Recent Progress on Topology of Plane Curves: A Quick Trip
Part II:
The Cohomology Algebra of a Plane Curve

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Settings

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- Prove Formality of X.

The Line Arrangement Case

$$\mathcal{C} = \ell_0 \cup \ell_1 \cup \ldots \cup \ell_r \subset \mathbb{P}^2,$$

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Consider $X = \mathbb{C}^2 \setminus (\ell_1 \cup \ldots \cup \ell_r)$.

The Line Arrangement Case

Theorem (Arnold, Brieskorn, Orlik-Solomon)

The ring $H^*(X)$ is generated by $H^1(X)$, that is, by:

$$\sigma_i := \frac{d\ell_i}{\ell_i}.$$

A complete set of relations is given by:

$$\sigma_i \wedge \sigma_j + \sigma_j \wedge \sigma_k + \sigma_k \wedge \sigma_i = 0,$$

whenever $\ell_i \cap \ell_j \cap \ell_k \neq \emptyset$.

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Therefore,

$$\ell_i \ell_j \ell_k \cdot (\sigma_j \wedge \sigma_k + \sigma_k \wedge \sigma_i) = \ell_k (d\ell_j \wedge d\ell_i) = -\ell_i \ell_j \ell_k \cdot \sigma_i \wedge \sigma_j$$

The General Case

However,

$$\mathcal{C} = \ell_0 \cup q,$$

where $\ell_0 = \{z = 0\}$ and $q := \{z^2 = xy\}$.

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In fact,

$H^2(X) = \langle \frac{\omega}{\ell_0 q} \rangle_{\mathbb{C}}$, where $\omega := zdx \wedge dy + xdy \wedge dz + ydz \wedge dx$.

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The sheaf $\pi_*\mathcal{E}^*_S(\log \bar{\mathcal{C}})$ is the sheaf of *log-resolution logarithmic forms* of $\mathcal{C}$ w.r.t. π.

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- Denote it by $\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})$.
- $\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})$ inherits a weight filtration W_*.

$$H^i(S; W_i\mathcal{E}^*_S(\log \bar{\mathcal{C}}))$$

$$\begin{array}{c} H^i(\mathbb{P}^2; W_i\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})) \\ \| \\ H^i(S; W_i\mathcal{E}^*_S(\log \bar{\mathcal{C}})) \end{array}$$

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$$\begin{array}{ccc} H^i(\mathbb{P}^2; W_i\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})) & & \\ \| & & \\ H^i(S; W_i\mathcal{E}^*_S(\log \bar{\mathcal{C}})) & \longrightarrow & H^i(S; W_i/W_{i-1}) \\ \| & & \\ H^i(X) & & \end{array}$$

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Such a residue map will be denoted by $\mathrm{Res}^{[i]}$.

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Such a residue map will be denoted by $\mathrm{Res}^{[i]}$.
In more generality:

$$H^i(\mathbb{P}^2; W_k\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})) \xrightarrow{\mathrm{Res}^{[i,k]}} H^{i-k}(\bar{\mathcal{C}}^{[k]}).$$

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1. $\operatorname{Res}^{[1,1]}$ *is injective.*
2. *If* $\psi \in \mathcal{E}^2(\mathbb{P}^2)(\log \mathcal{C})$ *is such that* $\operatorname{Res}^{[2,2]} \psi = 0$ *and* $\operatorname{Res}^{[2,1]} \psi = 0$, *then* $\psi = 0$.

Example

Consider $f = y^2 - x^3$, $\mathcal{C} = \{f = 0\}$, and the 2-form $\frac{dx \wedge dy}{f}$.

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However, if $\psi = \varphi \frac{dx \wedge dy}{f}$, then

- $\varphi \in (x, y) \Rightarrow \psi \in \mathcal{E}_0^2(\log \mathcal{C})$.
- Moreover, if $\varphi \in (y) \Rightarrow \left(\operatorname{Res}^{[2,2]} \psi\right)_P = 0$ at all $P \in \bar{\mathcal{C}}^{[1]}$ infinitely near 0.

Theorem

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- *Generators in degree 1*: σ_i, $i = 1, ..., r$,

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$$\begin{array}{cc} \psi_P^{\delta_1,\delta_2}, & P \in \mathcal{C}_i \cap \mathcal{C}_j, \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j) \\ \psi_\infty^{i,k_i}, & i = 1, \ldots, r, k_i = 1, \ldots, d_i - 1 \\ \eta^{i,s_i}, \bar{\eta}^{i,s_i}, & i = 1, \ldots, r, s_i = 1, \ldots, g_i. \end{array}$$

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- *Relations:*

$$\psi_P^{\delta_1,\delta_2} = -\psi_P^{\delta_2,\delta_1}$$
$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

for any $P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k$ and $\delta_1 \in \Delta_P(\mathcal{C}_i)$, $\delta_2 \in \Delta_P(\mathcal{C}_j)$, $\delta_3 \in \Delta_P(\mathcal{C}_k)$.

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$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

- *Product:*

$$\sigma_i \wedge \sigma_j = \sum_{P\in\mathcal{C}_i\cap\mathcal{C}_j} \mu_P(\delta_1,\delta_2)\psi_P^{\delta_1,\delta_2} + d_i \sum_{k_j=1}^{d_j-1} \psi_\infty^{j,k_j} - d_j \sum_{k_i=1}^{d_i-1} \psi_\infty^{i,k_i}.$$

Remark

Note that from the given presentation one can deduce that $H^*(X)$ only depends on the following invariants of $\mathcal{C}$:

$$(\{1,...,r\}, \mathcal{S} = \operatorname{Sing}\mathcal{C}, \{\Delta_P\}_{P\in\mathcal{S}}, \{\phi_P\}_{P\in\mathcal{S}}, \{\mu_P\}_{P\in\mathcal{S}})$$

such an ordered set of invariants of $\mathcal{C}$ will be referred to as the *Weak Combinatorics of $\mathcal{C}$*.

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such an ordered set of invariants of $\mathcal{C}$ will be referred to as the *Weak Combinatorics of $\mathcal{C}$*.

Hence

Theorem

The cohomology algebra of X only depends on its weak combinatorics.

Consider $\omega \in H^1(X)$.

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$$0 \to H^0(X) = \mathbb{C} \xrightarrow{\bullet\wedge\omega} H^1(X) \xrightarrow{\bullet\wedge\omega} H^2(X) \to 0 \qquad (H^*(X), \wedge\omega)$$

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Definition

The *i-th Resonance Variety of X* is defined as

$$\mathcal{R}^i(X) := \{\omega \in H^1(X) \mid h^1(H^*(X), \wedge\omega) \geq i\}$$

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Remark

Note that for any graded algebra A^* one can analogously define the i-th Resonance Variety $\mathcal{R}^i(A)$ of A^*.

Theorem

There is an Orlik-Solomon-like graded algebra A^* whose resonance varieties are isomorphic to $\mathcal{R}^i(X)$.

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$$A^1 := \textstyle\sum_{i=1}^r \sigma_i \mathbb{C} \quad A^2 := \sum_{P\in\mathcal{S}} \frac{\bigwedge^2 A_P}{I_P},$$

where

$$A_P := \sum_{\delta\in\Delta_P} \psi_P^\delta \mathbb{C}$$

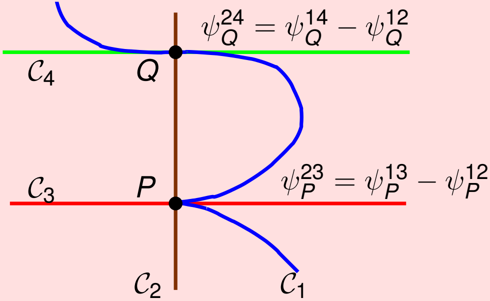
$$I_P := \langle \psi_P^{\delta_1} \wedge \psi_P^{\delta_2} + \psi_P^{\delta_2} \wedge \psi_P^{\delta_3} + \psi_P^{\delta_3} \wedge \psi_P^{\delta_1} \rangle_{\mathbb{C}}$$

and

$$\sigma_i \wedge \sigma_j := \sum_{P\in\mathcal{C}_i\cap\mathcal{C}_j} \mu_P(\delta_1,\delta_2)\psi_P^{\delta_1,\delta_2}$$

Example

Consider



$$
\begin{aligned}
\sigma_{12} &= 2\psi_P^{12} + \psi_Q^{12} \\
\sigma_{13} &= 3\psi_P^{13} \\
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\end{aligned}
\qquad
M := \begin{bmatrix}
2\beta & -2\alpha-\gamma & \beta & 0\\
3\gamma & \gamma & -3\alpha-\beta & 0\\
\beta & -\alpha-\delta & 0 & \beta\\
3\delta & \delta & 0 & -3\alpha-\beta\\
0 & 0 & \delta & -\gamma
\end{bmatrix}$$

- rank $M = 2 \Leftrightarrow (\lambda, -3(\lambda+\mu), 2\mu, \mu)$.

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- rank $M = 2 \Leftrightarrow (\lambda, -3(\lambda+\mu), 2\mu, \mu)$.
- Notice that $\mathcal{C}_1, 3\mathcal{C}_2, 2\mathcal{C}_2 + \mathcal{C}_4$ belong to a pencil of cubics.

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Definition

A differential space X is called *formal* if its algebra of differential forms $(\mathcal{E}(X), d)$ is formal.

Theorem (-,D.Matei,D.Macinic)

The complement of a plane curve X is a formal space.

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$$H^*(X) \quad \overset{e}{\to} \quad \mathcal{E}^*(\mathbb{P}^2)(\log \mathcal{C}) \tag{2}$$

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$$\begin{array}{ccc} H^*(X) & \stackrel{e}{\longrightarrow} & \mathcal{E}^*(\mathbb{P}^2)(\log \mathcal{C}) \\ [\sigma_i] & \mapsto & \sigma_i \\ \left[\psi_P^{\delta_1,\delta_2}\right] & \mapsto & \psi_P^{\delta_1,\delta_2} \\ \left[\psi_\infty^{i,k_i}\right] & \mapsto & \psi_\infty^{i,k_i} \\ \left[\eta^{i,s_i}\right] & \mapsto & \eta^{i,s_i} \end{array} \qquad (2)$$

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Can we choose forms so that e is *well-defined*?

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

Choose δ_P at each $P \in \mathcal{S}$, then

$$\psi_P^{\delta_1,\delta_2} = \psi_P^{\delta_P,\delta_2} - \psi_P^{\delta_P,\delta_1}$$

$$\sigma_i \wedge \sigma_j =$$

$$= \sum_{P \in \mathcal{C}_i \cap \mathcal{C}_j} \mu_P(\delta_i, \delta_j) \psi_P^{\delta_i, \delta_j} +$$

$$+ d_i \sum_{k_j=1}^{d_j-1} \psi_\infty^{j,k_j} - d_j \sum_{k_i=1}^{d_i-1} \psi_\infty^{i,k_i}.$$

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$$+ d_i \sum_{k_j=1}^{d_j-1} \psi_\infty^{j,k_j} - d_j \sum_{k_i=1}^{d_i-1} \psi_\infty^{i,k_i}.$$

Let $\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k$ be such that:

- $d_i = d_j = d_k$
- $\mu_P(\delta_i, \mathcal{C}_j) = \mu_P(\delta_i, \mathcal{C}_k)$,
- $\mu_P(\delta_j, \mathcal{C}_i) = \mu_P(\delta_j, \mathcal{C}_k)$,
- $\mu_P(\delta_k, \mathcal{C}_i) = \mu_P(\delta_k, \mathcal{C}_j)$,

then

$$\sigma_i \wedge \sigma_j + \sigma_j \wedge \sigma_k + \sigma_k \wedge \sigma_i = 0 \tag{3}$$

Note that if $\mathcal{C}_k = \alpha\mathcal{C}_i + \beta\mathcal{C}_j$, then (3) is trivial.

Theorem (Max-Noether Fundamental Theorem (M.Noether,...,Fulton))

Let F, G, and H be three plane curves with no common components. If $H_P \in (F_P, G_P)$ at any $P \in V(F) \cap V(G)$, then there exist two forms $A, B \in \mathbb{C}[x, y, z]$ such that

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Remark

The conditions $H_P \in (F_P, G_P)$ at any $P \in V(F) \cap V(G)$ are commonly known as the *Noether Conditions*.

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Three curves F, G, and H satisfying ( )
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Theorem (-,M.A.Marco)

If F, G, and H belong to a primitive combinatorial pencil, then they belong to an algebraic pencil ($H = \alpha F + \beta G$).

Remark

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Any combinatorial pencil admits a primitive refinement.

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Proposition

Any combinatorial pencil admits a primitive refinement.

This proves the formality of X.

Open Problems

- Are there also *nice* combinatorial descriptions of $H^*(X)$ in higher dimensions?

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- Are the complements of hypersurfaces in the projective space formal?
- What about toric varieties, or weighted projective spaces?
- Study the resonance varieties of Abstract Curve Combinatorics. This could lead to criteria for non-quasiprojective groups.