Recent progress on topology of plane curves: A quick trip
Part V:
Orbifolds and Quasi-projective Groups

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Starting point

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1. *If* $\dim \Sigma > 0$ *then there exists a primitive surjective morphism* $\rho : X \to C$, C *algebraic curve, and a torsion element* σ *such that* $\Sigma = \sigma\rho^*(H^1(C; \mathbb{C}^*))$.

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- $\mathcal{C}$ a sextic with six cusps on a conic: $\mathcal{C} = \{f_2^3 - f_3^2 = 0\}$. Note that $\mathbb{T}_H = \mathbb{C}^*$ and $\Sigma_{\mathcal{C},1} = \{\zeta_6, \zeta_6^{-1}\}$, $\Sigma_{\mathcal{C},2} = \emptyset$.

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An *orbifold* X_φ is a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ such that $\operatorname{Sing}(X_\varphi) := \{x \in X \mid \varphi(x) > 1\}$ is a finite set. Assume the following interpretation: the angle of a disk centered at x equals $\frac{2\pi}{\varphi(x)}$.

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Definition

For an orbifold $X_\varphi = X_{\varphi(x), x \in \operatorname{Sing}(X_\varphi)}$ we define:

$$\pi_1^{\text{orb}}(X_\varphi) := \pi_1(X \setminus \operatorname{Sing}(X_\varphi)) / \langle \mu_x^{\varphi(x)} = 1, \forall x \in \operatorname{Sing}(X_\varphi) \rangle, \mu_x \text{ a meridian of } x.$$

Orbifold morphism

Example

$G_{p,q} := \pi_1^{\text{orb}}(\mathbb{C}_{p,q}) = \mathbb{Z}/p * \mathbb{Z}/q$. If $\gcd(p,q) = 1$ then $H = \mathbb{Z}/pq$ and it is not hard to check that $\Sigma_{G_{p,q},1}$ is composed by the primitive pq-roots of unity.

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Definition

Let X_φ be an orbifold and Y a smooth algebraic variety. A dominant algebraic morphism $\rho : Y \to X$ defines an *orbifold morphism* $Y \to X_\varphi$ if for all $x \in X$, $\frac{1}{\varphi(x)}\rho^*(x)$ is a divisor.

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Remark

Such a morphism induces a mapping $\pi_1(Y) \twoheadrightarrow \pi_1^{\text{orb}}(X_\varphi)$ if it is primitive. Note that if we choose a tranversal disk to a smooth point of the regular part of $\rho^*(x)$ then for suitable local coordinates, this map is of the form $t \mapsto t^n$ for n a multiple of $\varphi(x)$.

Orbifolds and characteristic varieties

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- The mapping $[x:y:z] \mapsto [y^2z : x^3]$ from $\mathbb{P}^2 \setminus \mathcal{C}$ misses only $\{[1:0],[1:1]\}$ and the pull-back of $[0:1]$ is a double divisor. Then, it defines an orbifold morphism onto $\mathbb{C}_2^*$.

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- This map pulls back the 1-dimensional component of $\Sigma_{G_n^*,1}$ into $\Sigma_{\mathcal{C},1}$.

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- $G_n^{**} := \pi_1^{\text{orb}}(\mathbb{C}_n^{**}) = \mathbb{F}_2 * \mathbb{Z}/n$ where $\mathbb{C}^{**} := \mathbb{C}^* \setminus \{1\}$. The torus $\mathbb{T}_H$ has equation $t_3^n = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_n^{**},1} = \mathbb{T}_H$ and $\Sigma_{G_n^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1,1)\}$.

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Example

- Let Y be an elliptic curve and let $G_{2,2}^0 := \pi_1^{\text{orb}}(Y_{2,2})$ with presentation

$$\langle a, b, u, v \mid u^2 = v^2 = [a,b]uv = 1\rangle.$$

The torus $\mathbb{T}_H$ has equation $t_3^2 = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_{2,2}^0,1} = ((\mathbb{C}^*)^2 \times \{-1\}) \cup \{(1,1)\}$.

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- We can never have an orbifold map from the complement of a projective curve onto $\mathcal{C}$.

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- Note that H is the kernel of the natural mapping $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m$, where $m := \operatorname{lcm}(p, q, r)$. For example $H = \mathbb{Z}/6$ for $(2, 3, 6)$, $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ for $(2, 4, 4)$ and $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $(3, 3, 3)$.

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- $\Sigma_{G_{p,q,r},1} = \{\xi \mid \ell(\xi) = 3\}$. These data will be used in the last lecture.

Main result

Theorem (Arapura)

Let Σ be an irreducible component of $\Sigma_{G,1}$, $G = \pi_1(X)$, X quasi-projective surface. Then,

1. *If $\dim \Sigma > 0$ then there exists a primitive surjective morphism $\rho : X \to C$, C algebraic curve, and a torsion element σ such that $\Sigma = \sigma \rho^*(H^1(C; \mathbb{C}^*))$.*
2. *If $\dim \Sigma = 0$ then Σ is unitary.*

In particular, positive dimensional irreducible components are subtori translated by torsion elements.

Main result

Claim

- X *a quasi-projective smooth variety*
- $\Sigma := \Sigma_k(X)$ *the* k^{th} *characteristic variety of* X, V *an irreducible component of* Σ.

Then, there exists:

- *a primitive surjective orbifold morphism* $\rho : X \to C_\varphi$ *and*
- *an irreducible component* V_1 *of* $\Sigma_k(\pi_1^{\text{orb}}(C_\varphi))$

such that $V = \rho^*(V_1)$.
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Remark

The claim is not correct. The Degtyarev curve has as characteristic variety four points of torsion 10 which cannot be obtained as pull-back from an orbifold (–,Cogolludo).

Main result

Theorem (–,Cogolludo,Matei)

Let X be a quasi-projective smooth variety and let Σ be the k^{th} characteristic variety of X. Let V be an irreducible component of Σ. Then one of the two following statements holds:

1. *There exists a primitive surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component V_1 of $\Sigma_k(\pi_1^{\mathrm{orb}}(C_\varphi))$ such that $V = \rho^*(V_1)$.*
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2. *V is an isolated torsion point.*

Remark

The proof uses Deligne-Timmerscheidt theory and follows ideas of Beauville, Arapura and Delzant. One essential ingredient is that for non-unitary characters, some non-trivial elements of the twisted cohomology are represented by twisted logarithmic 1-forms, defining foliations.

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- If Σ is an irreducible component of Σ_1 of dimension $k > 2$ then it is also the case for $\operatorname{Sh} \Sigma$.
- For rational orbifolds the same can be assumed for $k \geq 2$.

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- Except $\mathbf{1}$, irreducible components of Σ_k are connected components of $\mathbb{T}_H$.
- Given a translated subtorus V, its shadow $\operatorname{Sh} V$ is the *parallel* subtorus passing through $\mathbf{1}$.
- An irreducible component Σ of Σ_1 of dimension $k > 0$ is also an irreducible component of Σ_{k-2}, if $\mathbf{1} \in \Sigma$, or Σ_k if not.
- If Σ is an irreducible component of Σ_1 of dimension $k > 2$ then it is also the case for $\operatorname{Sh} \Sigma$.
- For rational orbifolds the same can be assumed for $k \geq 2$.
- An irreducible component of dimension 1 never passes through $\mathbf{1}$.

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- Let Σ_1 be an irreducible component of $\Sigma_k(G)$ and let Σ_2 be an irreducible component of $\Sigma_\ell(G)$, both of positive dimension. If $\xi \in \Sigma_1 \cap \Sigma_2$ then it is a torsion point and $\xi \in \Sigma_{k+\ell}(G)$.

An Artin group

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- *If p, q, r are even, not all of them equal and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ then the groups $G_{p,q,r}$ are not quasiprojective.*