Towards a higher dimensional generalization of the Bennequin theory

(Atsuhide Mori, Osaka Univ.)

Setup. $S^{2n-1} \subset \mathbb{R}^{2n} (\approx \mathbb{C}^n)$: the unit hypersphere

$$\lambda := \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i), \qquad z_i = x_i + y_i\sqrt{-1}$$

then $\quad d\lambda$: the standard symplectic form on $\mathbb{R}^{2n}$

$\lambda|S^{2n-1}$: the standard contact form on S^{2n-1}

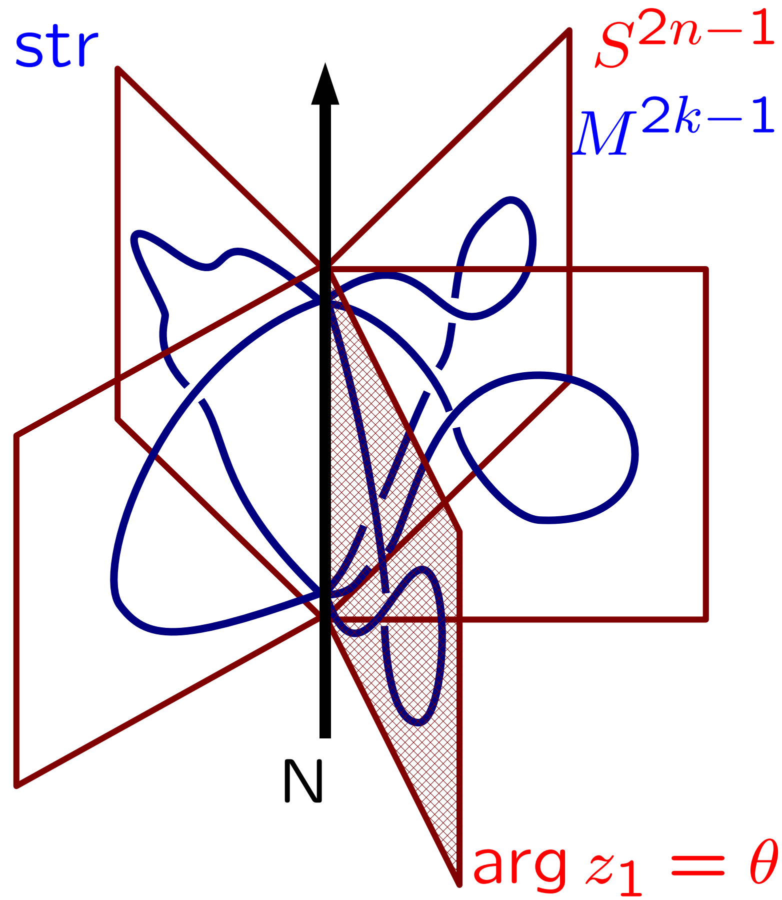
Take $M^{2k-1} \subset S^{2n-1}$: compact submanifold w/o ∂.

M is called a **contact spinning** if $\ker(\lambda|M)$ is contact and $\arg z_1|M$ defines a supporting† open-book.

†open-book supporting cont str
⇔ $\exists X$: cont vecter field s.t.

1) X is positively $\pitchfork$ to the cont str, and

2) $\forall$pages of the open-book is a Birkhoff section of X. (positive, ∂ is also positive)



For cont spinning, we may assume $\exists X$:close to the rotation around $N = \{z_1 = 0\}$.

Remark. The binding is then $M^{2k-1} \cap N$ ($\emptyset$ for $k = 1$). It is a cont submfd tangent to $X (\neq 0)$.

Well-known examples

i) For a given $\pitchfork$-link (1-dim cont submfd) $L \subset S^3$,

Bennequin's lemma. L is cont-topic to $\exists$closed braid.

Remarks. 1) $\forall$closed braid placed near the great circle $\{|z_1| = 1\} \cap S^3$ is an embedded 1-dim cont spinning.

2) "Bennequin's lemma" (Mitsumatsu & M). $\forall$link $\pitchfork$ to the cont str supported by a given open-book on a closed 3-mfd is cont-topic to a link $\pitchfork$ to each page of the open-book, i.e., cont-topic to a braid position.

ii) $\{z_{k+1} = \cdots = z_n = 0\} \cap S^{2n-1}$: standard spinning

iii) $L = \{f_j(\varepsilon z_1, \ldots, \varepsilon z_n) = 0 : \text{holomorphic}\} \cap S^{2n-1}$ is a cont spinning (under a "moderate" assumption).

Realizability

Theorem (M'03, generalized by Martínez Torres'11). $\forall$closed cont $(2k-1)$-mfd can be immersed in S^{4k-3} and embedded in S^{4k-1} as **cont spinnings**.

This improves Giroux theorem ($\exists$ of supporting open-book). It is also an application of approximately holomorphic geometry (Donaldson-Auroux). It recovers Gromov's result without appealing to h-principle.

A smooth spinning, which lacks contactness, is still improtant in high codimensional smooth knot theory. It is a special case of Litherland's 'deformation spin'. And I prefer Tamura's 'spinnable str' to 'open-book'.

— If you are intersted in this object, see also the papers of TAKASE

More examples.

Lemma. Let $M \subset S^{2n-1} (\subset \mathbb{C}^n)$ be a submfd s.t.
i) $M \cap \{\arg z_1 = \theta\}$ are symplectic pages, and
ii) it is nearly conic near the binding, i.e., roughly

$$\{(re^{\sqrt{-1}\theta}, \sqrt{1-r^2}p) \,|\, 0 \le r < \varepsilon,\, \theta \in S^1,\, (0,p) \in M\}.$$

Then we can deform M to a spinning cont submfd.

Eg. hyperelliptic cont spinning from braiding curve

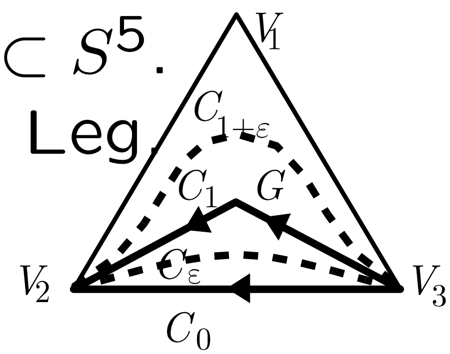
$$y^2 = (x - b_1(\theta)) \cdot \cdots \cdot (x - b_m(\theta))$$

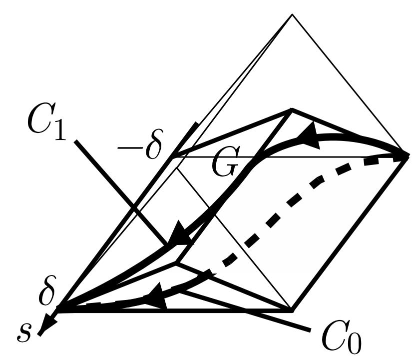
$(\{b_i(\theta) | i = 1, \ldots, m\}_{\theta \in S^1})$ on $\mathbb{C}^2 \times S^1 \approx \mathrm{int} B^4 \times S^1$.

More examples.

Theorem. Consider $\Delta = \{(r_1^2, r_2^2, r_3^2) | r_i^2 + r_2^2 + r_3^2 = 1\}$ of $S^5 = \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3$. Take a curve $C : [-\delta, \delta] \to \Delta$ which is parametrized by $\theta_1 + \theta_2 + \theta_3 \in [-\delta, \delta]$. This defines T^2-fibration $M(\subset S^5) \to [-\delta, \delta]$ possibly **degenerated**. Then (the sign of) $\lambda \wedge d\lambda | M$ coincides with (that of) the negative areal velocity of C with respect to the barycenter G of Δ.

C_ε: cont spinning $S^3 \subset S^5$.
C_1: Reeb foliation by Leg.
$C_{1+\varepsilon}$: negative cont.
(V_i: $\{|z_i| = 1\} \approx S^1$)





Dehn twist in higher dimension

Definition. A Dehn-Seidel twist is the following symplectomorphism τ of T^*S^m supported near the zero section S^m (well-defined up to symplectic isotopy). First we fix the restriction $\tau|S^m$ as the antipodal map. Then extend it using geodesic flow so that it quiets down to the identity map away from S^m.

Each page P of open-book supporting a contact str on M^{2k-1} can be considered as a symplectic manifold by taking $d\alpha|P$ of the cont form α with $\alpha(X) = 1$.

Conjecture(Giroux). The symplectic monodromy is a product of Dehn-Seidel twists and their inverses.

Hopf plumbing

Fix a properly embedded Lagrangian ball $B^{k-1} \subset P$. We consider the disjoint union $B^{k-1} \sqcup$ (a copy of S^{k-1}), and identify $B^{k-1} \subset P$ with the hemisphere of S^{k-1}. We slightly extend the quotient to a new symplectic page on which S^{k-1} becomes a Lagrangian sphere. We extend the symplectic monodromy φ identically, and then compose it with the Dehn-Seidel twist τ (resp τ^{-1}) along S^{k-1}. The new symplectic open-book is called a positive (resp. negative) Hopf plumbing. It supports a new cont str on the same mfd.

Conjecture(Giroux). $\forall$positive Hopf plumbing does not change the contact structure up to cont-topy.

3-dim case

($\pm$)-Hopf plumbing on a closed braid is equivalent to the Murasugi sum of ($\pm$)-Hopf band along (a square nbhd of) an proper arc B^1 which connects two points on a pege $\{\arg z_1 = \text{const}\}(\approx D^2)$.
Torisu'00 and Giroux proved that a positive Hopf plumbing does not change the contact str.

Theorem (Giroux'03). Two supporting open-books of a cont str on a closed 3-mfd can be related by a sequence of positive Hopf plumbings/deplumbings.

Indeed we can obtain a common open-book from them by a suitable sequence of positive Hopf plumbings.

Overtwistedness of 3-dim contact structure

We can modify a given cont str near a (certain) codim-2 cont submfd. Lutz define such modification and I generalized it to higher dimension. Bennequin proved that the standard cont str of S^3 is not equivalent to the modified one. Indeed he found a property, called **tightness**, of the standard cont str which is spoiled by Lutz-modification. Lutz-modified cont str is said to be **overtwisted**. Eliashberg proved that 3-dim (Bennequin) tightness is equivalent to unovertwistedness. He also proved that $\forall$homotopy class of plane fields on a 3-mfd contains a unique overtwisted cont str.

Hopf plumbing and overtwistedness

We see(?) that a negative Hopf plumbing produces an overtwisted contact structure on S^3. Conversely

Theorem (Giroux'03). ∀overtwisted cont str can be supported by ∃negatively Hopf plumbed open-book. This is the definition of overtwistedness in higher dim. So the theorem is Lutz OT=Giroux OT if dim$=3$.

The proof is based on Eliashberg's classification of 3-dim (Lutz) overtwisted cont strs.

Remark. As an application, Giroux deduced Harer's conjecture from Eliashberg's classification of cont strs on S^3 including the uniqueness of tight cont str on S^3.

The work of Loi-Piergallini

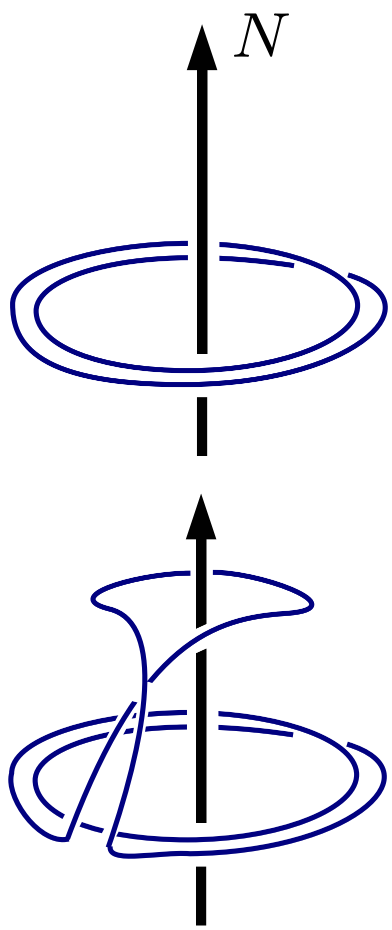
(Montesinos-Morton'91) A simple branched covering $\pi : M^3 \to S^3$ defines an open-book O on M by $\pi^* \arg z_1$ if the ramification locus B forms a closed braid. Then for $\forall$stabilization B' of B, $\exists$simple branched covering $\pi' : M^3 \to S^3$ which defines Hopf plumbing O' of O.

We say that π is **simple** if $\forall$critical level contains a single multiple point and further it is a double point.

Interpreting the Eliashberg theorem on the topology of Stein manifolds, Loi and Piergallini showed that

$\forall D$: Stein fillable contact str, $\exists B$: quasipositive and $\exists \pi$: simple 3-fold s.t. O: positive and supporting D.

Markov stabilization for closed braid



(±)-stabilization. Pull out a small segment from a closed braid and hang it on the axis N so that an additional (±)-interchange appears.

Here an 'interchange' in closed braid is a mapping class of m-punctured disk conjugate to the standard one.

Quasipositivity. A quasipositive closed braid is one presenting a composition of (+)-interchanges.

Orevkov theory

Theorem(Orevkov'00). A braid is quasipositive if (and only if) its positive stabilization is quasipositive.
Is $\forall$open-book supporting a Stein fillable str pisitive?

Theorem(Orevkov'00, due to Buckel'97 and Laver'96). A negative stabilization is never quasipositive.
Want to associate it with Giroux overtwistedness.

Theorem(Orevkov-Shevchishin'03, Nancy Winkle'02). Two braid presentations of the same $\pitchfork$-link can be related by a sequence of positive (de)stabilizations.

Want to associate it with the Giroux theorem.

High dim stabilization (Main dish I)

We can stabilize $\Sigma = \{z_{k+1} = \cdots = z_n = 0\} \cap S^{2n-1}$ in $S^{2k+1} = \{z_{k+2} = \cdots = z_n = 0\} \cap S^{2n-1}$ to

$\Sigma^+ = \{\varepsilon z_1 = z_2^2 + \cdots + z_{k+1}^2,\ z_{k+2} = \cdots = 0\} \cap S^{2n-1}$,

$\Sigma^- = \{\varepsilon \overline{z_1} = z_2^2 + \cdots + z_{k+1}^2,\ z_{k+2} = \cdots = 0\} \cap S^{2n-1}$.

(+)-stabilization can be interpolated by a cont-topy.
(−)-stabilization can be interpolated by a diffeotopy, but (−)-one makes the cont str (Giroux) overtwisted.
Indeed (±)-stabilization involves (±)-Hopf plumbing.

Problem. Generalize Orevkov theory to 5-dim so that it implies results on positive open-books (Akbulut, Ozbagci,...) and positive Hopf plumbings.

What is Bennequin theory ?(side dish)

The Bennequin thm = his inequality for standard S^3

Eliashberg, using J-curves, proved & generalized it.

however, as is reconsidered by Birman-Menasco,

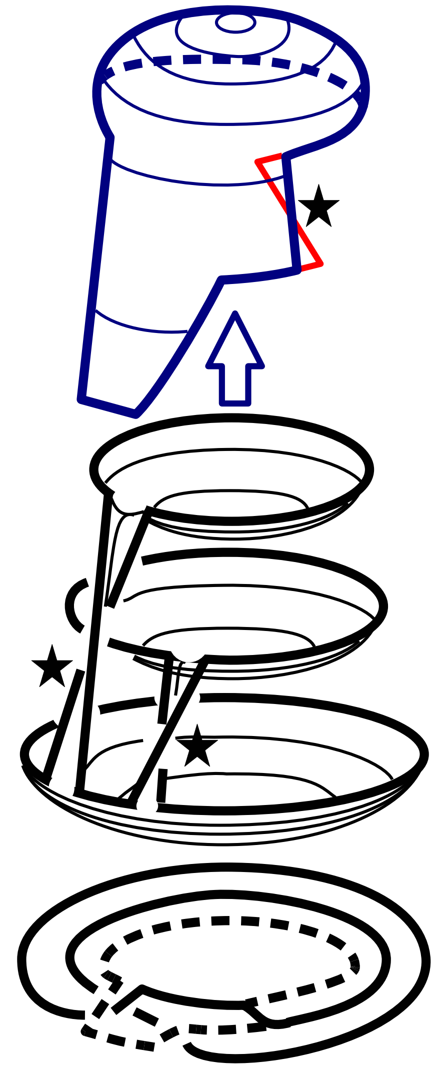
Bennequin theory = his proof of the Markov thm
and its application to cont geom
(Entrelacements et équations de Pfaff)

Bennequin considered a Seifert surface of a given braid which realizes the maximal χ, tangents to the page $\{\arg z_1 = \text{const}\}$ only at saddles, and transversely (and essentially) intersects with the binding $\{z_1 = 0\}$.

Then he observed ....

Eliminating stabilization

The figure (not ideal but real) presents a positive stabilization of a closed negative 2-braid which produces "the simplest pseudo Anosov closed braid".
Then we can obtain the standard (1-)braid by an ovbious pair of positive & negative destabilizations.
Surprisingly, we can eliminate saddle tangencies to $\arg z_1 = \text{const}$ (marked by $\star$) in each step.
Example of Bennequin's "poche"



Convex Seifert hypersurface(Main dish II)

1. Closed convex hypersurface (Giroux '91)

Σ: closed hypersurf embedded in a cont $(2k-1)$-mfd
Suppose $\exists X$: cont vect field $\pitchfork$ to Σ and orient it.
Then we say that Σ is **convex**.

Giroux lemma. For $\forall\Sigma$:convex, $\exists$ cont form α of the cont str s.t. we can decompose Σ as $\Sigma = \Sigma_+ \cup (-\Sigma_-)$ accoding to the sign of $(d\alpha)^{k-1}|\Sigma$. Then the dividing set $\Gamma = \{(d\alpha)^{k-1}|\Sigma = 0\} = \partial\Sigma_\pm$ is a cont submfd.

Lemma(M'09). Given two strong exact symplectic fillings Σ_i of a cont $(2k-3)$-mfd, we can construct a $(2k-1)$-dim cont nbhd of convex $\Sigma_1 \cup (-\Sigma_2)$.

2. Convex Seifert surface (M '09)

Bennequin's inequality (I omit the precise, but it is an inequality between relative characteristic numbers) can be naturally generalized in higher dim. However I proved that no cont mfd of dim>3 satisfies it.

Perhaps this is beacuse any surf in cont 3-mfd can be smoothly approximated by a convex one (Giroux). In higher dim, I found a small hypersurf far from convex (as an application of Eliashberg-Floer-McDuff thm.)

Definition. 1) A Seifert hypersurf Σ is said to be convex if $\exists X, \exists \alpha$ s.t. $\partial\Sigma \subset \partial\Sigma_+$:contact-type.
2) Then Bennequin's inequality becomes $\chi(\Sigma_-) \leq 0$.

Tight vs Overtwisted

Theorem.(M '09) 1) (E,G+ε) A cont 3-mfd is tight iff any convex Seifert surf satisfies the inequality.
2) The (local) modification of Lutz & I produces a convex Σ violating Bennequin's ineqality.
3) The modification also produces a "bounded Legendrian open-book" which is a high dim generalization of OT disk (see Massot-Niederkrüger-Wendl'11).

Probrems. 1)(tightness) Prove that S^{2n-1} satisfies Bennequin's inequality at least for convex Seifert hypersurfaces spanning contact spinnings.
2)(overtwistedness) Show that negatively stabilized contact submanifold violates the inequality.

Again on realizability

Conjecture. $\forall$closed contact $(2k-1)$-manifold could be embedded in S^{4k-3} as a contact spinning.

Want to associate to it a family of Legendrian submfd (other than Reeb fol). $S^{4k-3}\backslash$(1-point) is contactomorphic to the 1-jet space $J^1(\mathbb{R}^{2(k-1)}, \mathbb{R})$. Then M^{2k-1} presents a system of $2(k-1)$ first order PDEs for a function with $2(k-1)$ variables. If it may define a codim-1 (possibly singular) foliation by Legendrians. Such a foliation arises as a wall between the spaces of cont submfds and reverses their orientations. We can understand negative stabilization as a “round trip” beyond the wall. ——Thank you for careful reading.