits a non-singular orbifold covering.

An orbifold to be uniformizable, must be LOCALLY UNIFORMIZABLE.
This is easy to check:

1. the valence of $x \in V_B$ is 3.

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   - $(2, 3, 3)$
   - $(2, 3, 4)$
   - $(2, 3, 5)$

Definition

An orientable n-orbifold U is said to be UNIVERSAL iff every closed, orientable n-manifold is the underlying space of an orbifold that is an orbifold-covering of U.

We can reformulate the

Theorem

Every closed, orientable surface is a covering of S^2 branched over three points A, B and C. The branching indexes on top of A (resp. B; C) are all 2 *(resp. all* 3*; all* 4 *or* 8*).*

as follows

Theorem

The 2-orbifold $S238$ is universal.

Example

The 2-orbifold $S236$ is not universal.

Proof

- $S236$ is a euclidean orbifold. In fact $S236$ is the result of pasting together along their boundary two euclidean triangles of angles 30^o, 60^o and 90^o.

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- $S236$ is a euclidean orbifold. In fact $S236$ is the result of pasting together along their boundary two euclidean triangles of angles 30^o, 60^o and 90^o.
- Any orbifold Q covering $S236$ is euclidean except at some cone points with angles $\alpha < 2\pi$.

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- Therefore the underlying surface $|Q|$ has a metric of non-negative curvature.

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- Thus $S236$ is not universal.

The three levels

- **Combinatorial level**: There is a universal branching set L for closed, orientable 2-surfaces S. (Every S is a branched covering of S^2 branched over L).

- **Orbifold level**: There is a universal 2-orbifold. (Every S is a branched covering of S^2 branched over L and the branching indexes are bounded).

- **Geometric level**: There is a **hyperbolic** universal 2-orbifold.

- The universal 2-orbifold S_{238} is a hyperbolic orbifold, quotient of the hyperbolic plane H^2 under the action of a Fuchsian group U. This group can be called UNIVERSAL because for every closed, orientable surface S there is a subgroup $\Gamma \leq U$ of finite index such that H^2/Γ is homeomorphic to S .

Definition

A subgroup U of direct isometries of H^3 is called a **universal group** iff given a closed, orientable 3-manifold M there is a finite index subgroup Γ of U such that H^3/Γ is homeomorphic to M.

We know that universal branching sets exist in dimension 3 (combinatorial level):

Problem

Do universal 3 -orbifolds exist? Do universal groups exist in dimension three?

Three-manifolds.

Universal 3-orbifolds do exist.

Theorem (Lozano-M)

Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the 2-(standard) cable of the Borromean rings and the branch indexes are 1 an 2. That is, the double of the Borromean rings BB with isotropy 2 in each component is a universal 3-orbifold, denoted $(BB, 2)$.

Theorem

Let M^3 be a closed orientable 3–manifold. Then M^3 is a branched covering of S^3 with branch set the 2-(standard) cable of the Whitehead link WhWh. That is $(WhEWh, 2)$ is a universal 3-orbifold.

These cables are not hyperbolic links. But one can even construct a a universal 3-orbifold $(L, 2)$ where the link L is hyperbolic.

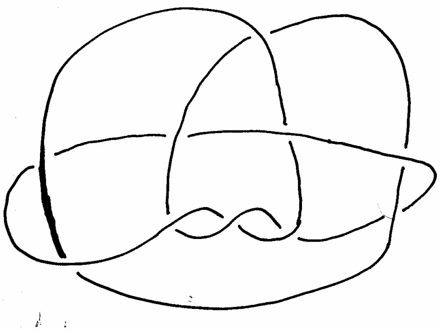
Theorem (Brumfield, H-L-M, Ramirez-Losada, Short, Tejada, Toro)

There are universal 3-orbifolds $(K, 2)$ *where* K *is a knot.*

The knot is very complicate.

Problem

Is the orbifold $(10_{161}, 2)$ *a universal 3-orbifold?*



10_{161} knot

The knot has been selected because it has the necessary condition of being the singular set of a cone-structure which is hyperbolic between the angle 0^o and an angle (computable) $> 180^o$.

But there is an important universal 3-orbifold which is hyperbolic and which has many interesting properties.

Universal groups

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- Here v associates integers $m, n, p > 1$ to the components of B.
- We will write B_{mnp} to denote this orbifold.

Fact

B_{222} is not a universal 3-orbifold.

Proof

- Tessellate the euclidean space E^3 by $2 \times 2 \times 2$ cubes all of whose vertices have odd integer coordinates.

- Let $\widehat{U}$ be the group generated by 180° rotations in the axes a, b, and c (the cube here is centered at the origin):

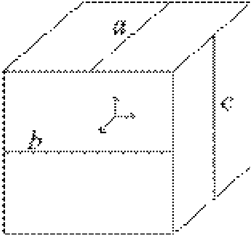


Figure: $2 \times 2 \times 2$ cube

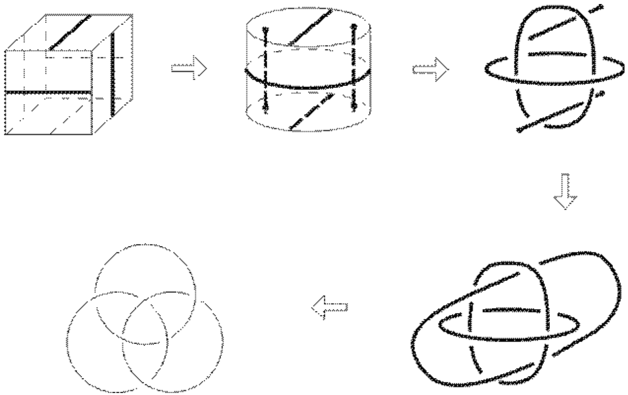
- The group $\widehat{U}$ is a well known Euclidean crystallographic group that preserves the tessellation.

- The group $\widehat{U}$ is a well known Euclidean crystallographic group that preserves the tessellation. A fundamental domain for $\widehat{U}$ is the $2 \times 2 \times 2$ cube centered at the origin.

- The map $p : E^3 \to E^3/\widehat{U} \approx S^3$ is an orbifold covering of B_{222}, where the orbifold E^3 is non-singular (p is a uniformization of B_{222}).

- This gives S^3 the structure of a Euclidean orbifold with singular set the Borromean rings and singular angle 180°.

- We can see that $E^3/\widehat{U}$ equals S^3 with singular set the Borromean rings by making face identifications in the fundamental domain:



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- $|Q|$ cannot be a hyperbolic manifold

- Thus B_{222} is not universal.

However:

Theorem (Hilden-Lozano-M-Whitten)

B_{444} is a universal 3-orbifold. Moreover B_{444} is hyperbolic and its holonomy group U is a universal Kleinian group.

[New Proof (Brumfield,H-L-M,Ramirez-Losada, Short,Tejada,Toro)]

- Take a tessellation of E^3 by $6 \times 6 \times 6$ cubes with integer coordinates that are odd multiples of three.

- Let $\widetilde{U}$ be the group generated by 180° rotations in the axes a', b', c' where $a' = (t, 0, 3)$, $b' = (3, t, 0)$ and $c' = (0, 3, t)$; $-\infty < t < \infty$.

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- Of course $E^3/\widetilde{U} = S^3$.

- As the rotations about a', b' and c' belong to $\widehat{U}$ then $\widetilde{U} \subset \widehat{U}$ and $\left[\widehat{U} : \widetilde{U}\right] = 27$ by comparing the size of fundamental domains.

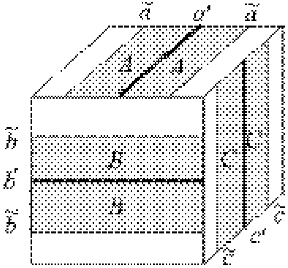
- $\widetilde{U}$ is not a normal subgroup of $\widehat{U}$. We are in fact really interested in the map $t : S^3 = E^3/\widetilde{U} \to E^3/\widehat{U} = S^3$ induced by the inclusion of $\widetilde{U}$ in U.

- $\widetilde{U}$ is not a normal subgroup of $\widehat{U}$. We are in fact really interested in the map $t : S^3 = E^3/\widetilde{U} \to E^3/\widehat{U} = S^3$ induced by the inclusion of $\widetilde{U}$ in U.
- t is a 27 to 1 irregular covering branched over the Borromean rings. The branch indexes are all 1 or 2.

- The doubled Borromean rings occur as a sublink of the preimage of the branch set. The doubled Borromean rings consist of three pairs of components:

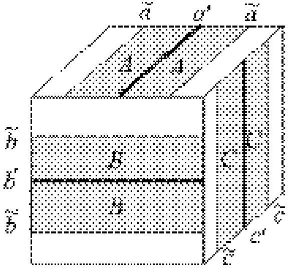
Fact

- *Each pair bounds an annulus disjoint from the other pairs.*



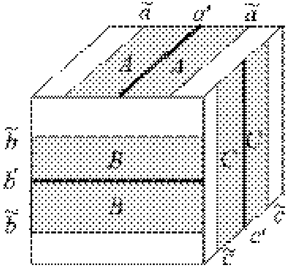
Fact

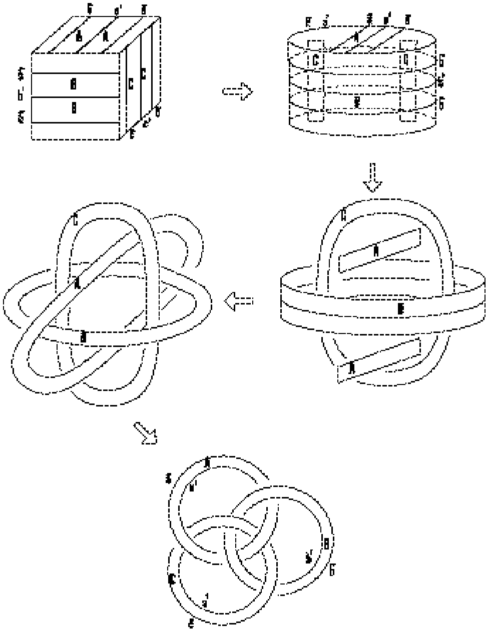
- *Each pair bounds an annulus disjoint from the other pairs.*
- *Each pair is mapped under* $t : S^3 = E^3/\widetilde{U} \to E^3/\widehat{U} = S^3$ *to the same component of the Borromean rings.*



Fact

- *Each pair bounds an annulus disjoint from the other pairs.*
- *Each pair is mapped under* $t : S^3 = E^3/\widetilde{U} \rightarrow E^3/\widehat{U} = S^3$ *to the same component of the Borromean rings.*
- *And each pair contains one component of the branch cover and one component of the pseudo branch cover.*





So far we have:

Fact (1)

Let M^3 be a closed orientable 3–manifold. Then there is a branched covering $p : M^3 \to S^3$ branched over the double Borromean rings and with branching indexes 1 and 2.

Fact (2)

There is a branched covering $t : S^3 = E^3/\widetilde{U} \to E^3/\widehat{U} = S^3$ *branched over the borromean rings* B *with branching indexes* 1 *and* 2 *such that the double of the borromean rings* BB *is a subset of* $t^{-1}(B)$. *Moreover the sublink* B *of* BB *is part of the branching cover. The remaining of* BB *(also a sublink* B*) is part of the pseudo-branching cover.*

Corollary

The composition $t \circ p : M^3 \longrightarrow S^3$ *is a branched covering branched over the Borromean rings and with branching indexes* 1, 2 *and* 4. *Therefore the* 3*-orbifold* B_{444} *(more generaly* $B_{4a,4b,4c}$, *for any positive integers* a, b, c*) is universal.*

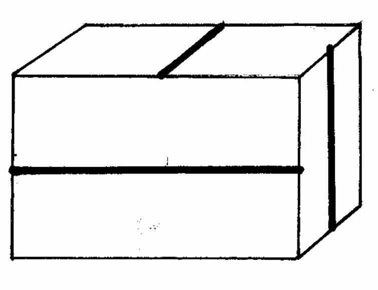
Theorem (Hilden,Lozano,M, Whitten)

B_{444} is hyperbolic. Thus the holonomy group U of B_{444} is universal.

Proof

- Only remains to see that B_{444} is hyperbolic (W. Thurston):

- Consider the following combinatorial dodecahedron:



Pasting faces in pairs, by reflection on the thickened edges (there are 6 of them, not visible in the picture, but the ones in opposite faces of the paralelepipedon are parallel) we get S^3.

- The boundary of the dodecahedron is sent to the three golden ratio cards, and the thickened edges go to the borromean link B. If we think on the above dodecahedron as a euclidean parallelepipedon, then S^3 inherits a euclidean structure with singular set B. Here the cone angle is π. Thus $(B, 2)$ is a euclidean orbifold.

- But if we take a regular dodecahedron D inside a sphere S, both centered at the origen of R^3, then the interior of S is the projective model of hyperbolic 3-space H^3. The dodecahedron D is also regular in H^3 but its dihedral angles depend on the radius of the sphere S.

- If the vertexes of D lie on S the dihedral angles are of 60^o and when the radius of S tends to infinite then D tends to be euclidean with angles of approximately 116^o. In between there is a radius for which the angles are of 90^o. After the identifications, S^3 inherits a hyperbolic structure with singular set B. The cone angle is $\pi/2$. Thus $(B, 4)$ is a hyperbolic orbifold.

Corollary (Brumfield, H-L-M, Ramirez-Losada, Short, Tejada, Toro)

Geometric branched covering space Theorem Let M^3 be a closed orientable 3–manifold. Then there are subgroups G and G_1 of the universal group U such that $[G_1 : G] = 3$ and $[U : G] < \infty$ and $M^3 = H^3/G$ and $S^3 = H^3/G_1$. The map induced by the inclusion of groups $H^3/G \to H^3/G_1$ is a 3–fold simple branched covering of S^3 by M^3.

Proof

- The branched covering $t : S^3 = E^3/\widetilde{U} \to E^3/\widehat{U} = S^3$ branched over B is an orbifold covering $t : S^3 = Q \to B_{444} = S^3$ (because the branch indexes 1 and 2 divide 4).

- The orbifold Q has singular set formed by the curves of the seudo-branch set (with isotropy 4) of t and the curves of the branch set of t with isotropy 2.

- $p : M^3 \rightarrow S^3 = Q$ is an orbifold covering of Q.

- $p : M^3 \to S^3 = Q$ is an orbifold covering of Q.
- (Because $p : M^3 \to S^3$ branches over part of the singular set of Q and the branch indexes of p divide 2).

- The domain of p is an orbifold M^3_o with underlying space M^3 and singular set part of $(t \circ p)^{-1}(B)$ and valoration 2

- The universal orbifold covering of B_{444} is H^3 and the group of automorphisms of it is U.

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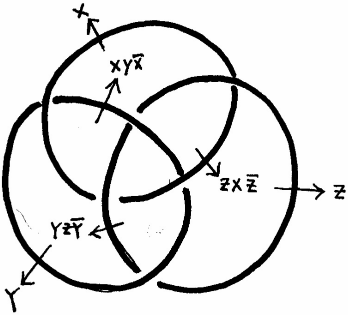
- **The Theorem follows.**

The universal group U is the group of automorphisms of the universal covering $p : H^3 \longrightarrow B_{444}$. Then U is isomorphic to the fundamental group $\pi_1^o(B_{444})$ of the orbifold B_{444}

- The group $\pi_1^o(B_{444})$ comes from $\pi_1(S^3 \setminus B)$ by killing the fourth powers of the meridians of B. The group $\pi_1(S^3 \setminus B)$ has a presentation with three generators x, y, z (meridians of the components of B) and three relations (anyone of which is unnecessary) that declare the commutativity of each meridian with its corresponding longitud.



Meridians

- Then we have the following presentation for $U = \pi_1^o(B_{444})$ is:

$$U = \left| x, y, z : \left[x, \left[z^{-1}, y\right]\right] = \left[y, \left[x^{-1}, z\right]\right] = \left[z, \left[y^{-1}, x\right]\right] = x^4 = y^4 = z^4 \right.$$

- Under the isomorphism from the group U of automorphisms of $p : H^3 \to B_{444}$ and the fundamental group $\pi_1^o(B_{444})$ of the orbifold B_{444} the meridians x, y, z correspond to the 90^o rotations around the three thickened edges of the dodecahedron. Thus the group U is generated by these three rotations (that we denote x, y, z) subject to the above relations.

- The group U acts on H^3 and the regular dodecahedron D with 90^o dihedral angles is the Voronoi domain of this action with respect to the center of D. Thus H^3 is tessellated by replicas of D. There are 4 replicas around every edge and 8 replicas around every vertex. The dual tessellation is formed by cubes with $2\pi/5$ dihedral angles.

Some consequences of U being universal.

- Let M be an arbitrary closed, orientable manifold.

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- Let M be an arbitrary closed, orientable manifold.
- Then there is some $\Gamma \leq U$ of finite index such that H^3/Γ is homeomorphic to M.
- In the original proof showing that U is universal by H-L-M-Whitten, it was shown that Γ can always be supposed to contain a 90^o rotation. We will assume this.

Look to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & U & \longrightarrow & C_4 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow\approx & & \\
0 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & C_4 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow\approx & & \\
0 & \longrightarrow & S & \longrightarrow & t(\Gamma) & \longrightarrow & C_4 & \longrightarrow & 0
\end{array}$$

- L =kernel of the epimorphism sending x, y, z to $1 \in C_4 = Z/4Z$;

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\end{array}$$

- L =kernel of the epimorphism sending x, y, z to $1 \in C_4 = Z/4Z$;
- $N = L \cap \Gamma$ is a normal subgroup of Γ;

- $$\begin{array}{ccccccccc} 0 & \to & L & \to & U & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & N & \to & \Gamma & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & S & \to & t(\Gamma) & \to & C_4 & \to & 0 \end{array}$$

- $$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & U & \longrightarrow & C_4 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & C_4 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \longrightarrow & S & \longrightarrow & t(\Gamma) & \longrightarrow & C_4 & \longrightarrow & 0 \end{array}$$
- $t(\Gamma)$ is the subgroup of Γ generated by rotations (it is a normal subgroup);

- $$\begin{array}{ccccccccc} 0 & \to & L & \to & U & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & N & \to & \Gamma & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & S & \to & t(\Gamma) & \to & C_4 & \to & 0 \end{array}$$
- $t(\Gamma)$ is the subgroup of Γ generated by rotations (it is a normal subgroup);
- S is the subgroup $N \cap t(\Gamma)$.

- $$\begin{array}{ccccccccc} 0 & \to & L & \to & U & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & N & \to & \Gamma & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & S & \to & t(\Gamma) & \to & C_4 & \to & 0 \end{array}$$

- $$\begin{array}{ccccccccc} 0 & \to & L & \to & U & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & N & \to & \Gamma & \to & C_4 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \to & S & \to & t(\Gamma) & \to & C_4 & \to & 0 \end{array}$$
- Since we are assuming that $t(\Gamma) \to C_4$ is onto, the vertical arrows in the third column are isomorphisms.

Corollary (HLM)

M is simply connected iff Γ is generated by rotations, that is, iff $\Gamma = t(\Gamma)$.

Variation of a Theorem by Sakuma:

Corollary (HLM)

Every closed, orientable 3-manifold has a 4-fold cyclic branched covering which is a hyperbolic manifold. The cyclic action is by isometries.

Corollary (HLM)

The fundamental group of a closed, orientable 3-manifold acts freely as a group of isometries of a hyperbolic manifold.

Corollary (HLM)

Every closed, orientable 3-manifold is the underlying space of a hyperbolic orbifold with singular set a link, and isotropy cyclic of orders 2 or 4

Corollary (HLM)

Every closed, orientable 3-manifold has a euclidean cone manifold structure with a link as singular set. The cone angles are either π or 4π.

Corollary (HLM)

There exists a hyperbolic manifold M which is a F_5-bundle over S^1, such that the quaternion group acts on it as a subgroup of isometries, giving the orbifold B_{444} as quotient. The manifold M has infinitely many different surface-bundle structures over S^1.

Theorem (HLM)

The universal group is an arithmetic group

- The problem of finding automorphic functions for the universal coverings of B_{444} is still open. The case $B_{\infty\infty\infty}$ has been solved by K. Matsumoto:

- The problem of finding automorphic functions for the universal coverings of B_{444} is still open. The case $B_{\infty\infty\infty}$ has been solved by K. Matsumoto:
- Automorphic functions with respect to the fundamental group of the complement of the Borromean rings. J. Math. Sci. Univ. Tokyo 13 (2006), no. 1, 1–11.