Deformations of product-quotient surfaces and reconstruction of Todorov surfaces via $\mathbb{Q}$-Gorenstein smoothing

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$/\mathbb{C}$

A projective surface S is called a <u>product-quotient surface</u> if $\exists$ finite group G acting faithfully on two smooth curves C_1, C_2 and diagonally on their product, so that S is isomorphic to the <u>minimal resolution</u> of

$$X = C_1 \times C_2 / G$$

X is called a <u>singular product-quotient</u> surface

Recently, Bauer, Catanese, Pignatelli constructed many interesting examples of surfaces of general type with $p_g = 0$ by using product-quotient surfaces.

- Surfaces with $p_g = q = 1$ are constructed by Polizzi & others.

- In this talk, I discuss on ($\mathbb{Q}$-Gorenstein) deformations on some special singular product-quotient surfaces.

We focus on the case

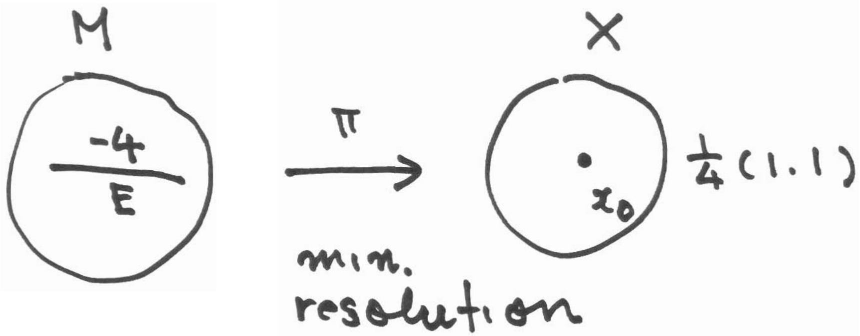
$$g(C_1) = g(C_2) = 3, \qquad G = \mathbb{Z}_4$$

$$C_1 \xrightarrow{G} \mathbb{P}^1 \quad \text{simple cyclic } \mathbb{Z}_4\text{-cover}$$

$X = C_1 \times C_2 / G$ contains precisely 16 cyclic quotient singularities, $\frac{1}{4}(1,1)$ or $\frac{1}{4}(1,3)$

$\rightsquigarrow$ A_3-singularity

$\mathbb{Q}$-Gorenstein smoothing of $\frac{1}{4}(1,1)$-singularity



$$K_M = \pi^* K_X - \frac{1}{2} E$$

K_X is not Cartier

but $2K_X$ is Cartier

$2K_X \simeq \mathcal{O}_X$ locally

(local index one cover)

$\mathrm{Spec}\ \mathbb{C}[[x,y,z]]/(xy-z^2) \xrightarrow[\text{étale except } x_0]{2:1} \hat{\mathcal{O}}_{X,x_0}$

étale except x_0

$\mathrm{Spec}\ \mathbb{C}[[x,y,z,u]]/(xy-z^2+u)$ gives a smoothing

$\mathbb{Z}_2 = \langle \sigma \rangle$ acts

$\sigma(x,y,z,w) = (\alpha x, \alpha^{-1} y, \alpha z, u)$ $\alpha = -1$

And it gives a smoothing of $\hat{\mathcal{O}}_{X,x_0}$

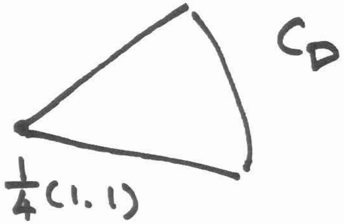
" $\mathbb{Q}$ - Gorenstein smoothing "

$$V = \nu_2(\mathbb{P}^2) \qquad \subset \quad \mathbb{P}^5$$

$$F = \mathbb{P}^1(\mathcal{O}(2) + \mathcal{O}(2\,1))$$

making cones $\quad C_V, \quad C_F \subset \mathbb{P}^6$

hyperplane section is a cone C_D of rational curve of deg 4



C_V, C_F give smoothings of C_D

$$\begin{tikzcd} V \arrow[r, squiggly, "C_V"] & C_D & F \arrow[l, squiggly, "C_F"'] \end{tikzcd}$$

$K_V^2 = K_{C_D}^2$, but $K_F^2 \neq K_{C_D}^2$

special cyclic singularities

$\frac{1}{r^2 d}(1, dra-1)$ RDPs $\rightsquigarrow$ $\mathbb{Q}$-Gorenstein

$\gcd(r, a) = 1$ smoothing

These singularities were observed originally by J. Wahl.

In [Kollár – Shepherd-Barron], these singularities were studied in the purpose of construction of compact moduli space of surfaces of general type.

"singularity of class T"

Manetti, Hacking $\rightsquigarrow$ Use them to construct compact moduli space of pairs of $(\mathbb{P}^2, C)$

$\Downarrow$

Use them to construct examples of surfaces to show Def equiv $\not\sim$ Diff equiv.

[—, Park. Invent. math. 2007]

Develop the method of construction of surfaces of general type$/\mathbb{C}$ with $p_g = 0$, $\pi_1 = 1$ by using $\mathbb{Q}$-Gorenstein smoothings of singular rational surfaces.

[—, Nakayama math. AG. arXiv 1103.5185]

Develop the method in [—, Park] purely algebraic way.
(characteristic free)

$\Rightarrow$ Possible to construct algebraically simply connected surfaces of general type with $p_g = q = 0$, $1 \leq K^2 \leq 4$ in any characteristic.

Let $Z = C_1 \times C_2$

$G = \mathbb{Z}_4 = \langle \xi \rangle$ $\qquad$ $H = \langle \xi^2 \rangle = \mathbb{Z}_2$

$$
\begin{tikzcd}
Y \arrow[rr, "H"] \arrow[dd, "G/H"'] & & E_1 \times E_2 \arrow[dd, "G/H \times G/H"] \\
& Z \arrow[ul, "H"'] \arrow[ur, "H \times H"] \arrow[dl, "G"'] \arrow[dr, "G \times G"] & \\
X \arrow[rr, "G"] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}
$$

E_i elliptic curve

$$
\begin{tikzcd}
S \arrow[d, "\text{min. resol.}"'] \\
X
\end{tikzcd}
$$

$K_S^2 = 8 - b$

$e(S) = 4 + \frac{15a}{4} + \frac{7b}{4}$

$q(S) = 0$

$p_g(S) = \frac{15a}{48} + \frac{3b}{48}$

if $\quad a \times \frac{1}{4}(1,3)$

$\qquad b \times \frac{1}{4}(1,1)$

A) $16 \times \frac{1}{4}(1.\ 3)$

B) $16 \times \frac{1}{4}(1.\ 1)$

c) $8 \times \frac{1}{4}(1.\ 3)$, $8 \times \frac{1}{4}(1.\ 1)$

$$X \xrightarrow{G\text{-cover}} \underset{\substack{\shortparallel \\ Q}}{\mathbb{P}^1 \times \mathbb{P}^1}$$

$G = \mathbb{Z}_4$ - cover is determined by the following data.

$$4L_{x_1} = D_1 + 3D_3$$

$$2L_{x_2} = D_1 + D_3$$

$$4L_{x_3} = 3D_1 + D_3$$

χ_0. χ_1 χ_2. $\chi_3 \in \hat{G}$ D_1. D_3 branch divisors

trivial character

characters

A) $L_{x_1} \in |\mathcal{O}_Q(1,1)|$, $L_{x_2} \in |\mathcal{O}_Q(2,2)|$, $L_{x_3} \in |\mathcal{O}_Q(3,3)|$

$D_1 \in |\mathcal{O}_Q(4,4)|$ $D_3 = \mathcal{O}_Q$

'simple $\mathbb{Z}_4$-cover'

B) $L_{x_1} \in |\mathcal{O}_Q(1,3)|$, $L_{x_2} \in |\mathcal{O}_Q(2,2)|$, $L_{x_3} \in |\mathcal{O}_Q(3,1)|$

$D_1 \in |\mathcal{O}_Q(4,0)|$, $D_3 \in |\mathcal{O}_Q(0,4)|$

C) $L_{x_1}, L_{x_2}, L_{x_3}, D_1, D_3 \in |\mathcal{O}_Q(2,2)|$

Understand

$$0 \to H^1(\Theta_X) \to \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \underset{\substack{\| \\ H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X))}}{\mathcal{T}^1_X} \xrightarrow{ob} H^2(\Theta_X)$$

$$ob^* : H^2(\Theta_X)^* \to (\tau^1_X)^* \qquad p : Z \to X$$

$$H^2(\Theta_X)^* = H^0(\Omega^1_Z \otimes \Omega^2_Z)^G \qquad (\tau^1_X)^* = (p_* \Omega^1_Z)^G / \Omega^1_X$$

$$H^0(\Omega^1_Z \otimes \Omega^2_Z)^G = \bigoplus_{\chi \in \hat{G}} \left[\left(H^0(g_{1*} \omega_{C_1})^\chi \otimes H^0(g_{2*} \omega^2_{C_2})^\chi \right) \oplus \left(H^0(g_{1*} \omega^2_{C_1})^\chi \otimes H^0(g_{2*} \omega_{C_1})^\chi \right) \right]$$

case A

case B χ^{-1}

A) $\left[H^0(\omega_{1} \otimes M_1^2) \otimes H^0(\omega^2_{\mathbb{P}^1}(B_2) \otimes M_2^2 \right. \oplus \left[H^0(\omega^2_{\mathbb{P}^1}(B_1) \otimes M_1^2) \otimes H^0(\omega_{\mathbb{P}^1} \otimes M_2^2) \right]$

$M_i = \mathcal{O}_{\mathbb{P}^1}(1) \qquad B_i = \mathcal{O}_{\mathbb{P}^1}(4)$

$h^2(\Theta_X) = 6$

B) $h^2(\Theta_X) = 14$

C) $h^2(\Theta_X) \neq 0$ by using different method

A) By an explicit local computation ob^* is injective in A)

$\Rightarrow$ ob is surjective

'RDPs'

$(p_* \Omega^1_Z)^G / \Omega^1_X$ loc generated by $x^i y^{i+1} dx - y^i x^{i+1} dy$ for $i = 0, 1, 2$.

[Catanese]

$$0 \to \underset{\overset{\|}{\mathbb{C}^2}}{H^1(\Theta_X)} \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \mathbb{C}^{42} \to 0$$

$$H^1(\Theta_S) \cong H^1(\Theta_X) \oplus \mathbb{C}^{48}$$
$$H^2(\Theta_S) \cong H^2(\Theta_X)$$
$$h^1(\Theta_S) - h^2(\Theta_S) = 10\,\chi(\mathcal{O}_S) - 2K_S^2 = 44$$
$$\therefore\ h^1(\Theta_S) = 50$$

$\therefore \dim_{[X]} \mathrm{ESDef}(X) = 2$ By varying $B_i \subset \mathbb{P}^1$ one obtains 2-dim'l deformation.

Using $P_g(S) = 5$, & $X \to Q$ is simple $\mathbb{Z}_4$-cover

general deformation of X is c.i. of type $(2, 4)$ in $\mathbb{P}^4$

$\therefore$ $\mathrm{Def}(X)$ is smooth at $[X]$ of dim 44.

$\mathrm{Def}(S)$ is singular at $[S]$

($16 \times A_3$-cycles of S does not have independent in deform)

<u>natural deformations</u> [Pardini]

of G-cover $X \to Q$ is the subvariety X' in $W = \bigoplus_{\hat{G}\setminus\{x_0\}} V(L_x^{-1})$

locally defined by

$$W \to Q$$

$$w_1^2 = \tau_3 w_2$$
$$w_1 w_2 = \tau_3 w_3$$
$$w_1 w_3 = \tau_1 \tau_3 w_0$$
$$w_2^2 = \tau_1 \tau_3 w_0$$
$$w_2 w_3 = \tau_1 w_1$$
$$w_3^2 = \tau_1 w_2$$

w_i local coord on $V(L_{x_i}^{-1})$

τ_i local equation D_i

natural deformation parametrized by

$$\bigoplus_{x \in \{x_0, x_1, x_2\}} H^0(\mathcal{O}_Q(D_1) \otimes L_x^{-1}) \qquad \bigoplus_{x \in \{x_0, x_2, x_3\}} H^0(\mathcal{O}_Q(D_3) \otimes L_x^{-1})$$

A) $H^0(\mathcal{O}_Q(4.4)) \oplus H^0(\mathcal{O}_Q(3.3)) \oplus H^0(\mathcal{O}_Q(2.2)) \cong \mathbb{C}^{50}$

$$z^4 - \alpha_4 = 0$$

$$z^4 - a_2 z^2 - a_3 z^1 - a_4 - \alpha_4 = 0$$

natural deformation gives a smoothing

B) $H^0(\mathcal{O}(4.0)) \oplus H^0(\mathcal{O}(0.4))$

all natural deformation preserve $16 \times \frac{1}{4}(1.1)$

C) $H^0(\mathcal{O}(2\ 2)) \oplus H^0(\mathcal{O}(2.2)) \oplus H^0(\mathcal{O}) \oplus H^0(\mathcal{O}) \oplus H^0(\mathcal{O}) \oplus H^0(\mathcal{O}$

natural deformations give a smoothing $8 \times \frac{1}{4}(1.3)$

but preserves $8 \times \frac{1}{4}(1.1)$

B). c)

$\omega_0 = 1$

$$\operatorname{rank} \begin{pmatrix} \tau_3 & \omega_1 & \omega_2 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_3 & \omega_2 & \tau_1 \end{pmatrix} \leq 1$$

$$\begin{pmatrix} \tau_3 & \omega_1 & \omega_2 \\ \omega_1 & \omega_2 + s & \omega_3 \\ \omega_3 & \omega_2 & \tau_1 \end{pmatrix}$$ gives a $\mathbb{Q}$-Gorenstein smoothing

$s \in H^0(L_{x_2})$

More explicit way (geometric & global)

B)

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & E_1 \times E_2 = A \arrow[d] \\ X \arrow[r] & \mathbb{P}^1 \times \mathbb{P}^1 = Q \end{tikzcd}$$

- Def (Y) is smooth at $[Y]$ of dim 18
- general deformation Y_t of Y is a double cover $Y_t \to A_t$ branched over Ξ (4.4)-type

$$Y_t \longrightarrow A_t^{\circlearrowleft i_t} \longrightarrow \mathrm{Kum}(A_t)$$

lifting i_t on Y_t if branching locus is invariant.

the divisors in $|\Xi|$ which are invariant under i_t form a family of dim $\frac{1}{2}(h^0(\mathcal{O}_A(\Xi)) + 2 - 1 = 9$

$\Rightarrow$ 12-dim'l $\{Y_t\}$ of deform of Y which admit a lifting of i_t

$Y_t \circlearrowleft i_t$

$\downarrow$ branched over 16 nodes & Ξ/i_t

$X_t = Y_t/i_t \longrightarrow \mathrm{Kum}(A_t)$

$p_g(X_t) = 1, \quad q(X_t) = 0 \quad K^2 = 8$

X_t is a $\mathbb{Q}$Gorenstein smoothing of X.

Remark

Fix abelian surface A and the embedding $Kum(A) \hookrightarrow \mathbb{P}^3$

Choice of the deformation parameter $s \in H^0(L_{x_2})$ corresponds to the choice of $\Phi \in |\mathcal{O}_{\mathbb{P}^3}(2)|$

By [Todorov], there is a $\Phi_k \in |\mathcal{O}_{\mathbb{P}^3}(2)|$ which contains exactly k $(0 \leq k \leq 6)$ of the nodes of $Kum(A)$

Pull back in A of $D_k := Kum(A) \cap \Phi_k$ is a polarization of type (4.4) which contains exactly k of the nodes.

Partial $\mathbb{Q}$-Gorenstein smoothing X whose general fiber X_t is isom to the double cover $Kum(A)$ branched over D_k and $16-k$ nodes of $Kum(A)$.

Its minimal resolution of sings of X_t is a Todorov surface with $p_g = 1$, $q = 0$. $K^2 = 8 - k$ $(0 \leq k \leq 6)$.

c) natural deformations of X

$w_0 = 1$

$$w_1^2 = (g_3 + d_2 w_2 + d_3 w_3)\, w_2$$
$$w_1 w_2 = (g_3 + d_2 w_2 + d_3 w_3)\, w_3$$
$$w_1 w_3 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3)$$
$$w_2^2 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3)$$
$$w_2 w_3 = (g_1 + c_1 w_1 + c_2 w_2)\, w_1$$
$$w_3^2 = (g_1 + c_1 w_1 + c_2 w_2)\, w_2 \qquad c_i, d_i \in H^0(\mathcal{O}) = \mathbb{C}$$

For a general choice of the parameters $\bar{X} \to Q$ is not a Galois cover.

Natural deformations $\bar{X}$ of X which keep the G-action is parameterized by $H^0(\mathcal{O}_Q(2,2)) \oplus H^0(\mathcal{O}_Q(2,2))$

i.e $c_1 = c_2 = d_2 = d_3 = 0$

$\overline{X} \to Q$ factors into $\overline{X} \xrightarrow[2:1]{} K \xrightarrow[2:1]{p} Q$

K — K3 surface with $8 \times A_1$-sings

$p : K \to Q$ branched over $D_{G,x_1} + D_{G,x_3}$

Let $\overline{D}_{G.x_2} = p^* D_{G.x_2}$

$2\overline{D}_{G.x_i} = p^* D_{G.x_i}$ for $i = 1.3$

$D_{G.x_2} \equiv D_{G.x_i} \Rightarrow \overline{D}_{G.x_2} \equiv \overline{D}_{G.x_1} + \overline{D}_{G.x_3}$

- Double cover $\tilde{X}$ of K branched over $\overline{D}_{G.x_2}$ is deformation equiv. to $\overline{X}$.
- $\tilde{X}$ can be realized as the bidouble cover of Q branched over $D_{G.x_1}$, D_{G,x_3} and $D_{G.x_2}$

$$\begin{tikzcd} X & \bar{X} \arrow[l, squiggly] \arrow[d, squiggly] \\ & \tilde{X} \\ \tilde{X}_t \arrow[ur, squiggly] & \end{tikzcd}$$

X: $8 \times \frac{1}{4}(1.3)$, $8 \times \frac{1}{4}(1.1)$

$\bar{X}$: $8 \times \frac{1}{4}(1.1)$ $\quad (\bar{X} \xrightarrow{\mathbb{Z}_4} Q)$

$\tilde{X}$: $8 \times \frac{1}{4}(1.1)$ $\quad (\tilde{X} \xrightarrow{\mathbb{Z}_2^{\oplus 2}} Q)$

$\tilde{X}_t$: $\mathbb{Q}$-Gorenstein smoothing by deforming $D_{G.\mathbb{Z}_2}$ to a general divisor of bidegree (2.2)

Remark

By using partial $\mathbb{Q}$-Gorenstein smoothing $\tilde{X}$ and minimal resolution, one can construct surfaces of general type with $P_g = 3$, $q = 0$, $K^2 = \alpha$ $(2 \leq \alpha \leq 8)$