Non-Galois triple coverings of projective plane branched along quintic curves and cubic surfaces in projective space

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Definition

Let X and Y be normal projective varieties. We denote the function fields of X and Y by $\mathbf{C}(X)$ and $\mathbf{C}(Y)$, respectively. We call a finite surjective morphism $\pi : X \to Y$ with the non-Galois cubic extension $\mathbf{C}(X)/\mathbf{C}(Y)$ of these function fields induced by π a non-Galois triple covering. Then

- $\Delta_\pi := \{y \in Y \mid \sharp(\pi^{-1}(y)) < 3\}$: the branch locus of π.
- $D \subset \Delta_\pi$: an irreducible component.
  - ⋆ π: totally branched along $D \overset{\text{def}}{\iff} {}^\forall p \in D$, $\sharp\pi^{-1}(p) = 1$.
  - ⋆ π: simply branched along $D \overset{\text{def}}{\iff} {}^\exists U \subset D$: Zariski open set s.t. ${}^\forall p \in U$, $\sharp\pi^{-1}(p) = 2$.
- $\pi : X \to \mathbf{P}^2$: non-Galois triple covering of QL-type (for simply QL-type) $\overset{\text{def}}{\iff} {}^\exists Q$: a quartic and ${}^\exists L$: a line such that $\Delta_\pi = Q + L$ and π is totally (resp. simply) branched along L (resp. Q).

QL-type

Form H. Tokunaga [1], we obtain that the branch locus of a non-Galois triple covering of QL-type falls into one of the following:

Δ_π	Q	$Q \cap L$	Δ_π	Q	$Q \cap L$
Δ_1	Q_1	(i)	Δ_{10}	Q_5	(ii)
Δ_2	Q_2		Δ_{11}	Q_6	(iii), a_3
Δ_3	Q_3		Δ_{12}	Q_{12}	
Δ_4	Q_4		Δ_{13}	Q_7	(iii), a_6
Δ_5	Q_5		Δ_{14}	Q_8	(v), a_4
Δ_6	Q_9		Δ_{15}	Q_{10}	(iv), $2a_3$
Δ_7	Q_1	(ii)	Δ_{16}	Q_{13}	
Δ_8	Q_2		Δ_{17}	Q_{11}	(v), a_7
Δ_9	Q_4		Δ_{18}	Q_{14}	(v), ordinary 4-ple point

Q	irreducible components	singular points
Q_1	irreducible	$2a_2$
Q_2	irreducible	$a_1 + 2a_2$
Q_3	irreducible	$3a_2$
Q_4	irreducible	a_5
Q_5	irreducible	e_6
Q_6	irreducible	$a_2 + a_3$
Q_7	irreducible	a_6
Q_8	irreducible	$a_2 + a_4$
Q_9	two conics	$a_1 + a_5$
Q_{10}	two conics	$2a_3$
Q_{11}	two conics	a_7
Q_{12}	a cuspital cubic and a line	$a_1 + a_2 + a_3$
Q_{13}	a conics and two lines	$2a_3 + a_1$
Q_{14}	four lines	ordinary 4-ple point

(i) L is a bitangent line of Q at two smooth points.

(ii) L is a tangent line of Q at a smooth point with multiplicity four.

(iii) L is tangent to Q at one smooth point and passes through one singular point of Q.

(iv) L passes through two distinct singular points of Q.

(v) L meets Q at just one singular point.

Let $\pi_i : X_i \to \mathbf{P}^2$ be a non-Galois triple covering such that Δ_{π_i} is of type Δ_i $(1 \leq i \leq 18)$. Let $\gamma : \overline{X_i} \to X_i$ be the minimal resolution of X_i. If $1 \leq i \leq 17$ (resp. $i = 18$), then we see that the topological Euler number $\chi_{top}(\overline{X_i})$ is 9 (resp. 0) and that the self intersection number of the canonical divisor $K_{\overline{X_i}}$ of $\overline{X_i}$ is 3 (resp. 0).

Facts

Using the following three facts, we obtain that X_i $(1 \leq i \leq 17)$ are cubic surfaces in $\mathbf{P}^3$.

Lemma 0.1. *Let $\pi : X \to \mathbf{P}^2$ be a triple covering of QL-type and $\gamma : \overline{X} \to X$ the minimal resolution of X. If $\chi_{top}(\overline{X}) = 9$ and ${K_{\overline{X}}}^2 = 3$ then*

$$-K_{\overline{X}} \sim (\gamma \circ \pi)^* l,$$

where l is a line on $\mathbf{P}^2$. □

Proposition 0.1. *Under the assumption of Lemma 0.1, $|-K_{\overline{X}}|$ induces a morphism $\varphi_{|-K_{\overline{X}}|} : \overline{X} \to \mathbf{P}^3$ such that $\overline{X}$ is birationally equivalent to the image $\operatorname{Im}\varphi_{|-K_{\overline{X}}|}$ and $\operatorname{Im}\varphi_{|-K_{\overline{X}}|}$ is a normal cubic surface whose singular points are rational double points.* □

Proposition 0.2. *Under the assumption of Lemma 0.1, $X = \operatorname{Im}\varphi_{|-K_{\overline{X}}|}$ and $\pi : X \to \mathbf{P}^2$ is a restriction of a projection $\mathbf{P}^3 \cdots \to \mathbf{P}^2$ from a point.* □

Centers of projections

To obtain non-Galois triple coverings of QL-types, the centers fall one of the following:

Sing S	Δ_{π_p}	normal forms of S	centers of the projections
$A_1 + 2A_2$	Δ_2	$WYZ + WX^2 + X^3 = 0$	$[1 : a : b : 0]$, $ab \neq -1, 0, 3$
	Δ_5		$[1 : a : b : 0]$, $ab = -1$
	Δ_6		$[1 : a : b : 0]$, $ab = 3$
	Δ_{12}		$[1 : a : b : 0]$, $ab = 0$, $a + b \neq 0$
	Δ_{16}		$[1 : 0 : 0 : 0]$
$A_1 + A_5$	Δ_8	$WXY + WZ^2 + X^3 = 0$	$[1 : a : b : 0]$, $a + b^2 \neq 0$
	Δ_{10}		$[1 : a : b : 0]$, $a + b^2 = 0$
$2A_2$	Δ_1	$W^3 + kWX^2 + WYZ + X^3 = 0$ $(4k^3 + 27 \neq 0)$	$[1 : a : b : 0]$, $ab \neq 0$, $a^4b^4 - 6a^2b^2k^2 - 8abk^3 - 108ab - 3k^4 \neq 0$
	Δ_4		$[1 : a : b : 0]$, $ab \neq 0$, $a^4b^4 - 6a^2b^2k^2 - 8abk^3 - 108ab - 3k^4 = 0$
	Δ_{11}		$[1 : a : b : 0]$,$k \neq 0$, $ab = 0$, $a + b \neq 0$
	Δ_{13}		$[1 : a : b : 0]$, $k = 0$, $ab = 0$, $a + b \neq 0$
	Δ_{15}		$[1 : 0 : 0 : 0]$, $k \neq 0$
$3A_2$	Δ_3	$WYZ + X^3 = 0$	$[1 : 0 : a : b]$, $ab \neq 0$
			$[1 : a : 0 : b]$, $ab \neq 0$
			$[1 : a : b : 0]$, $ab \neq 0$
A_5	Δ_7	$W^2Z + WXY + WZ^2 + X^3 = 0$	$[1 : a : b : 0]$, $27 + 4a^3 + 12a^2b^2 + 12ab^4 + 4b^6 \neq 0$
	Δ_9		$[1 : a : b : 0]$, $27 + 4a^3 + 12a^2b^2 + 12ab^4 + 4b^6 = 0$
E_6	Δ_{14}	$W^2Y + WZ^2 + X^3 = 0$	$[1 : a : b : 0]$, $a \neq 0$
	Δ_{17}		$[1 : 0 : b : 0]$, $b \neq 0$
$\widetilde{E_6}$	Δ_{18}	$kW^3 + lW^2X + WY^2 + X^3 = 0$ $(4l^2 + 27k^3 \neq 0)$	$p \in H_1 \setminus H_2$ $H_1, H_2 \in H_k$, $H_1 \neq H_2$

$a, b, k, l \in \mathbf{C}$.

if $l = 0$, $H_k = \{H_w, H_t, H_{su} \mid t^2 + k = 0, 2u^3 + k = 0, 3ks^2 = u^2, s \neq 0, (s, t, u \in \mathbf{C})\}$.

if $l \neq 0$, $H_k = \{H_w, H_{su} \mid 3lu^4 - 6us^2 - 1 = 0, 6lus^2 + 9ks^2 - 3u^2 + l = 0, s \neq 0, (s, t, u \in \mathbf{C})\}$.

$(H_W := V(W) \setminus V(X)$, $H_t := V(Y + tW) \setminus V(X)$.)

$(H_{su} := V(X - sY - uW) \setminus V(3s^3Y + (1 + 3us^2)W)$.)

References

[1] H. Tokunaga, *Dihedral covers and an elementary arithmetic on elliptic surfaces*, J. Math. Kyoto Univ. **44**, pp. 255–270, (2004).