Resolution of singularities of certain 4-fold covers in dimension 2

Tokyo Metropolitan University
Taketo Shirane

In Hiroshima, 2009

§1. Introduction

- Horikawa introduced a method of resolving singularities of double covers over a smooth surface (1975).

- Ashikaga generalized this method to triple covers over a smooth surface (1992).

In this talk, we will discuss resolution of singularities of certain 4-fold covers of surfaces over $\mathbb{C}$ by using $\mathfrak{S}_4$-covers.
($\mathfrak{S}_4$: the symmetric group of degree 4)

§2. Definitions

Def. (Galois covers)
Let $\pi : X \to Y$ be a finite surjective morphism of normal varieties.
Note that we can regard $\mathbb{C}(Y)$ as a subfield of $\mathbb{C}(X)$.

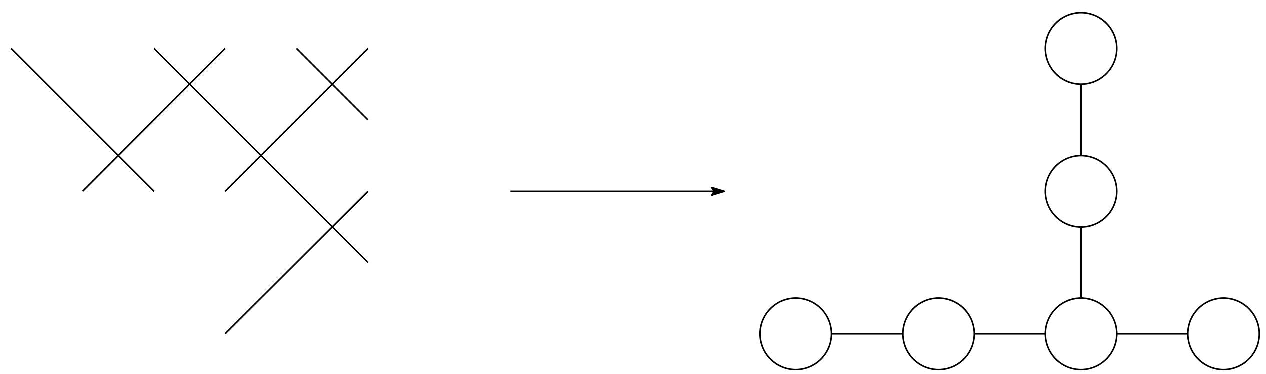
$$\pi \text{ : a Galois cover} \overset{\text{def}}{\Longleftrightarrow} \mathbb{C}(X)/\mathbb{C}(Y) \text{ : a Galois extension}$$

If $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$, we simply call π a G-cover.
(G : a finite group)

Def. (good resolutions)
Let $\nu : X' \to X$ be a resolution of singularities of X.

ν : a good resolution $\overset{\text{def}}{\iff}$ the exceptional set is a divisor with only simple normal crossings.

Example.
In dimension 2, the exceptional set is as follows, and its dual graph is the following:



§3. 4-fold covers and $\mathfrak{S}_4$-covers

Let $\pi : X \to Y$ be a 4-fold cover. There is an element z of $\mathbb{C}(X)$ such that $\mathbb{C}(X) = \mathbb{C}(Y)(z)$, and

$$z^4 + g_1 z^2 + g_2 z + g_3 = 0$$

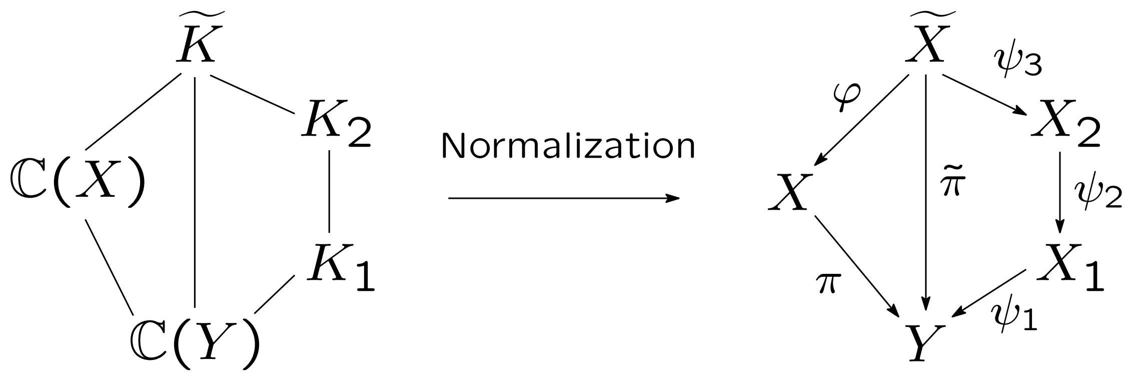
for some $g_1, g_2, g_3 \in \mathbb{C}(Y)$.

Let

$\widetilde{K}$: the Galois closure of $\mathbb{C}(X)/\mathbb{C}(Y)$.

Assume $\mathrm{Gal}(\widetilde{K}/\mathbb{C}(Y)) \cong \mathfrak{S}_4$.

By Lagrange's method to solve quartic equations, we have the following diagram of field extensions, as well as the following diagram of normalizations of Y :



where

$\tilde{\pi}$: an $\mathfrak{S}_4$-cover, φ : an $\mathfrak{S}_3$-cover,

ψ_1 : a double cover, ψ_2 : a cyclic triple cover, and

ψ_3 : a V_4-cover

($V_4 := \{id, (12)(34), (13)(24), (14)(23)\} \subset \mathfrak{S}_4$, the Klein group).

§4. Our strategy to resolve a singularity

Let $\pi : V \to W$ be a 4-fold cover of surfaces. Let $\mu : W_0 \to W$ be a composition of blowing-ups. To have the $\mathbb{C}(V)$-normalization V_0 of W_0, we construct the $\mathfrak{S}_4$-cover over W_0. We have V_0 as the quotient of $\widetilde{V}$ by $\mathfrak{S}_3$. Then we have a morphism $\nu : V_0 \to V$ by Stein factorization.

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$$\begin{array}{c} V \\ \pi \big\downarrow \\ W \end{array}$$

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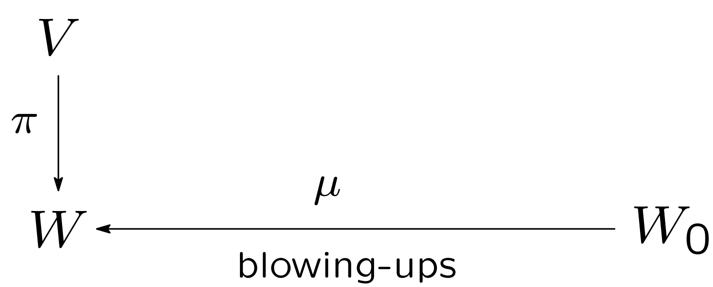

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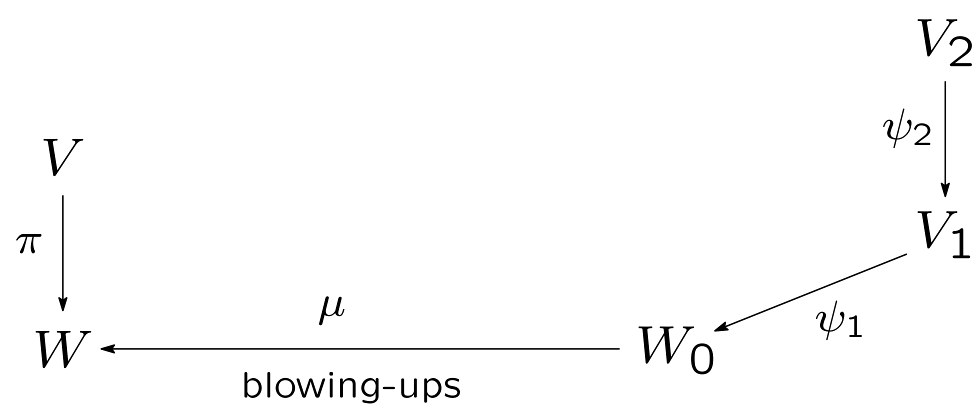

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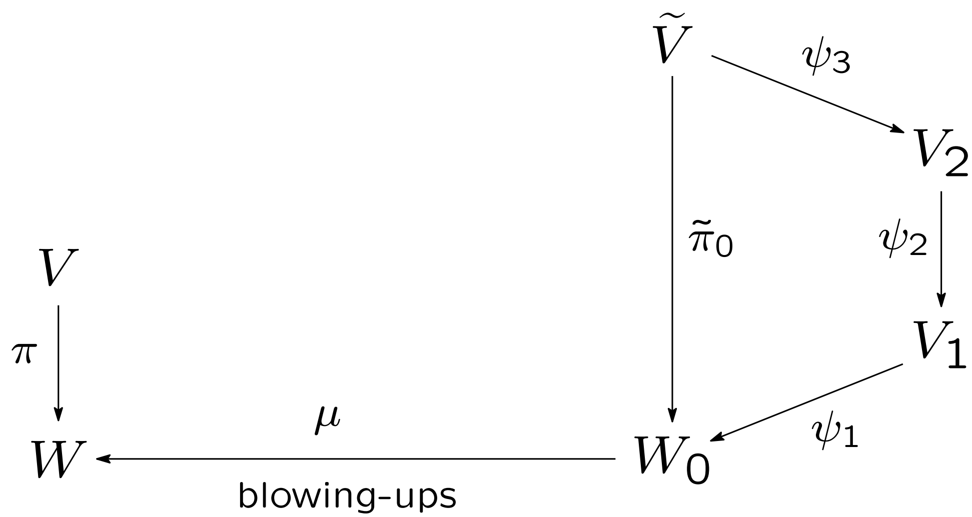

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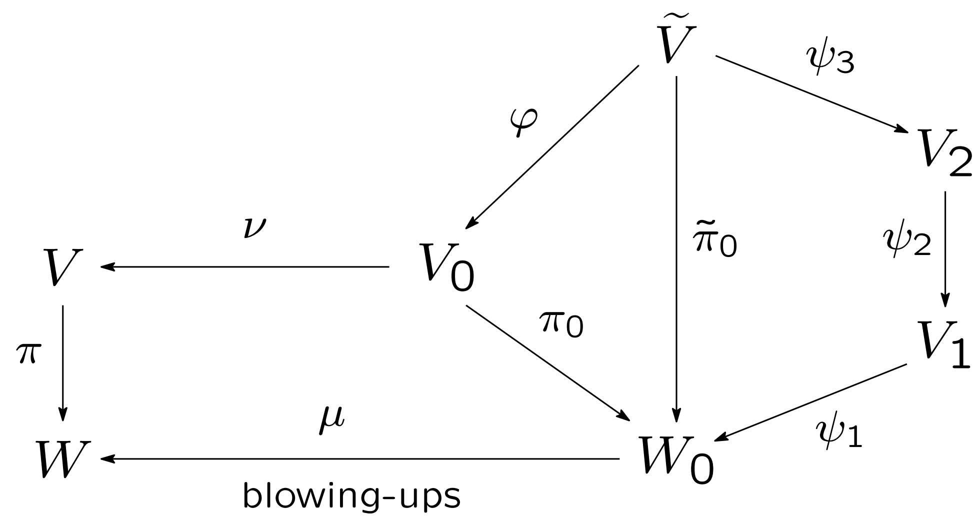

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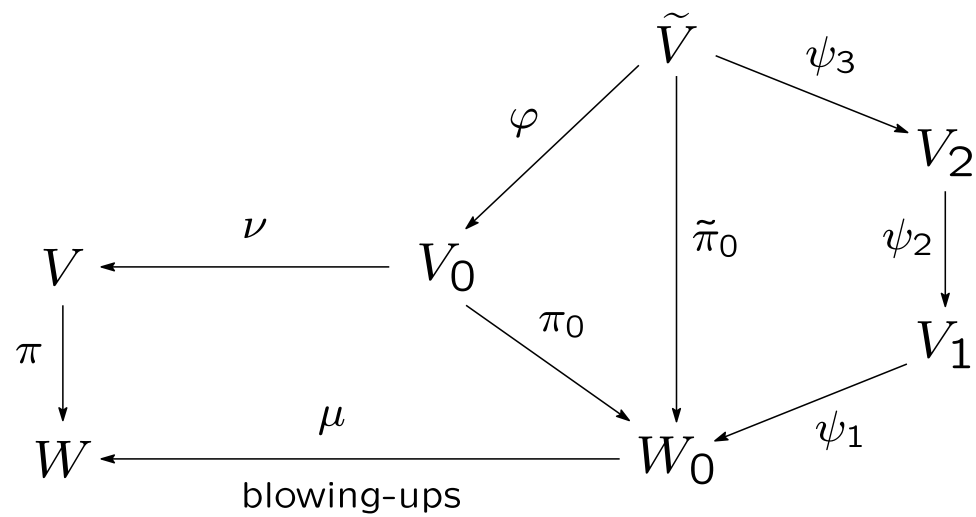


But V_0 may be singular for any μ. So we need to improve this strategy.

§5. Resolution of singularities of certain 4-fold covers

In this section, we resolve singularities of certain 4-fold covers.
We introduce some notations.

Let

W : a surface which is smooth at a point $P \in W$, and

$\pi : V \to W$: a 4-fold cover.

To analyze the singularities of the surface V, we will work locally on W.

In this talk, we consider the following 4-fold covers:

We put

$$f := z^4 + 4x^n z + 3y^m,$$

where $\{x, y\}$ is a system of local parameters at $P \in W$, and $n \geq 1$ and $m \geq 2$ are integers. Let

$$V_f \subset \mathbb{A}^1 \times W \ : \text{ the subvariety defined by the equation } f = 0.$$

Then we have a 4-fold cover π by restricting the projection $\mathbb{A}^1 \times W \to W$ to V_f:

$$\pi : V_f \to W.$$

In fact, there is an isolated singularity on V_f over P.

Let

D_f : the divisor on W defined by $x^{4n} - y^{3m} = 0$.

Since $x^{4n} - y^{3m}$ is the discriminant of f w.r.t. z, π is branched at D_f. We define integers as follows to describe our resolution of V_f:

$d := \mathrm{GCD}(4n, 3m)$,

n_1 : $4n = dn_1$,

m_1 : $3m = dm_1$,

$0 \leq e < m_1$: $n_1 e + 1 \equiv 0 \pmod{m_1}$,

$0 \leq e_1 < n_1$: $m_1 e_1 + 1 \equiv 0 \pmod{n_1}$.

We let

$$[a_1, \ldots, a_r] := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_r}}}$$

for integers $a_1, \ldots, a_r \geq 2$.
Let $\{a_i\}_{1 \leq i \leq r}$ and $\{b_i\}_{1 \leq i \leq s}$ be sets of bigger integers than 2 such that

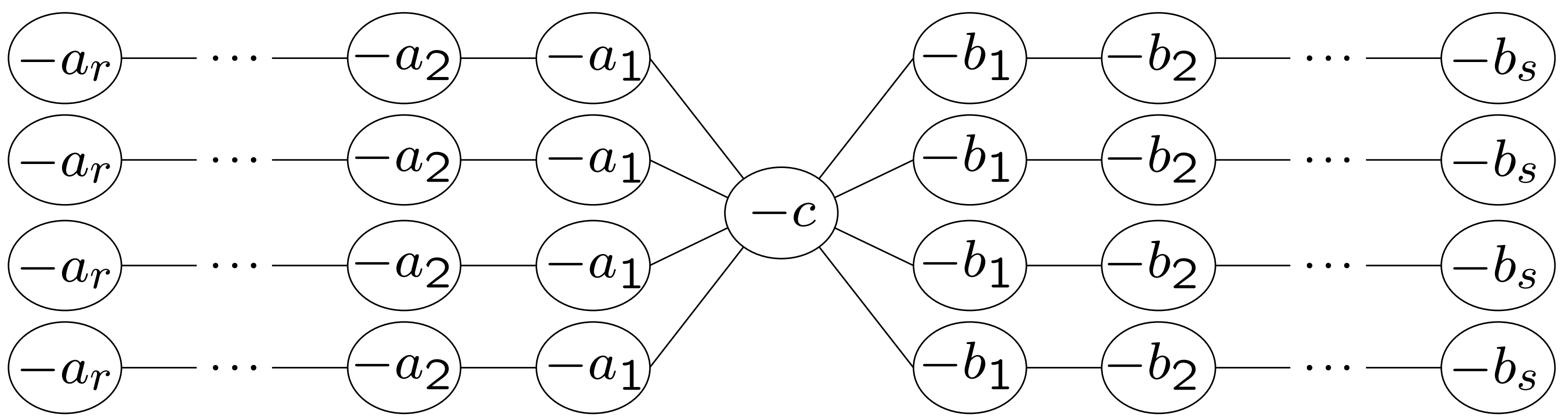
$$\frac{m_1}{e} = [a_1, \ldots, a_r],$$

$$\frac{n_1}{e_1} = [b_1, \ldots, b_s].$$

Proposition.

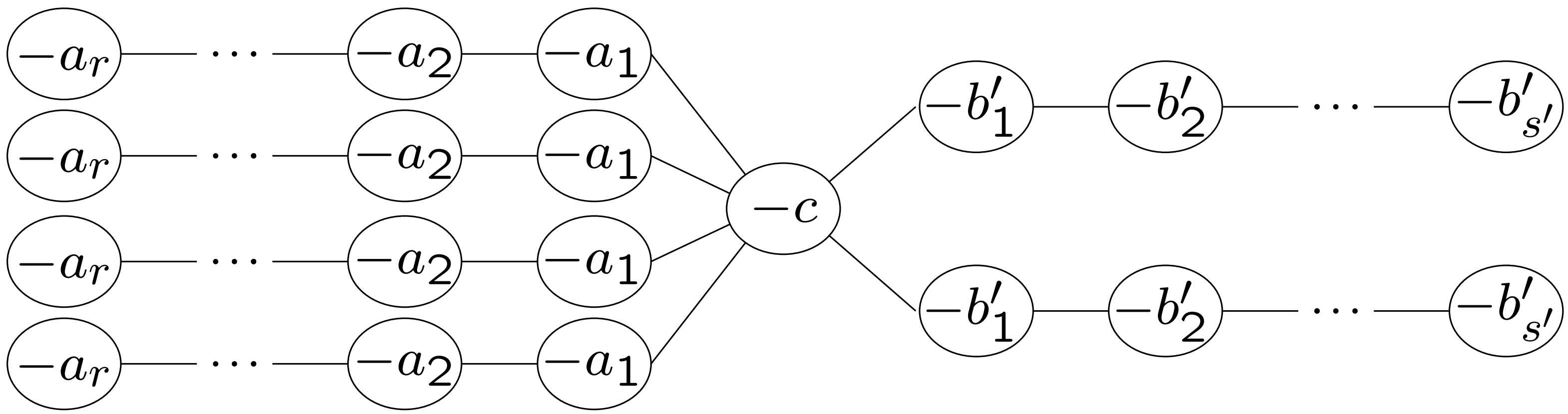
For each case (1),...,(6), there is a good resolution of the singularity of V_f over P such that the dual graph of the exceptional curves is as follows, and all exceptional curves except for the $(-c)$-curve are isomorphic to $\mathbb{P}^1$:

(1) If $d \equiv 0 \pmod{12}$,



where $c = 4$, and $g = d/2 - 3$
(g : the genus of the central $(-c)$-curve).

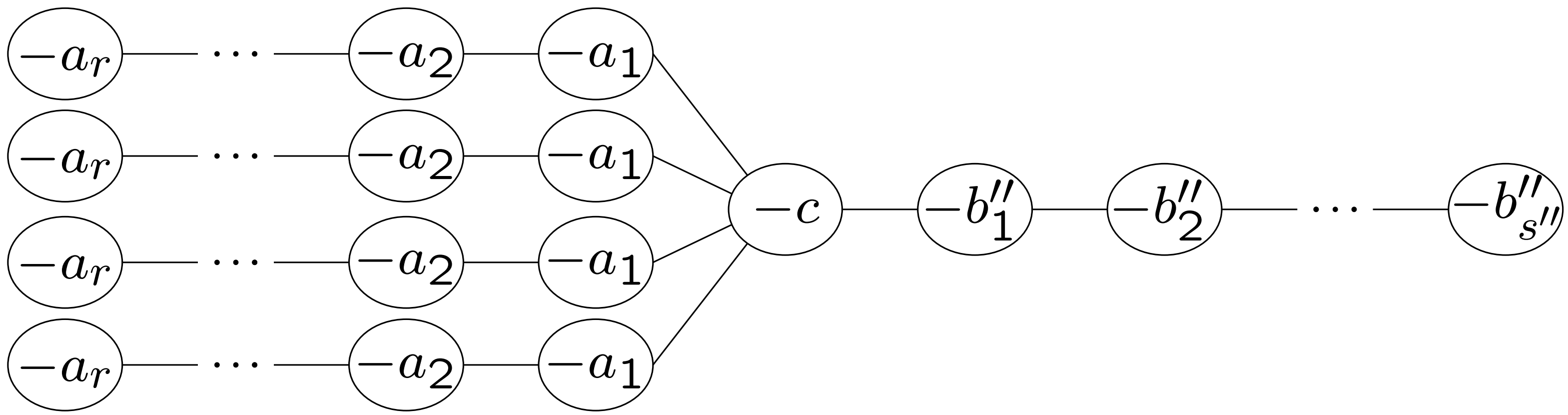
(2) If $d \equiv 0 \pmod 3$ and $d \equiv 2 \pmod 4$,



where $c = 4 - 2q'$, and $g = d/2 - 1$.

$$\begin{aligned} e'_1,\ q' \quad &: \ e_1 = q' \, \frac{n_1}{2} + e'_1 \quad (0 \leq e'_1 < \frac{n_1}{2}), \\ b'_1, \ldots, b'_{s'} \ &: \ \frac{n_1}{2e'_1} = [b'_1, \ldots, b'_{s'}]. \end{aligned}$$

(3) If $d \equiv 0 \pmod 3$ and $d \equiv 1 \pmod 2$,

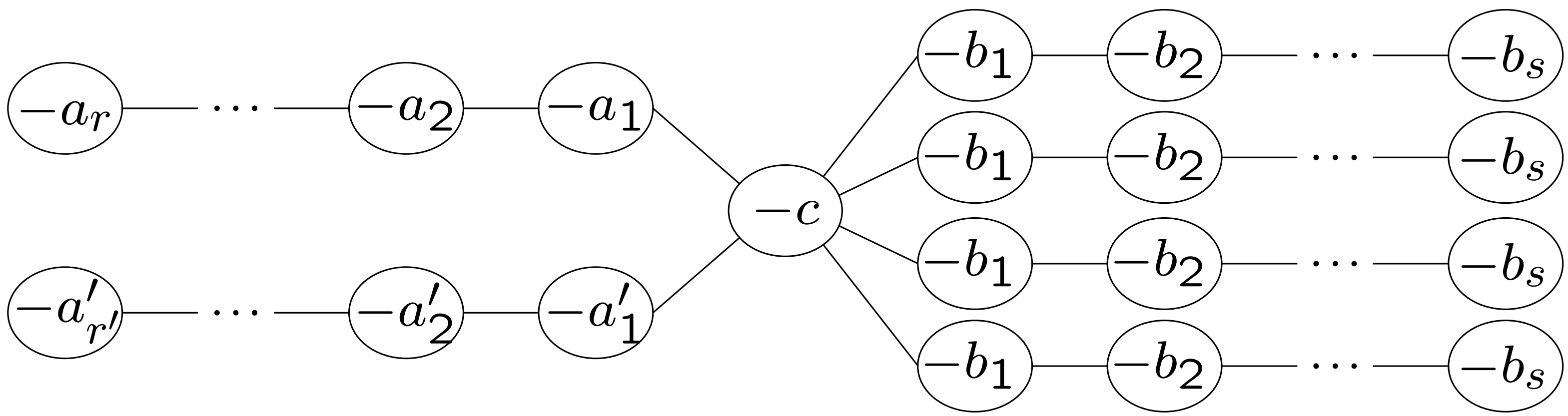


where $c = 4 - q''$, and $g = (d-3)/2$.

$$e''_1,\ q'' \quad : \quad e_1 = q''\,\frac{n_1}{4} + e''_1 \quad (0 \le e''_1 < \frac{n_1}{4}),$$

$$b''_1, \ldots, b''_{s''} \quad : \quad \frac{n_1}{4e''_1} = [b''_1, \ldots, b''_{s''}].$$

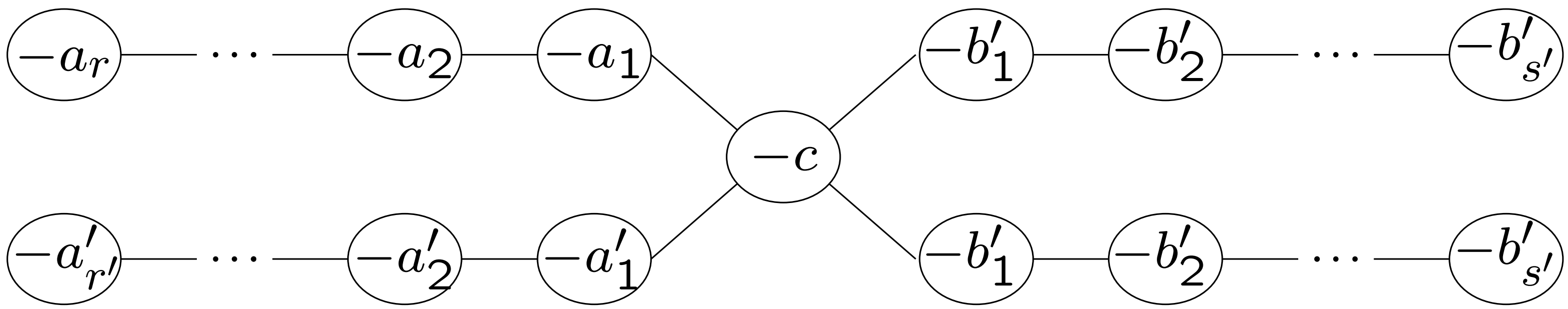
(4) If $d \not\equiv 0 \pmod 3$ and $d \equiv 0 \pmod 4$,



where $c = 4 - p'$, and $g = d/2 - 2$.

$$\begin{aligned} e',\ p' \quad &: \quad e = p' \frac{m_1}{3} + e' \quad (0 \le e' < \frac{m_1}{3}), \\ a'_1, \ldots, a'_{r'} \quad &: \quad \frac{m_1}{3e'} = [a'_1, \ldots, a'_{r'}] \end{aligned}$$

(5) If $d \not\equiv 0 \pmod 3$ and $d \equiv 2 \pmod 4$,

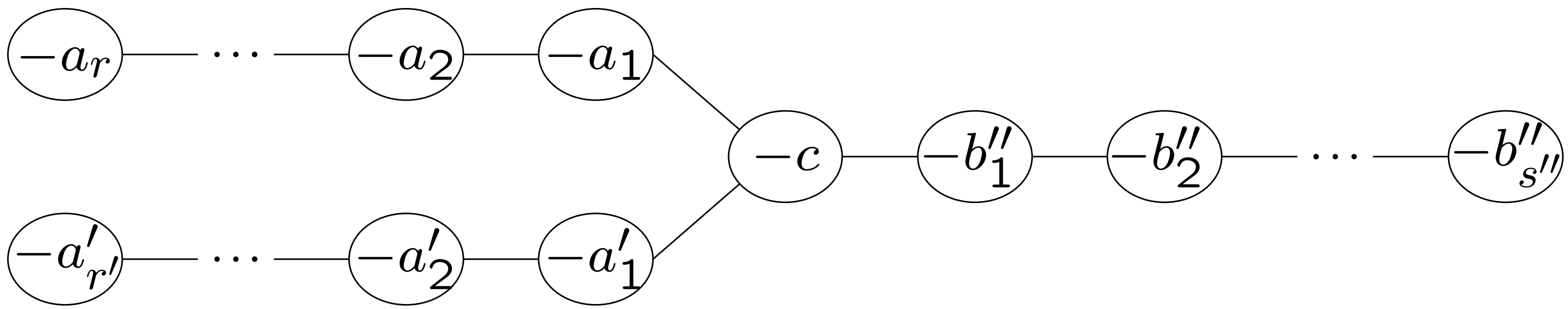


where $c = 4 - p' - 2q'$, and $g = d/2 - 1$.

$$e = p' \, \frac{m_1}{3} + e', \qquad e_1 = q' \, \frac{n_1}{2} + e'_1,$$

$$\frac{m_1}{3e'} = [a'_1, \ldots, a'_{r'}], \qquad \frac{n_1}{2e'_1} = [b'_1, \ldots, b'_{s'}].$$

(6) If $d \not\equiv 0 \pmod 3$ and $d \not\equiv 0 \pmod 2$,



where $c = 4 - p' - q''$, and $g = (d-1)/2$.

$$e = p' \ \frac{m_1}{3} + e', \qquad e_1 = q'' \ \frac{n_1}{4} + e''_1,$$

$$\frac{m_1}{3e'} = [a'_1, \ldots, a'_{r'}], \qquad \frac{n_1}{4e''_1} = [b''_1, \ldots, b''_{s''}].$$

Example ($n = 10$, $m = 30$).

If $n = 10$ and $m = 30$, then

$$d := \mathrm{GCD}(40, 90) = 10,$$

$$n_1 = 4, \qquad m_1 = 9.$$

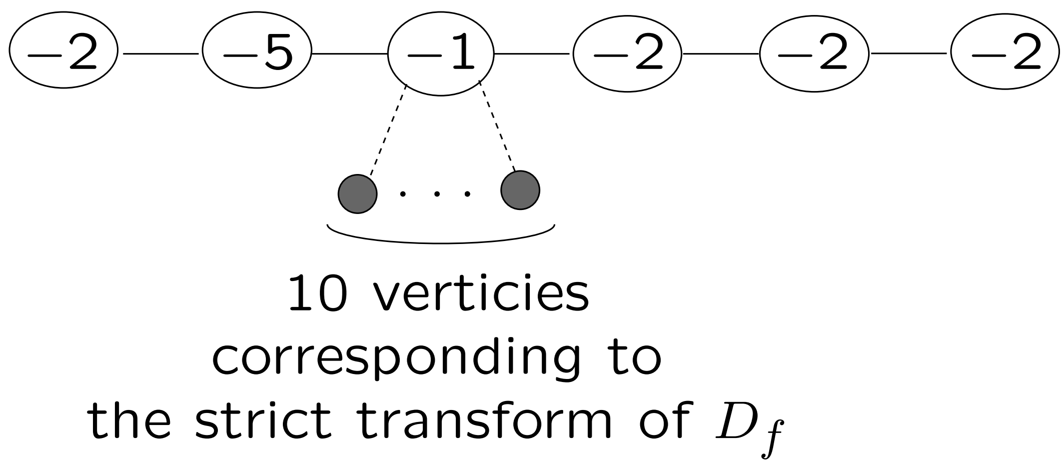
So this example is in the case (5).
Since $4 \cdot 2 + 1 \equiv 0 \pmod 9$, and since $9 \cdot 3 + 1 \equiv 0 \pmod 4$,

$$e = 2, \qquad e_1 = 3.$$

Recall that the branch divisor of $\pi : V_f \to W$ is $D_f : x^{40} - y^{90} = 0$. Since $9/2 = [5, 2]$, and since $4/3 = [2, 2, 2]$, by toric blowing-up,

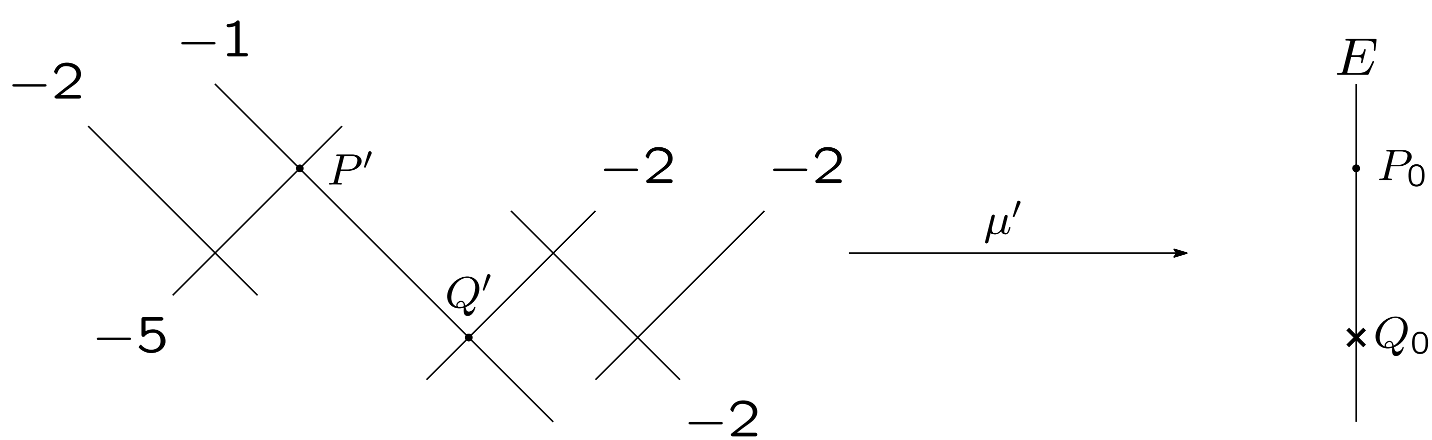
$$^{\exists}\mu : W' \to W \text{ : a resolution of singularity of } D_f \text{ (as a curve)}$$

such that the dual graph of the exceptional divisor of μ is the following:



Note that the $\mathbb{C}(V_f)$-normalization of W' may be singular over neighborhood of the exceptional set of μ.
So let

$\mu' : W' \to W_0$: the contract of the exceptional divisor except for the (-1)-curve.



Note that W_0 is singular at P_0 and Q_0, and E is isomorphic to $\mathbb{P}^1$.

And note that

the singularity at P_0 is $A_{9,2}$, and

the singularity at Q_0 is $A_{4,3}$.

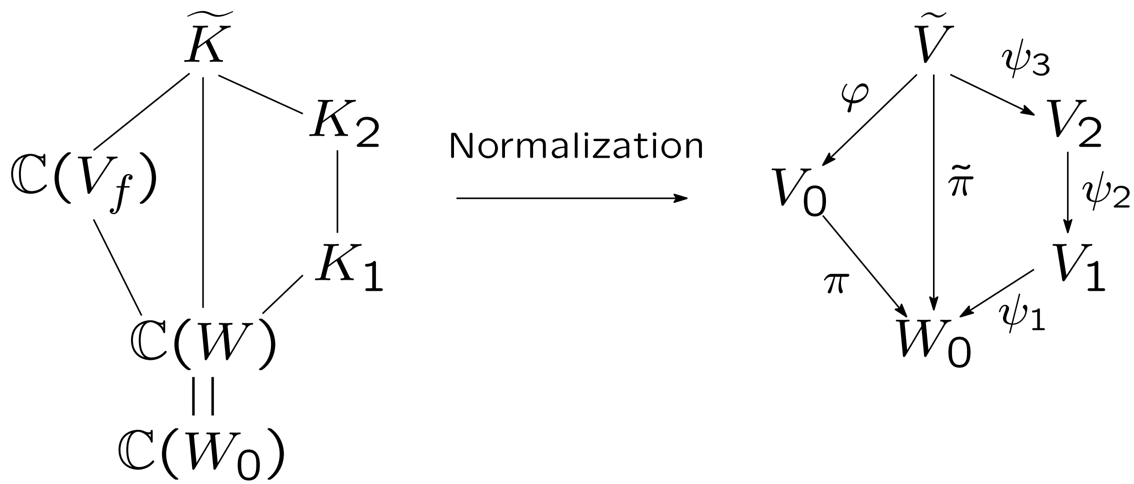
Here

$A_{a,b}$: the cyclic quotient singularity $(\mathbb{C}^2/G, \mathbf{0})$,

where GCD$(a, b) = 1$, and

$$G = \left\langle \begin{pmatrix} \zeta_a & 0 \\ 0 & \zeta_a^b \end{pmatrix} \right\rangle \quad (\zeta_a : \text{ a primitive } a\text{-th root of unity}).$$

Now we construct the $\mathfrak{S}_4$-cover over W_0 corresponding the Galois closure $\widetilde{K}$.

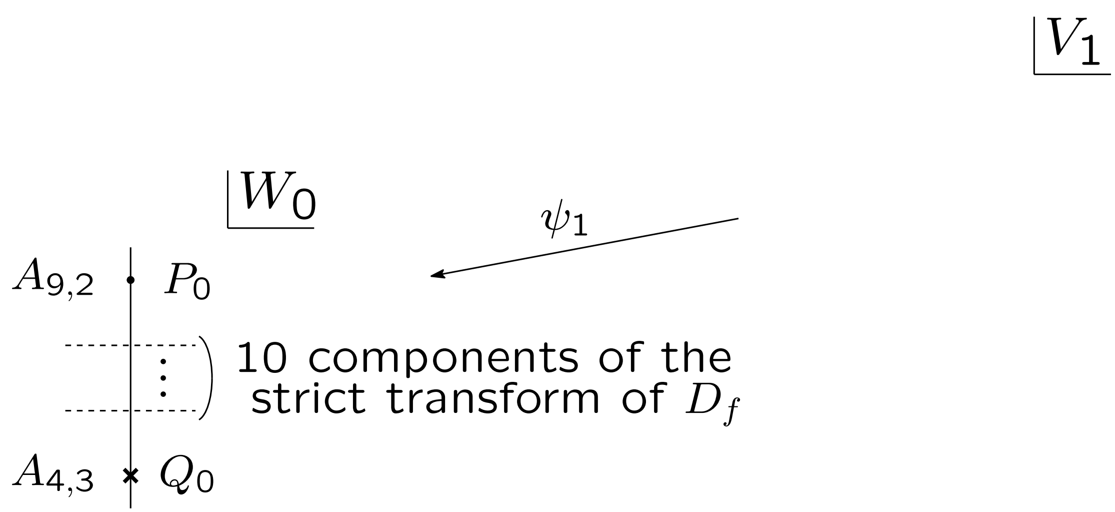


Note that E is not a branch divisor, and it depends on only d whether ψ_i is ramified over P_0 and Q_0 for each $i = 1, 2, 3$.
In this case ($d \not\equiv 0 \pmod 3$ and $d \equiv 2 \pmod 4$),

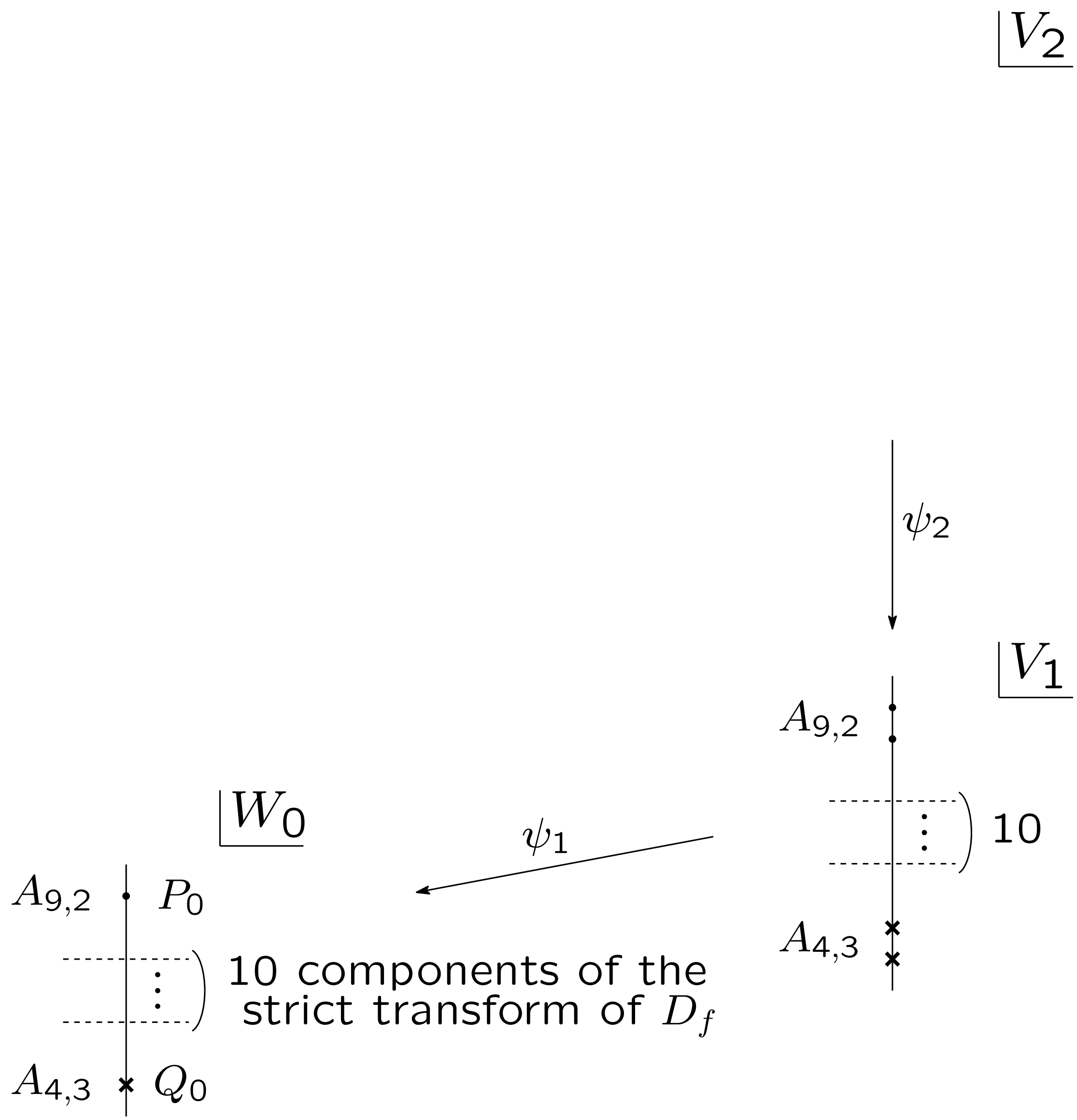
ψ_1 is ramified over the strict transform of D_f,

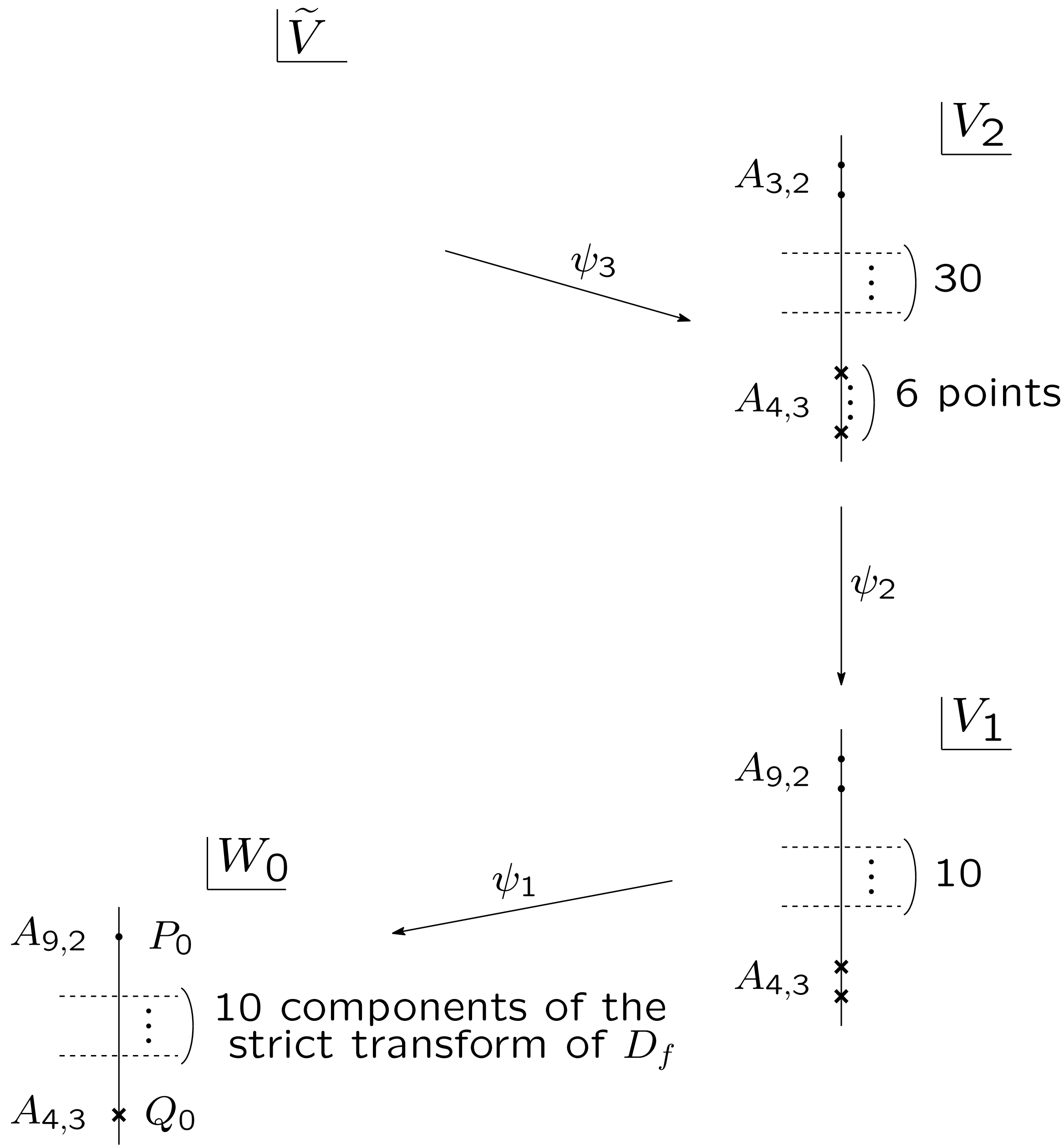
ψ_2 is ramified over P_0, and

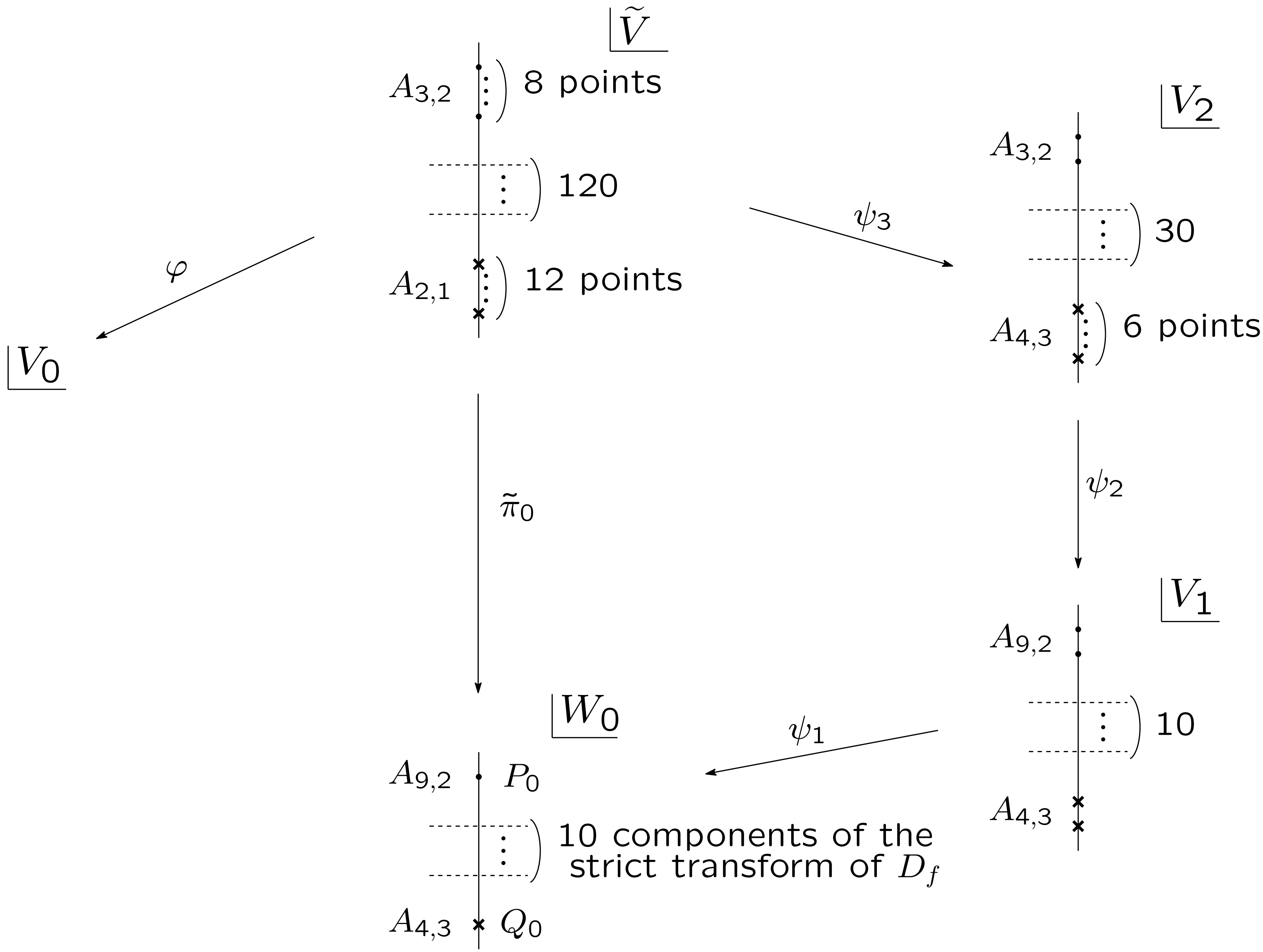
ψ_3 is ramified over Q_0.

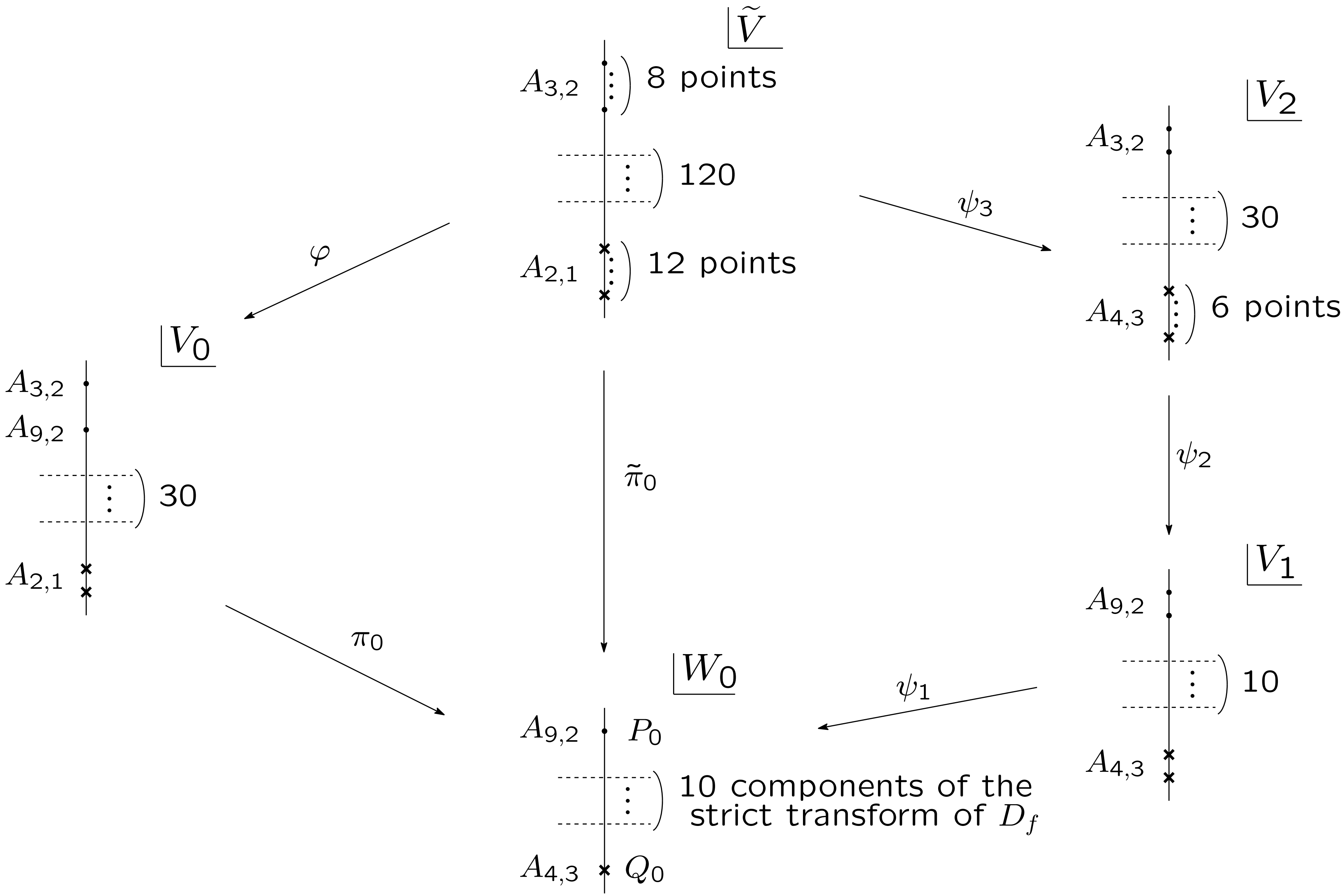


V_1
W_0
ψ_1
$A_{9,2}$
P_0
10 components of the strict transform of D_f
$A_{4,3}$
Q_0


V_2
ψ_2
V_1
$A_{9,2}$
10
$A_{4,3}$
ψ_1
W_0
$A_{9,2}$
P_0
10 components of the strict transform of D_f
$A_{4,3}$
Q_0


$\widetilde{V}$
V_2
$A_{3,2}$
ψ_3
30
$A_{4,3}$
6 points
ψ_2
V_1
$A_{9,2}$
10
$A_{4,3}$
ψ_1
W_0
$A_{9,2}$
P_0
10 components of the strict transform of D_f
$A_{4,3}$
Q_0


$\widetilde{V}$
$A_{3,2}$
8 points
120
$A_{2,1}$
12 points
φ
V_0
ψ_3
V_2
$A_{3,2}$
30
$A_{4,3}$
6 points
$\tilde{\pi}_0$
ψ_2
V_1
$A_{9,2}$
10
$A_{4,3}$
W_0
ψ_1
$A_{9,2}$
P_0
10 components of the strict transform of D_f
$A_{4,3}$
Q_0

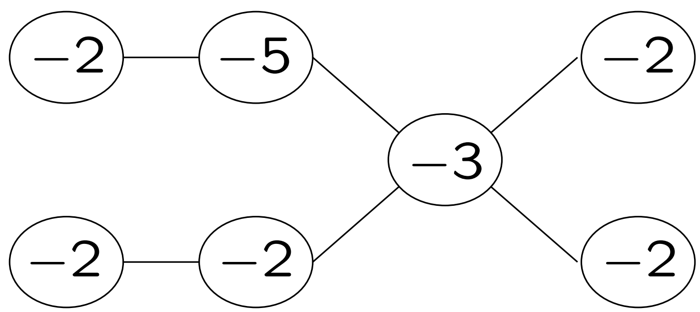


$\widetilde{V}$
$A_{3,2}$
8 points
120
$A_{2,1}$
12 points
V_2
$A_{3,2}$
30
$A_{4,3}$
6 points
ψ_3
φ
V_0
$A_{3,2}$
$A_{9,2}$
30
$A_{2,1}$
$\tilde{\pi}_0$
ψ_2
V_1
$A_{9,2}$
10
$A_{4,3}$
π_0
ψ_1
W_0
$A_{9,2}$
P_0
10 components of the strict transform of D_f
$A_{4,3}$
Q_0

Finally, let

$$\tilde{\mu} : \widetilde{V} \to V_0 : \text{ the resolution of quotient singularities on } V_0.$$

Then we have the following dual graph:



This is a resolution of V_f, and the genus of the (-3)-curve is equal to 4 by Hurwitz's formula.