Recent progress on topology of plane curves: A quick trip
Part III:
Characteristic varieties: a generalization of Alexander polynomial

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- The *Alexander polynomial* $\Delta_{G,\varepsilon}$ of G with respect to ε is the order of $H_1(\tilde{X}; \mathbb{C})$ as Λ-module (Λ is a PID).

Definition of Alexander Polynomial

Definition

Let $\mathcal{C}^{\text{aff}}$ be an affine curve defined by $f(x, y) = 0$ and let $\varepsilon : \pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) \to \mathbb{Z}$ the epimorphism determined by $f : \mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}} \to \mathbb{C}^*$. The *Alexander polynomial* of $\mathcal{C}^{\text{aff}}$ is $\Delta_{\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}), \varepsilon}$.

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In the same way one can define the Alexander polynomial for a non reduced curve: these polynomials are called Alexander-Oka polynomials.

Alexander polynomial of a curve whose group is unknown

Cyclic version of Sakuma's formula

Let $\varepsilon_d : G \to \mathbb{Z}/d\mathbb{Z}$ be the natural composition mapping, let $\rho_d : X_d \to X$ be the associated cyclic covering and let $t_d : X_d \to X_d$ the standard generator of the automorphism group of ρ_d.
For $\zeta \neq 1$ a d-root of unity, let m_ζ be the dimension of the ζ-eigenspace of $H^1(X_d; \mathbb{C})$ by the action of t_d. Then m_ζ is the multiplicity of ζ as root of $\Delta_{G,\varepsilon}$.

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- Let $\mathcal{C}$ be a projective curve defined by $F(x,y,z)=0$. Let $X_d := \{(x,y,z) \in \mathbb{C}^3 \mid F(x,y,z)=1\}$ and let $\rho_d : X_d \to \mathbb{P}^2 \setminus \mathcal{C}$ be the standard projection.

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- Then, the Alexander polynomial $\Delta_\mathcal{C}$ is determined by the action on cohomology $H^1(X_d; \mathbb{C})$ of the above multiplication.
- Moreover, if $\bar{X}_d$ is a smooth projective completion of X_d, all the computations can be done on $\bar{X}_d$ and Hodge structure can be used.

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Theorem (Zariski,Libgober,Esnault,Loeser-Vaquié,–)

Let $\mathcal{C}$ be a projective curve of degree d and let $k\in\{1,\dots,d\}$. Let $\sigma_k:H^0(\mathbb{P}^2,\mathcal{O}(k-3))\to\bigoplus_{P\in\operatorname{Sing}\mathcal{C}}\mathcal{O}_P/\mathcal{J}_{\mathcal{C},P,\frac{k}{d}}$ be the natural map. We set $a_k:=\dim\operatorname{coker}\sigma_k$.
Then, the multiplicity of ζ_d^k as root of $\Delta_{\mathcal{C}}$ equals a_k+a_{d-k}.

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- $\mathcal{C}$ an irreducible sextic with $\mathbb{A}_{17}$: $\Delta_{\mathcal{C}}=t^2-t+1$ (resp. 1) if there is (resp. not) a conic with intersection number 12 at $\mathbb{A}_{17}$.

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Let M be a finitely presented Λ-module and let $A \in \mathrm{Mat}(n \times m; \Lambda)$ be a presentation matrix, i.e., A is the matrix of a morphism $\Phi : A^m \to A^n$. Then, the k-Fitting ideal $\mathcal{J}_{M,k}$ of M is the ideal of H generated by the $(n-k)$-minors of A (it does not depend on A).

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The k-characteristic variety $\Sigma_{\mathcal{C},k}$ of a complex projective curve $\mathcal{C}$ is the zero locus of $\mathcal{J}_{H_1(\tilde{X},\mathbb{C}),k}$.

Alternative definition

- Let $\xi : G \to \mathbb{C}^*$ be a character. For the space of characters we have

$$\operatorname{Hom}(G;\mathbb{C}^*) = \operatorname{Hom}(H,\mathbb{C}^*) = \operatorname{Hom}(H_1(X;\mathbb{Z});\mathbb{C}^*) = H^1(X;\mathbb{C}^*) = \mathbb{T}_H.$$

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- This cohomology can be computed as follows. Consider the complex of $\mathbb{C}$-vector spaces defined by $C_*(X;\mathbb{C}_\xi) := C_*(\tilde{X};\mathbb{C}_\xi) \otimes_\Lambda \mathbb{C}_\xi$. It is isomorphic as (graded) $\mathbb{C}$-vector space with $C_*(X;\mathbb{C})$ but the differential is *twisted* by ξ. The cohomology of this complex is $H^*(X;\underline{\mathbb{C}}_\xi)$.

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- With this definition $\Sigma_{G_{\mathcal{C}},k}$ and $\Sigma_{\mathcal{C},k}$ coincide outside $\mathbf{1} \in \mathbb{T}_H$. This is due to the commutation of the operations *cohomology* and $\otimes_\Lambda \mathbb{C}_\xi$.

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- The resonance varieties are subspaces $R \subset H^1(X;\mathbb{C})$; the irreducible components of the characteristic varieties passing through $\mathbf{1}$ are obtained as $\exp(2i\pi R)$.

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- If we want to have non trivial homology the matrix must vanish: $s = -1$ and $t^4 = 1$. The case $(1, -1)$ cannot be obtained using quasi-adjunction ideals.

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