Fibered Faces and Dynamics of Mapping Classes

Branched Coverings, Degenerations, and Related Topics 2012

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I. Pseudo-Anosov mapping classes

1. Introduction
2. Visualizing pseudo-Anosov mapping classes
3. Train tracks
4. Minimum dilatation problem

II. Fibered Faces and Applications

1. Introduction
2. Fibered face theory
3. Alexander and Teichmüller polynomials
4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations

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2. Deformations of mapping classes on fibered faces
3. Penner sequences and applications
4. Quasiperiodic mapping classes

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Introduction: Overview of Lectures

Problem: Describe the small dilatation pseudo-Anosov elements $\mathcal{P}_S \subset \mathrm{Mod}(S)$ (e.g., invariants, constructions, families)

Method: Use *theory of Fibered Faces* (Thurston)

$$\mathcal{P} = \bigcup_S \mathcal{P}_S \hookrightarrow \bigcup_\alpha F_\alpha$$

where F_α are Euclidean convex polyhedra (fibered faces) and the image of $\mathcal{P}$ is the union of the interior rational points of F_α.

Focus of this lecture: The minimum dilatation problem for pseudo-Anosov mapping classes.

Preliminaries

Objects: $S = S_{g,n}$ compact oriented surface, $\chi(S) < 0$
$\mathrm{Mod}(S)$ mapping class group of S:

$\mathrm{Mod}(S) \overset{def}{=} \{\phi : S \overset{\text{homeo}^+}{\longrightarrow} S\}/\text{isotopy}$
(alternately, isotopy rel ∂S)

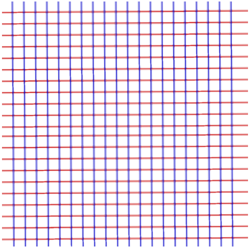
Nielsen-Thurston Classification: Let $\phi \in \mathrm{Mod}(S)$. Then ϕ is either *periodic, reducible or pseudo-Anosov*.

- *periodic:* $\exists k \geq 1$ such that $\phi^k \sim \mathrm{id}$
- *reducible:* $\exists k \geq 1$ and $\exists \gamma \subset S$ an essential simple closed curve such that $\phi^k(\gamma) \sim \gamma$
- *pseudo-Anosov:*
  $\exists$ *stable and unstable foliations* of ϕ: $(\mathcal{F}^s, \nu^s)$, $(\mathcal{F}^u, \nu^u)$, and
  $\exists$ *dilatation* of ϕ: $\lambda > 1$ where
  - $\mathcal{F}^s$ and $\mathcal{F}^u$ are singular ϕ-invariant foliations on S,
  - ν^s and ν^u are transverse measures, and
  - $\phi_*\nu^s = \frac{1}{\lambda}\nu^s$ and $\phi_*\nu^u = \lambda\nu^u$.

Local Flat Structure:

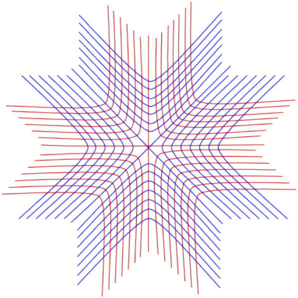
Let $\mathcal{P} = \bigcup \mathcal{P}_S$ be the set of pseudo-Anosov mapping classes, $(S, \phi) \in \mathcal{P}$.

Away from singularities $(\mathcal{F}^u, \nu^u)$ and $(\mathcal{F}^s, \nu^s)$ locally define a Euclidean structure on S.

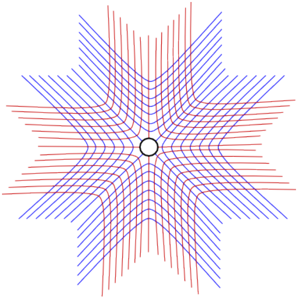


Near singularities and boundary components

Near a singularity:



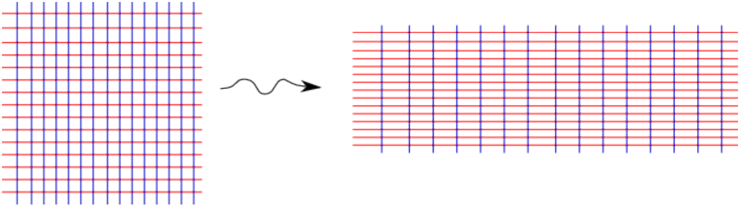
Near a boundary component:



These are called 4-pronged or degree 2.

Action of ϕ on local flat structure

An $r \times r$ square gets sent to a rectangle with sides $\lambda r \times \frac{1}{\lambda} r$.



Singularities (resp. boundary components) are permuted.

$\Rightarrow$ If boundary component is not 1-pronged, then fIlling in its orbit doesn't change dilatation.

Relation to closed Teichmüller geodesics on moduli space

- *Teichmüller space:* $\mathcal{T}(S)$ = marked Riemann surfaces (marking is a homeomorphism $f : S \to X$).
- Mod(S) acts on $\mathcal{T}(S)$ by pre-composition.
- *Moduli space:* $\mathcal{M}(S)$ = Riemann surfaces X homeomorphic to S.

$$\mathcal{M}(S) = \mathcal{T}(S)/Mod(S)$$

- Pseudo-Anosov elements correspond to closed geodesics on $\mathcal{M}(S)$
  (local flat structure determines points on the geodesic)
- *Length Spectrum*
  $\stackrel{def}{=}$ Teichmüller lengths of closed geodesic curves on $\mathcal{M}(S)$
  $= \{\log(\lambda(\phi)) \ : \ \phi \in \mathcal{P}_S\}$
  (i.e., study of dilatations has ties with study of geometry of $\mathcal{M}(S)$)

Visualizing pseudo-Anosov mapping classes

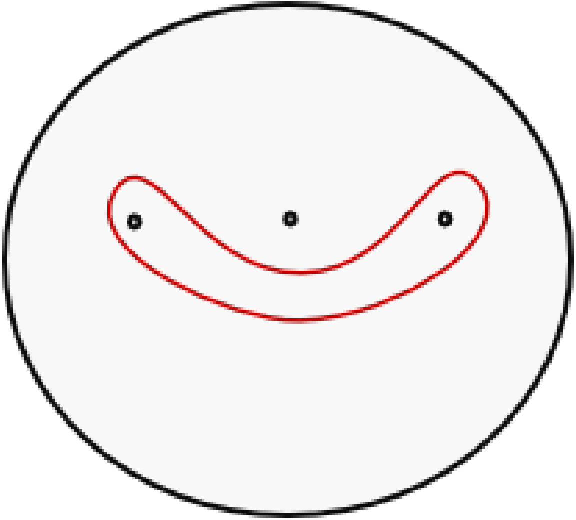
Look at actions on essential simple closed curves.

- Take any essential simple closed curve γ on S.
- If ϕ is not periodic or reducible,
$$\phi^m(\gamma) \rightsquigarrow (\mathcal{F}^s, \nu^s), \qquad \text{as } m \to \infty$$
where ν^s is a transverse measure (defined up to positive scalar multiple). $(\mathcal{F}^s, \nu^s)$ is called the *stable foliation* for ϕ.
- ϕ stretches along $\mathcal{F}^s$ and contracts ν^s by $\frac{1}{\lambda}$, where $0 < \frac{1}{\lambda} < 1$. This λ is the *dilatation* of ϕ.
- Similarly, the *unstable foliation* is determined by:
$$\phi^{-m}(\gamma) \rightsquigarrow (\mathcal{F}^u, \nu^u), \qquad \text{as } m \to \infty$$
- $(\mathcal{F}^s, \nu^s)$, $(\mathcal{F}^u, \nu^u)$ and λ are independent of the choice of γ.

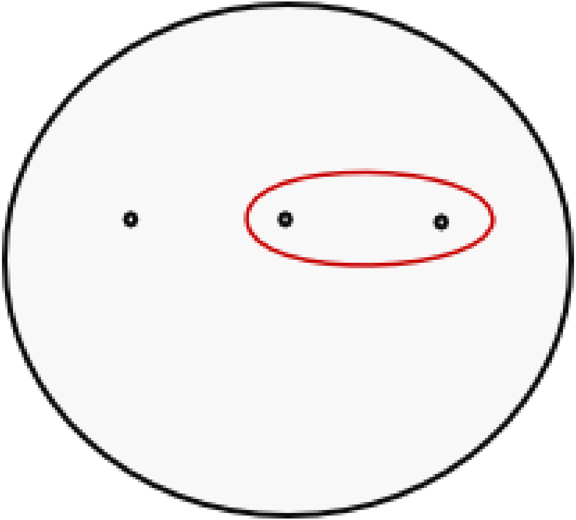
Action on essential simple closed curves

Example 1: a periodic map on the $S_{0,4}$, the sphere with 4 boundary components.

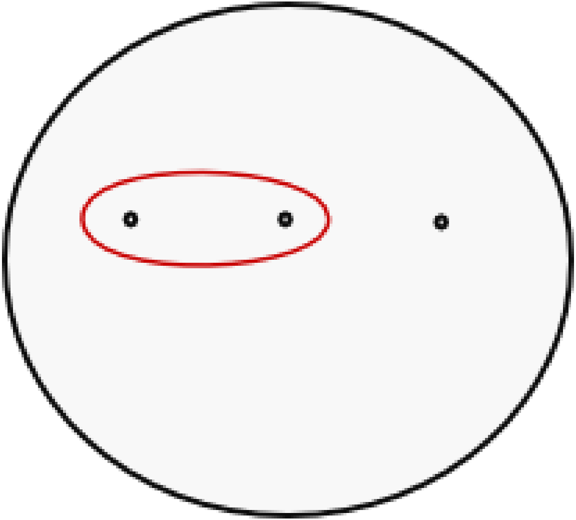
Action of periodic map on a simple closed curve (periodic map):



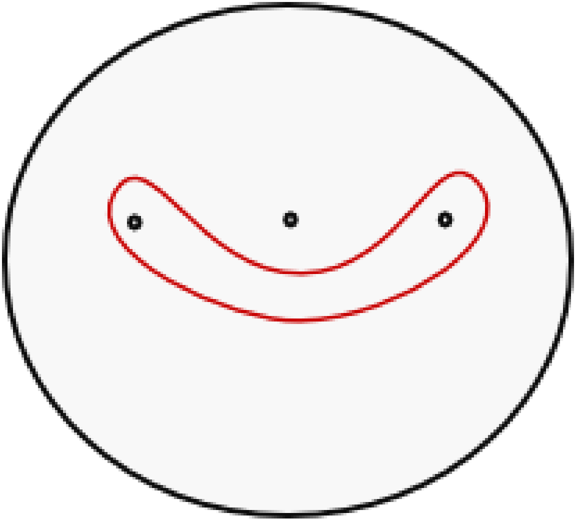
Action on a simple closed curve (one application):



Action on a simple closed curve (2nd application):



Action on a simple closed curve (3rd application):

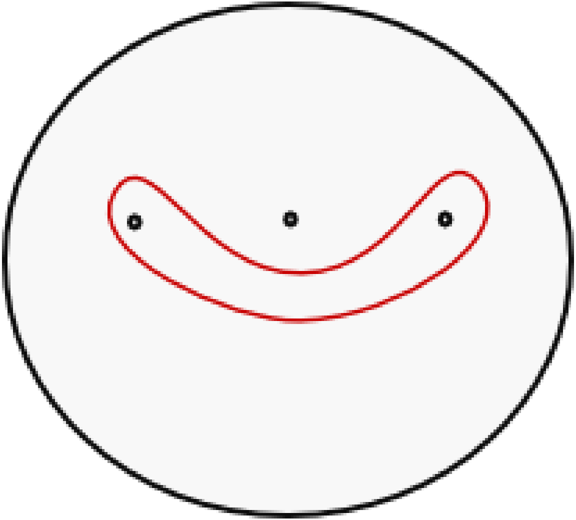


back to start

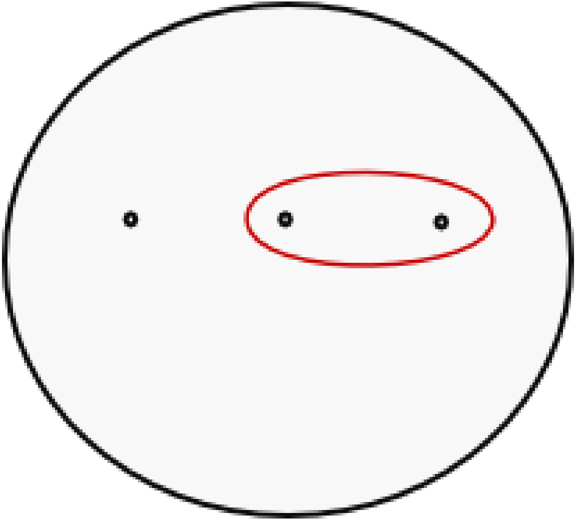
Action on essential simple closed curves

Example 2: simplest pseudo-Anosov braid monodromy

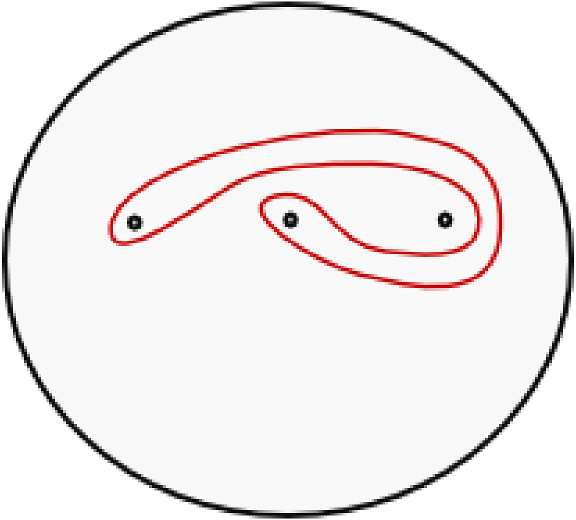
Action on a simple closed curve (simplest pA braid monodromy):



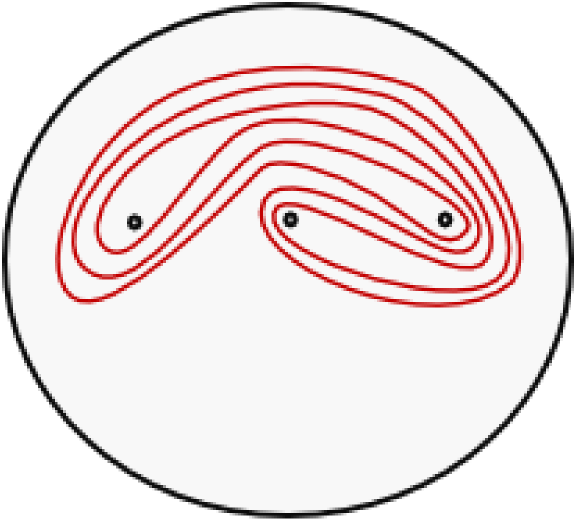
Action on a simple closed curve (one application of map):



Action on a simple closed curve (2 applications of map):



Action on a simple closed curve (3 applications of map):



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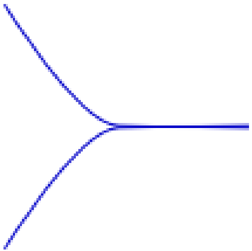
Train tracks

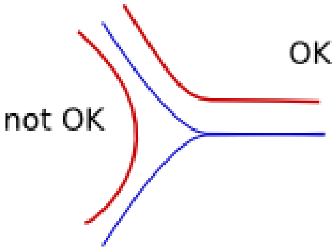
Train tracks are useful for capturing the dynamics of a mapping class.

A *train track* is an embedded graph on S with "smoothings" at the vertices.



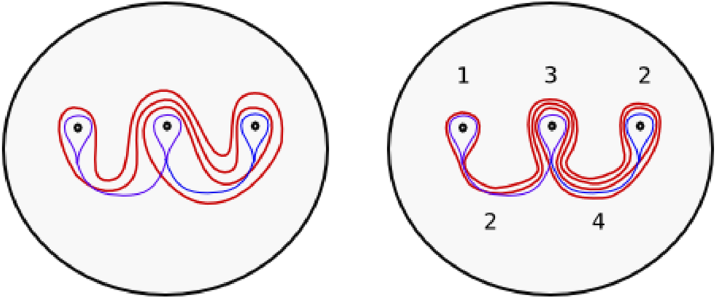
Train tracks

A curve is *carried* by the train track (or is an *allowable curve*) if it can be isotoped to the train track in a smooth way.



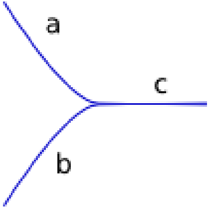
Train tracks

An allowable curve determines weights on the edges $\mathcal{E}_\tau$ (or *branches*) of the train track. For example,



Train tracks

When two branches meet, the edge weights corresponding to an allowable curve satisfy the *branching relation*



$\rightsquigarrow \qquad a + b = c$

The *weight space* of a train track is the vector space of maps that satisfy the branching conditions:

$$W_\tau = \{\mathcal{E}_\tau \to \mathbb{R}\}/\text{branching cond.} \;=\; \mathbb{R}[\mathcal{E}_\tau]^{\text{dual}}$$

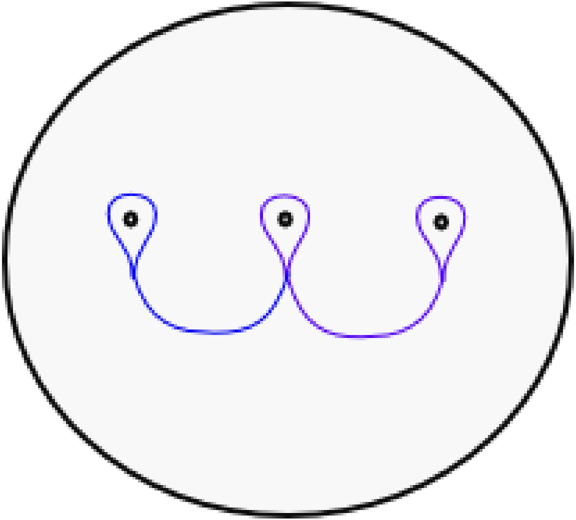
The non-negative elements of W_τ can be thought of as *virtual curves*.

Compatible Train Tracks and Stable Foliation

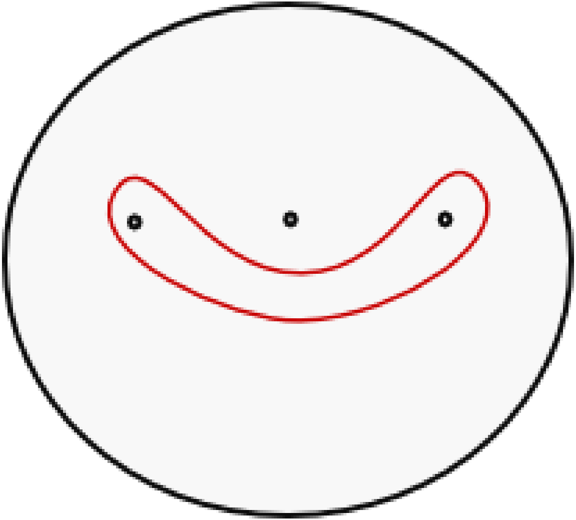
Given a pseudo-Anosov map (S, ϕ) and a compatible train track τ:

- For γ any ess. simple closed curve, $\phi^m(\gamma)$ is carried by τ for $m \gg 0$.
- The action of ϕ on S induces a transition matrix $T : W_\tau \to W_\tau$.
- "lengths" of virtual curves γ: $|\phi^m(\gamma)| \sim |T^m v_\gamma|$ for $m \gg 0$.
- The transition matrix restricted to W_τ is a Perron Frobenius map.
- Train track + PF eigenvector $\Rightarrow$ transverse measured singular foliatation.
- $\lambda =$ PF eigenvalue $= \lim_{m\to\infty} |T^m(v_\gamma)|^{\frac{1}{m}}$, for v_γ any virtual curve
- Singularities of $\mathcal{F}^s \Leftrightarrow$ complementary regions of the train track with $k \neq 2$ cusps.

Train track compatible with simplest pseudo-Anosov braid



Starting curve γ



Curve γ is not carried by train track

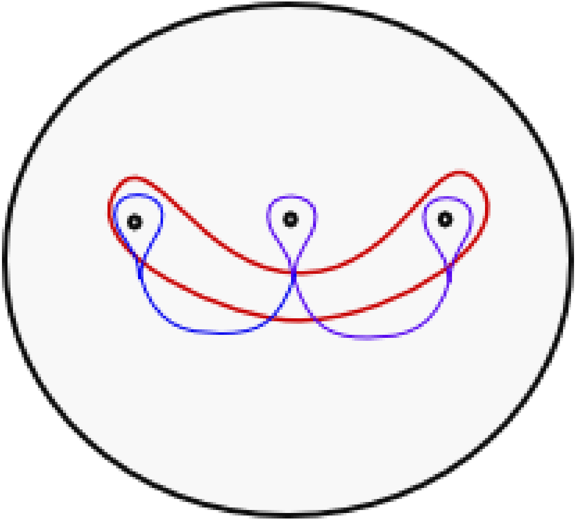
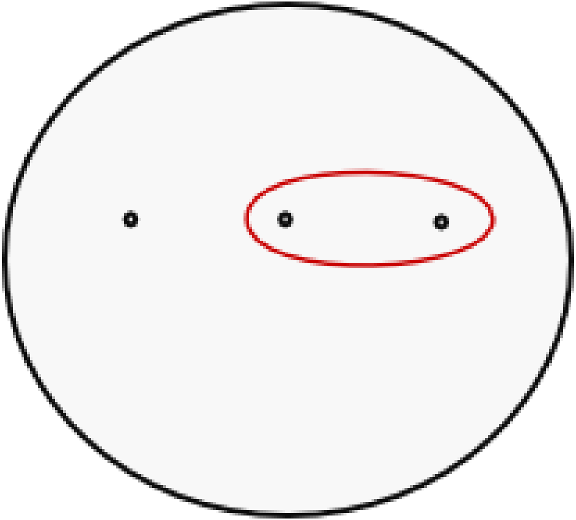
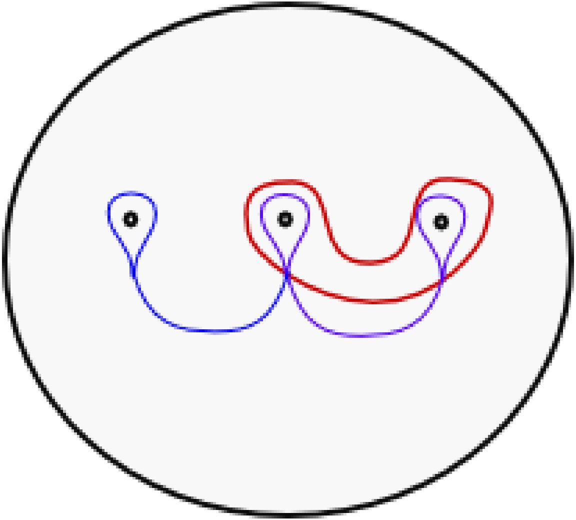


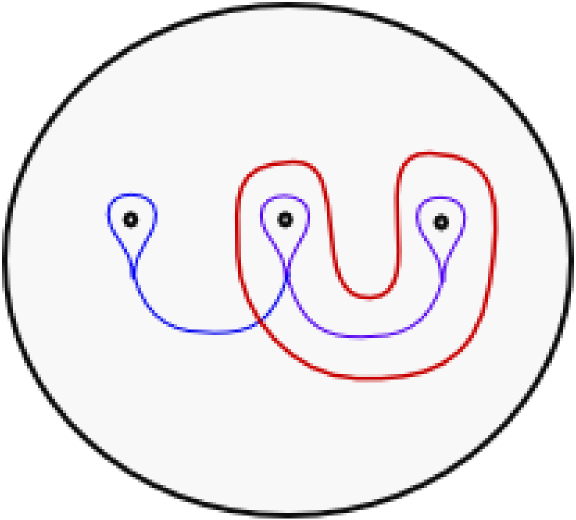
Image of γ after 1st application of map):



After 1st application of map (with train track):

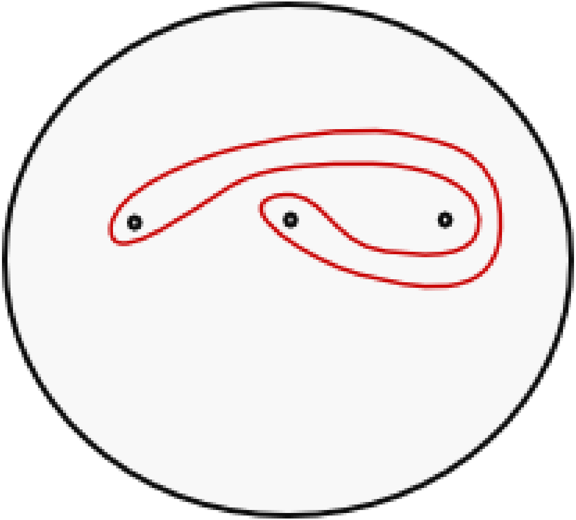


After 1st application of map (with train track):

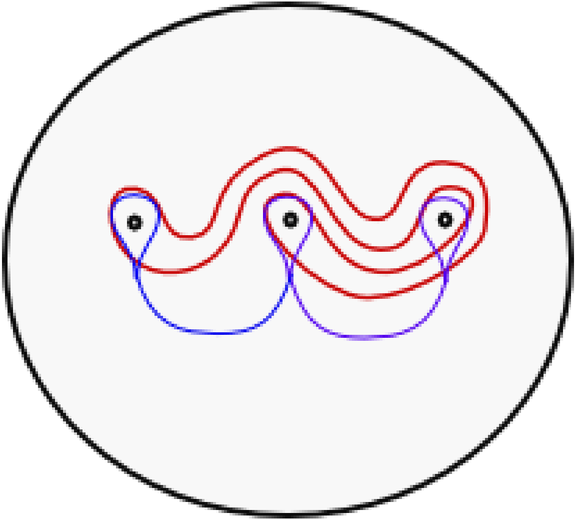


skip forward

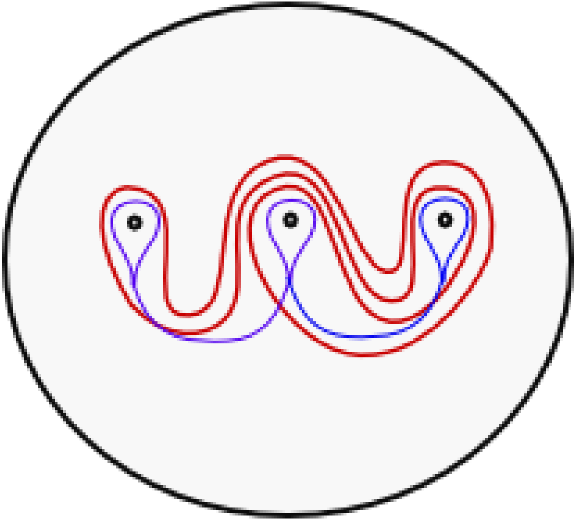
Curve γ after 2nd application of map:



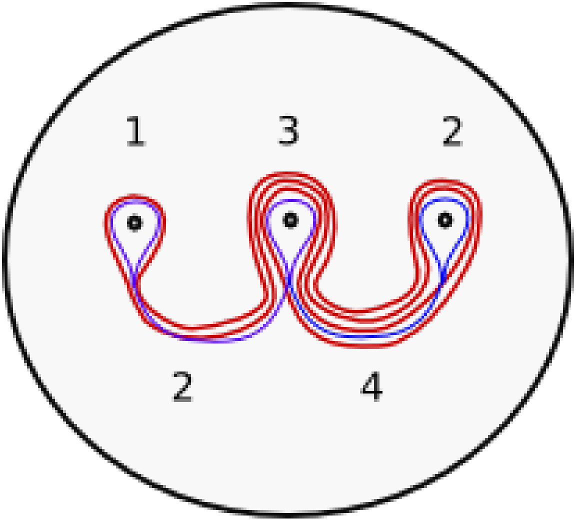
After 2nd application of map (with train track):



After 2nd application of map (with train track):

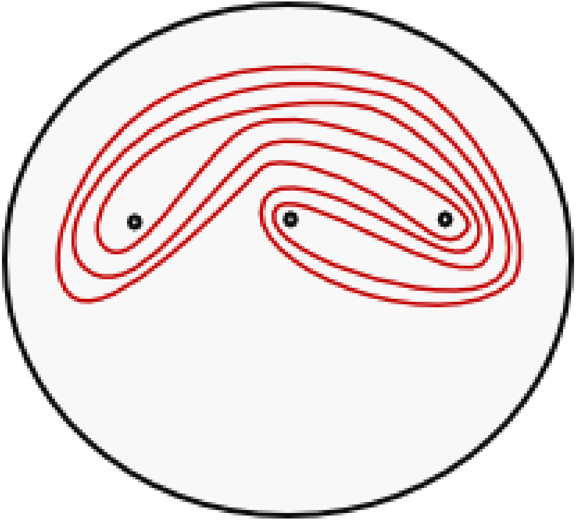


Train track with edge weights (after 2nd application of map):

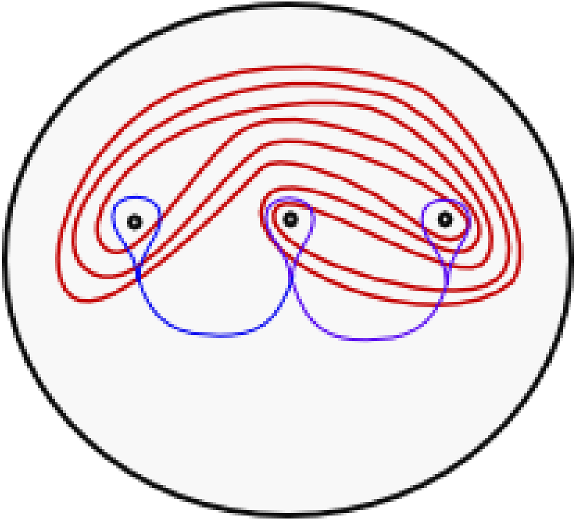


skip forward

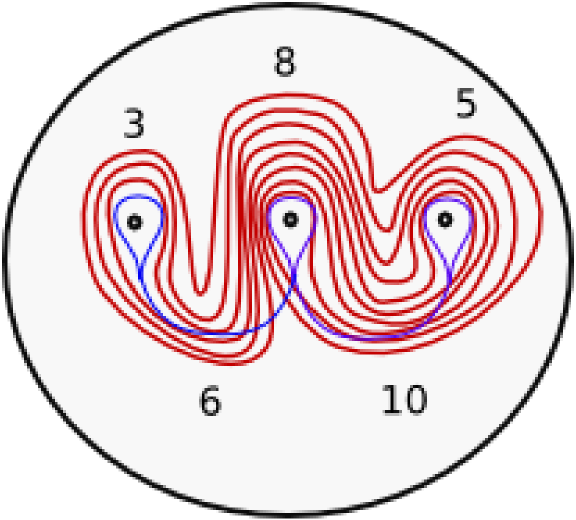
Curve γ after 3rd application of map:



After 3rd application of map (with train track):



Train track with edge weights (after 3rd application of map):



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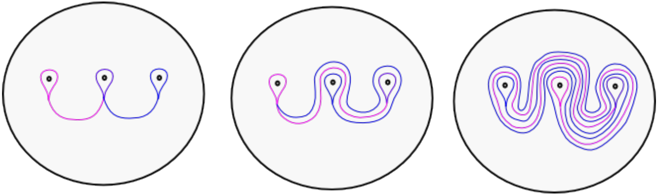
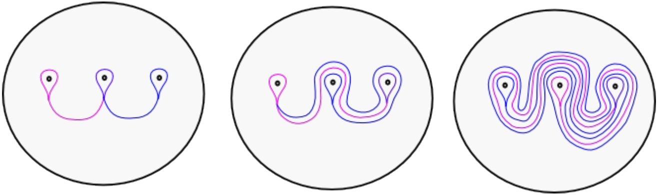
Transition matrix



$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 4 \end{bmatrix} \mapsto \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Transition matrix



$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Train track τ defines $\mathcal{F}^s$.
- Transition matrix T is a Perron-Frobenius map, and determines the transverse measure and dilatation
- PF-eigenvalue $\lambda = |x^2 - 3x + 1| = \frac{3+\sqrt{5}}{2}$ is the dilatation of ϕ

Number theoretic consequences for dilatations

Take $(S, \phi) \in \mathcal{P}$

$\lambda(\phi)$ is a *Perron unit*, degree $\leq 6g - 6 + 2n$. (Thurston, Penner)

- $\lambda(\phi)$ satisfies a monic integer polynomial
(the characteristic polynomial of the transition matrix)
- degree $\leq$ dimension of the space of allowable weights on a train track for ϕ
(related to the space of transverse measured singular foliatations)
- $\lambda(\phi)$ is an algebraic unit
(transition matrix on weight space preserves a symplectic form)
- $\lambda(\phi)$ is a Perron number, i.e., all other algebraic conjugates have strictly smaller norm
(transition matrices are Perron-Frobenius)

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Minimum dilatation problem I

Let $\delta(S) \stackrel{\text{def}}{=} \min\{\lambda(\phi) \ : \ \phi \in \mathcal{P}_S\}$

Property: $\delta(S) > 1$.

Minimum Dilatation Problem

- Find $\delta(S)$.
- Describe element (or elements) of $\mathcal{P}_S$ that realizes it.
- Study number theoretic properties of dilatations ("shape" of polynomials)

Small Euler characteristic examples

Applications of train track automata: (Ham, Ko, Los, Song)

Smallest n-strand braid monodromy,:

$n = 3$**:** (simplest pseudo-Anosov braid) $\delta = |x^2 - 3x + 1| = \frac{3+\sqrt{5}}{2}$.

$n = 4$**:** (Ko-Los-Song '02) $\delta = |x^4 - 2x^3 - 2x + 1| \approx 2.29663$.

$n = 5$**:** (Ham-Song '05) $\delta = |x^4 - x^3 - x^2 - x + 1| \approx 1.72208$.

Smallest genus:
$g = 1, n = 1$**:** double branched cover ($n = 3$) $\delta = |x^2 - 3x + 1|$.

$g = 2, n = 0$**:** (Cho-Ham '08) double branched covering ($n = 5$);
$\delta = |x^4 - x^3 - x^2 - x + 1|$

Minimum dilatation problem II

A subcollection $\mathcal{P}_0 \subset \mathcal{P}$ has *asymptotically small dilatation* if there is a constant C so that for all $(S, \phi) \in \mathcal{P}_0$, the *normalized dilatation* satisfies

$$L(S,\phi) \stackrel{\text{def}}{=} \lambda(\phi)^{|\chi(S)|} \leq C.$$

This implies

$$\log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.$$

Problem: How does the normalized dilatation behave for large g and n?

Asymptotic behavior in (g, n)-plane

- (Penner '91)

$$\log \delta(S_{g,n}) \geq \frac{\log(2)}{12g - 12 + 4n}, \qquad \log \delta(S_g) \asymp \frac{1}{g}.$$

- (H-Kin '06, Tsai '08, Valdivia '11)

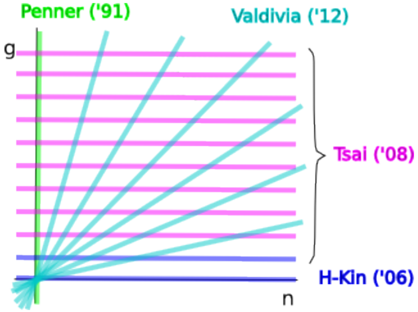
$$\log \delta(S_{g,n}) \asymp \frac{1}{|\chi(S_{g,n})|}$$

  for fixed $g = 0, 1$ and for (g, n) on positive rays through the origin with rational slope.

- (Tsai '08) For fixed $g \geq 2$,

$$\log \delta(S_{g,n}) \asymp \frac{\log(n)}{n}.$$

Asymptotic behavior in (g, n)-plane



$$\log(\delta(S_{g,n})| \asymp \frac{1}{|\chi(S_{g,n})|}(\text{blue/green}) \quad \text{vs.} \quad \frac{\log(|\chi(S_{g,n})|)}{|\chi(S_{g,n})|}(\text{red}).$$

Minimum dilatation problem III

Problem: Which naturally occurring subsets of $\mathcal{P}$ have asymptotically small dilatation?

Negative examples:

- Algebraic constraints, e.g., Torelli group (Farb-Leininger-Margalit '08)
- Geometric constraints e.g., on flat structure (Bossy, Lanneau '10)

Positive examples: Hyperelliptic mapping classes, orientable mapping classes (H-Kin '06, H '10)

...More special families in next two lectures...