Fibered Faces and Dynamics of Mapping Classes III

Branched Coverings, Degenerations, and Related Topics 2012

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2. Visualizing pseudo-Anosov mapping classes
3. Train tracks
4. Minimum dilatation problem

II. Fibered Faces and Applications

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3. Alexander and Teichmüller polynomials
4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations

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3. Second Application: Penner sequences
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Functions on $\mathcal{P}$

Let $F \subset \text{int}(\overline{\mathcal{P}})$, $G_F = H_1(M; \mathbb{Z})/\text{Torsion}$

Alexander polynomial
$\exists \Delta_F \in \mathbb{Z}[G_F]$ such that for $\phi \in F$, $\lambda_{\text{hom}}(\phi) = |\Delta_F^{\psi}|$.

Teichmüller polynomial (C. McMullen '00)
$\exists \Theta_F \in \mathbb{Z}[G_F]$ such that for $\phi \in F$, $\lambda(\phi) = |\Theta^{\psi}|$.

Normalized dilatation (D. Fried '82, C. McMullen '00)
$\exists L : \text{int}(\overline{\mathcal{P}}) \to \mathbb{R}$ continuous, convex function such that for $(S, \phi) \in \mathcal{P}$, $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$.

Goals in this lecture

- Understand these functions in terms of train tracks and coverings.
- Give examples of deformations of pseudo-Anosov mapping classes.

Fibrations and coverings

Fix M a hyperbolic 3-manifold, F a fibered face (i.e., a connected component of $\text{int}(\overline{\mathcal{P}})$).

The rational points in the interior of F correspond to the following equivalent objects:

- fibration $\psi : M \to S^1$;
- epimomorphism $\psi_* : H_1(M; \mathbb{Z}) \to \mathbb{Z}$;
- monodromy $\phi : S \to S$ (or (S, ϕ));
- an infinite cyclic covering $M_\psi \to M$, where $M_\psi = S \times \mathbb{R}$.

Maximal abelian covering

Let $\widetilde{\rho} : \widetilde{M}^{\mathrm{ab}} \to M$ be the maximal abelian covering.

This is determined by the Hurewicz map,
$\pi_1(M) \to H_1(M; \mathbb{Z})/\mathrm{Torsion} \quad (= G_F)$.

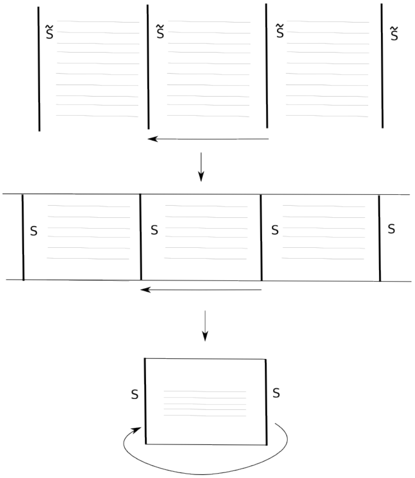
For all $\psi \in F$, we have a commutative diagram.

$$\begin{array}{ccc} \widetilde{M}^{\mathrm{ab}} = \widetilde{S} \times \mathbb{R} & \longrightarrow & M_\psi = S \times \mathbb{R} = \widetilde{M}^{\mathrm{ab}}/K_\psi \\ \downarrow & \swarrow & \\ M & & \end{array}$$

whereG_F acts as covering automorphisms of $\widetilde{M}^{\mathrm{ab}}$ over M, and
$K_\psi = \mathrm{Ker}(\psi_*) \subset G_F$.

Tower of abelian coverings

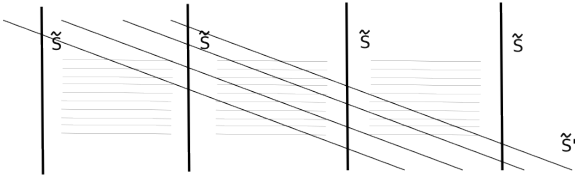
Picture:



$$\begin{array}{c} \widetilde{M}^{ab} \simeq \widetilde{S} \times \mathbb{R} \\ \downarrow \\ M_\psi \simeq S \times \mathbb{R} \\ \downarrow \\ M \end{array}$$

Suspension of flat structure on S lifts to M_ψ and $\widetilde{M}^{ab}$.

Deformation of $\widetilde{S}$ in $\widetilde{M}^{\mathrm{ab}}$



Let $(S', \phi') \in F$ be a deformation of (S, ϕ). The lift $\widetilde{S}' \subset \widetilde{M}^{\mathrm{ab}}$ has the property that it is preserved by the action of $K_{\psi'} = \mathrm{Ker}(\psi'_*)$.

Properties

Advantages of the picture:

- The homeomorphism type of the surface $\widetilde{S}$ does not depend on the choice of $(S, \phi) \in F$. Thus we can think of our deformations as happening on a single surface.
- The stable and unstable foliations and dilatation lift to $\widetilde{S}$.
- The local flat structure on S defined by ϕ lifts to $\widetilde{S}$, and the suspension defines a foliation on $\widetilde{M}^{\mathrm{ab}}$.
- Given any fixed $(S, \phi) \in F$, and $(S_a, \phi_a) \in F$, the local flat structure on the preimage $\widetilde{S}_a$ of S_a is the restriction of the suspension of $(\widetilde{S}, \widetilde{\phi})$ in $\widetilde{M}^{\mathrm{ab}}$.
- The action of $G_F = H_1(M; \mathbb{Z})/\text{Torsion}$ on $\widetilde{M}^{\mathrm{ab}}$ acts as isometry.

Properties

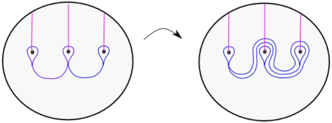
Caution:

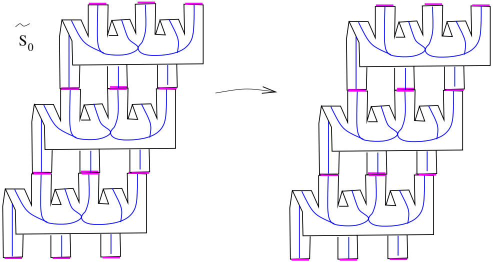
- The surface $\widetilde{S}$ is of infinite type (Moduli space theory for infinite surfaces?)
- The way $\widetilde{S}$ is embedded in M^{ab} changes, and hence so does its intersection with the suspended foliation on $\widetilde{M}^{\mathrm{ab}}$.

Difficulty: Describe the changing structure on $\widetilde{S}$ as you vary $\psi \in F$.

We can do this for special cases.

Application 1: Simplest pseudo-Anosov braid





Teichmüller polynomial

$$\begin{bmatrix} t & t \\ 1 & 1+t^{-1} \end{bmatrix} \quad \Rightarrow \quad \Theta(t,u) = u^2 - (t+1+t^{-1})u + 1$$

Alexander polynomial

$$\begin{bmatrix} -t & t \\ 1 & 1-t^{-1} \end{bmatrix} \quad \Rightarrow \quad \Delta(t,u) = u^2 - (-t+1-t^{-1})u + 1$$

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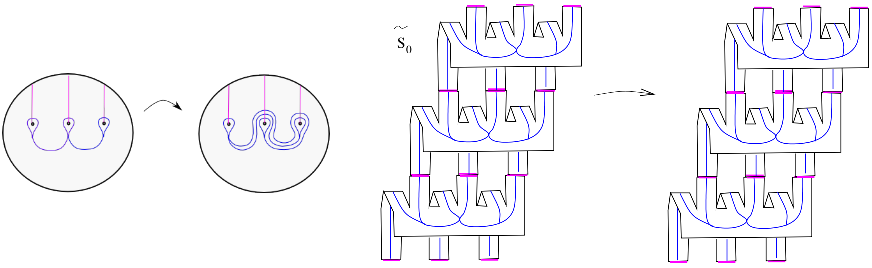

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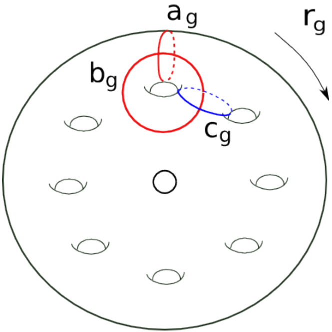
Penner's sequence of mapping classes



The mapping classes

$$\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}$$

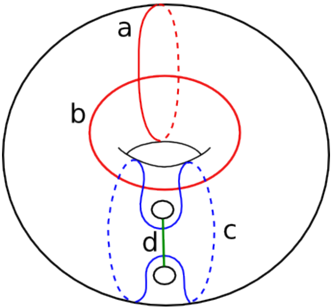
are pseudo-Anosov, and

$$\lambda(\phi_g)^g \leq 11.$$

Original proof:

- Penner's semi-group criterion;
- Train track transition matrix has bounded column sums.

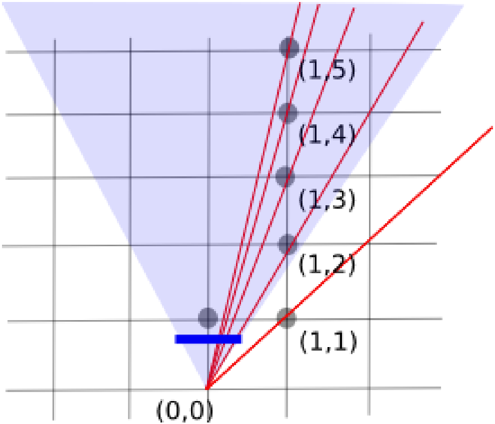
Alternate proof using fibered faces



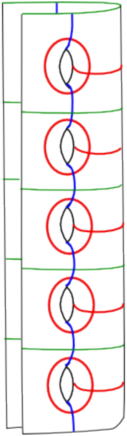
- Let $S = S_{1,2}$ and $\phi = \delta_c \delta_b^{-1} \delta_a$. This is pseudo-Anosov by semi-group criterion.
- The path d determines a ϕ-invariant element of $H^1(S; \mathbb{Z})$, and hence an element $\xi \in H^1(M; \mathbb{Z})$.

Deformation

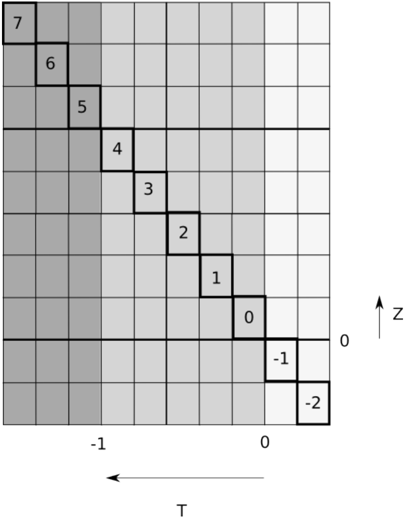
Penner's sequence (S_g, ϕ_g) *corresponds to the elements* $\psi_n = \xi + n\psi = (1, n)$ *in the fibered cone over* F_ψ*, for* n *large enough.*



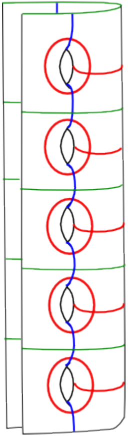
Deformation of $\widetilde{S}$



$\widetilde{S}$

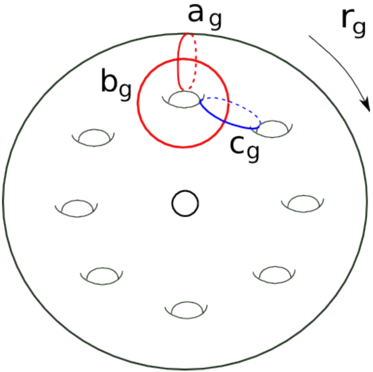


Deformations corresponding to the sequence ψ_g



$\widetilde{S}$

$\longrightarrow$



$(S_{1,n}, \phi_{1,n})$

Invariants

The Teichmüller polynomial for F_ϕ is given by

$$\Theta(u,t) = u^2 - u(5 + t + t^{-1}) + 1,$$

where u is dual to ψ and t is dual to ξ.

- (McMullen) $\Rightarrow \lambda(\phi_g) = |x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1|$.
- (Fried, McMullen) $\Rightarrow L(S_g, \phi_g) \to \left(\frac{7+3\sqrt{5}}{2}\right)^2$.

  or, equivalently,

$$\lim_{g\to\infty} \lambda(\phi_g)^g = \frac{7+3\sqrt{5}}{2} \approx 6.8541 < 11.$$

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Generalized Penner Sequences: Set up.

Let a, b be essential multi-curves on S, c a simple closed curve on S.
Let d be either a simple closed curve on S, or a relatively simple closed curve in $(S, \partial S)$.
Assume also:

- a, b and c meet pairwise minimally;
- a and c are disjoint;
- $a \cup b$ and d are disjoint;
- $i_{\text{alg}}(c, d) = 0$.

Let $\phi = \delta_c \delta_b^{-1} \delta_a$.

(Penner's semi-group criterion) $\Rightarrow$ If $a \cup b \cup c$ fill S, then (S, ϕ) is pseudo-Anosov.

Generalized Penner Sequences

Take m large.

Let $S_m \to S$ be the m-cyclic covering determined by d and let r_m be a generator for the group of covering automorphisms.

Let $\Sigma_m \subset S_m$ be a fundamental domain of r_m homeomorphic to $\Sigma = S \setminus d$.

Let $\widetilde{a}, \widetilde{b}, \widetilde{c}$ be lifts of a, b, c so that $\widetilde{a}$ and $\widetilde{b}$ are contained in Σ_m and

$$\widetilde{c} \subset \Sigma_m \cup r_m \Sigma_m \cup \cdots \cup (r_m)^k \Sigma_m,$$

for some $k < m$.

Theorem and consequences

Theorem (Valdivia - PhD. Thesis)

The maps (S_m, ϕ_m) are the monodromies of a single 3-manifold M.

(Fundamental groups, Mostow-Prasad rigidity)

Theorem (H_)

The maps (S_m, ϕ_m) are monodromy of the mapping torus M_ϕ corresponding to $\psi_n = \xi + n\psi$, where $\xi \in H^1(M; \mathbb{Z})$ is induced by d, and ψ is the fibration of M associated to ϕ.

Corollary:

- $L(S_m, \phi_m) \to L(S, \phi)$,
- $L(S_m, \phi_m)$ is bounded,
- $\chi(S_m)$ is proportional to m, and
- $\log(\lambda(\phi_m)) \asymp \frac{1}{m}$.

Applications of Penner sequences

Application 2: Handlebody subgroups

Background:

(S, ϕ) is a handlebody mapping classes if for some identification $S = \partial H$, ϕ extends to a homeomorphism of H.

Let $\mathcal{H}_S$ be the collection of handlebody mapping classes in $\mathcal{P}_S$, and $\mathcal{H} = \bigcup_S \mathcal{H}_S$.

(H. Masur '86) The limit set of the handlebody subgroup has measure zero in Thurston's sphere of measured foliations

In other words, $\mathcal{H}$ is small.

Application of Penner Sequences: $\mathcal{H}$ supports asymptotically small dilatations.

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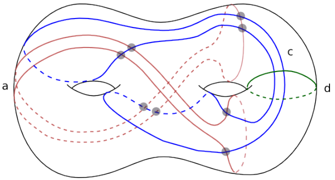
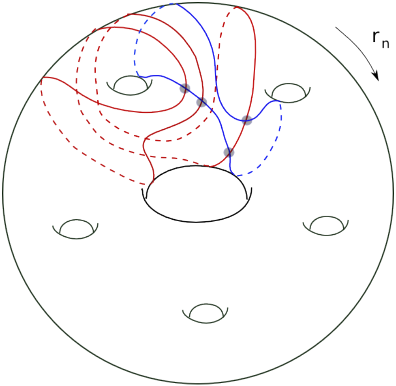
Handlebody map deformation





Application 3: Mapping classes with trivial homological dilatation

Background:

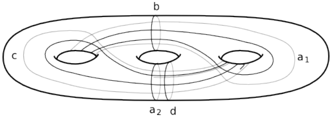
(Farb-Leininger-Margalit '08) The pseudo-Anosov elements in the Torelli groups have dilatation $\lambda(\phi) \geq c_0 > 1$.

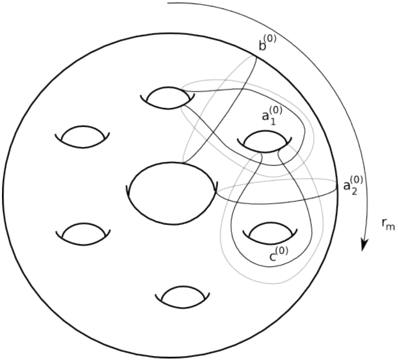
Application of Penner Sequences $\Rightarrow$ Let $\mathcal{F} \subset \mathcal{P}$ be the family of pseudo-Anosov mapping classes with trivial homological dilatation. Then $\mathcal{F}$ supports asymptotically small dilatation.

Lemma

The composition of a Torelli map and a periodic map has trivial homological dilatation.

Torelli map deformation





Quasiperiodicity Question

Question: (Farb-Leininger-Margalit) Given $C > 1$, does $L(S, \phi) < C$ imply that $(S, \phi) = (S_m, \phi_m)$ for some Penner sequence (S_m, ϕ_m) where the support Σ satisfies

$$|\chi(\Sigma)| < K_C?$$