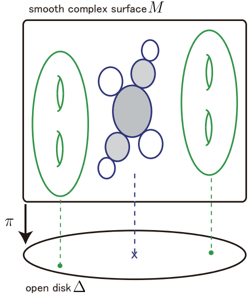
Splitting of singular fibers & topological monodromies

Takayuki OKUDA
(Kyushu University)

Tohoku Gakuin University

Feb. 24, 2015

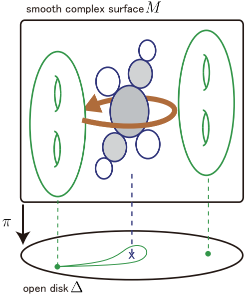
Degeneration
of Rieman surfaces



Topological monodromy

Splitting deformation
for degeneration of Riemann surfaces

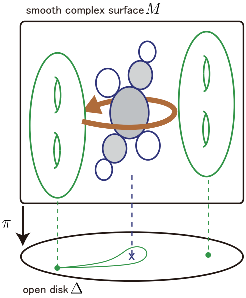
Degeneration
of Rieman surfaces



Topological monodromy

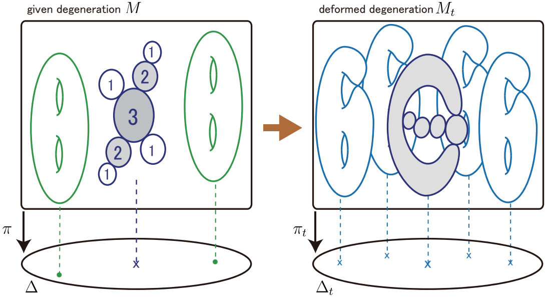
Splitting deformation
for degeneration of Riemann surfaces

Degeneration of Rieman surfaces

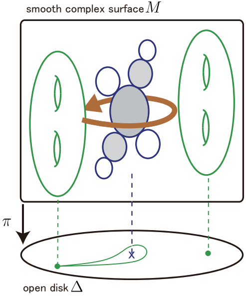


Topological monodromy

Splitting deformation for degeneration of Riemann surfaces

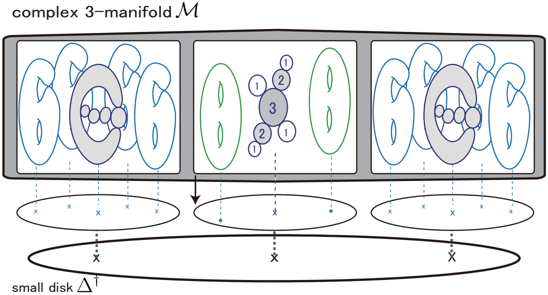


Degeneration
of Rieman surfaces



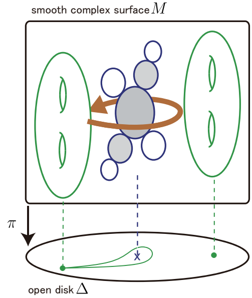
Topological monodromy

Splitting family
for degeneration of Riemann surfaces



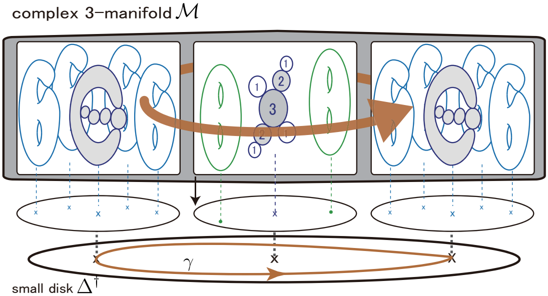
Topological monodromy

Degeneration
of Rieman surfaces



Topological monodromy

Splitting family
for degeneration of Riemann surfaces



Topological monodromy

Degeneration of Riemann surfaces

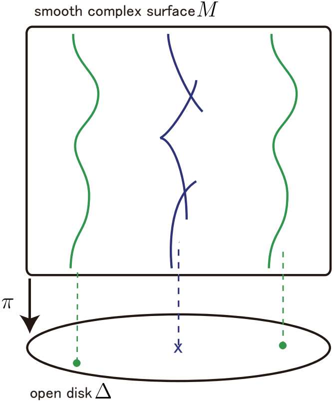
M : a smooth complex surface $\quad \Delta$: an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



- $X_s := \pi^{-1}(s)$ $(s \neq 0)$ **are all smooth fibers.**
- $X_0 := \pi^{-1}(0)$ **is a singular fiber.**

Local model

$\pi(z, w) = zw$ (or, $z^2 + w^2$)

: a **Lefschetz singular point**

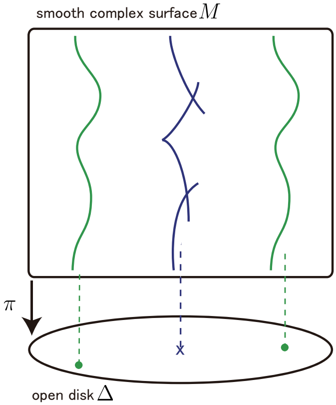
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\iff}$ it has a **unique singular value $0 \in \Delta$**.



- $X_s := \pi^{-1}(s)\ (s \neq 0)$ **are all smooth fibers**.
- $X_0 := \pi^{-1}(0)$ **is a singular fiber**.

Local model

$\pi(z, w) = zw$ (or, $z^2 + w^2$)

: a **Lefschetz singular point**

Degeneration of Riemann surfaces

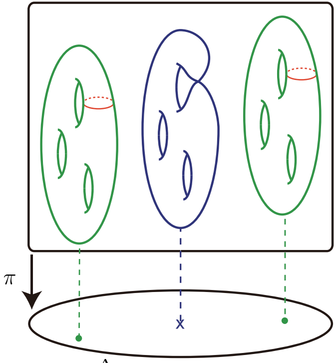
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



- $X_s := \pi^{-1}(s)$ $(s \neq 0)$ **are all smooth fibers**.
- $X_0 := \pi^{-1}(0)$ **is a singular fiber**.

Local model

$$\pi(z, w) = zw \quad \text{(or, } z^2 + w^2\text{)}$$

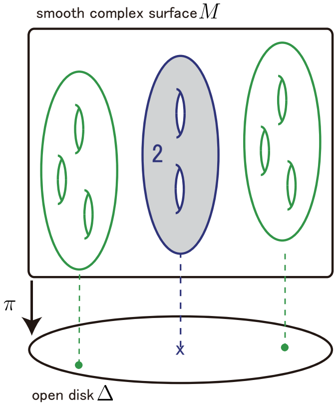
: a **Lefschetz singular point**

Degeneration of Riemann surfaces

M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map
i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**
$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.



- $\boldsymbol{X_s := \pi^{-1}(s)\ (s \neq 0)}$ **are all smooth fibers.**
- $\boldsymbol{X_0 := \pi^{-1}(0)}$ **is a singular fiber.**

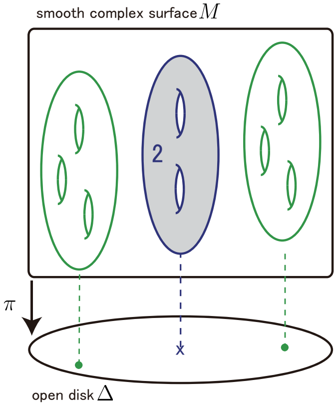
M : a smooth complex surface $\quad \Delta$: an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



Regard X_0 as the divisor defined by π and express as $X_0 = \sum m\Theta$, where Θ is an irreducible component with **multiplicity** m.

Degeneration of Riemann surfaces

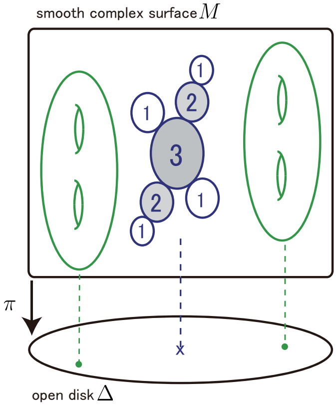
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.



Regard X_0 as the divisor defined by π and express as $X_0 = \sum m\Theta$, where Θ is an irreducible component with **multiplicity m**.

Local model

$$\pi(z, w) = z^2 w^3$$

Degeneration of Riemann surfaces

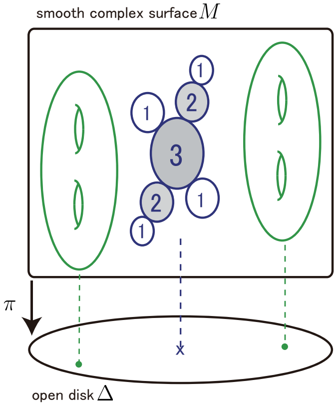
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



Regard $\boldsymbol{X_0}$ as the divisor defined by π and express as $\boldsymbol{X_0 = \sum m\Theta}$, where Θ is an irreducible component with **multiplicity** $\boldsymbol{m}$.

Note:

The self-intersection number of Θ is

$$(\Theta \cdot \Theta) = -\frac{\sum_{\Theta \cap \Theta_j \neq \emptyset} m_j}{m}.$$

Degeneration of Riemann surfaces

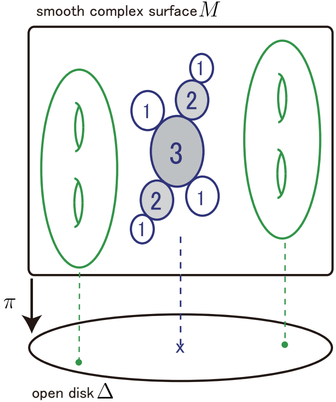
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.



X_0 is **relatively minimal**

$\overset{\text{df}}{\Longleftrightarrow}$ X_0 contains no (-1)**-curves**.

Degeneration of Riemann surfaces

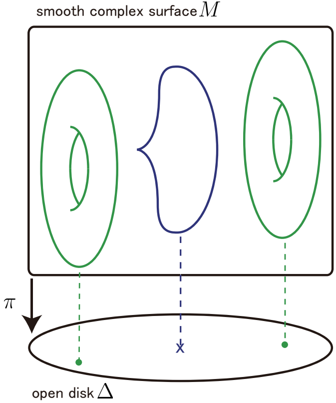
M : a smooth complex surface $\quad \Delta$: an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



X_0 is **relatively minimal**

$\overset{\text{df}}{\Longleftrightarrow}$ X_0 contains no (-1)-**curves**.

Local model

$$\pi(z, w) = z^3 + w^2$$

Degeneration of Riemann surfaces

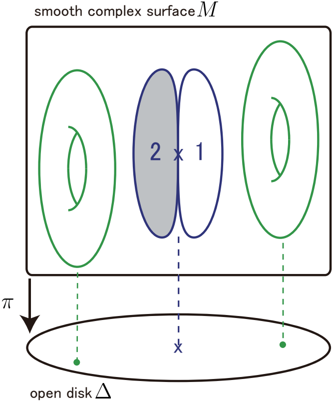
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.



X_0 is **normally minimal** $\overset{\text{df}}{\Longleftrightarrow}$

- X_0 has at most normal crossings.
- An irreducible component of X_0, if it is a (-1)-**curve**, intersects other components at at least 3 points.

Degeneration of Riemann surfaces

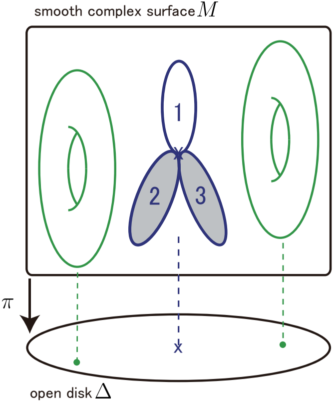
M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value** $0 \in \Delta$.



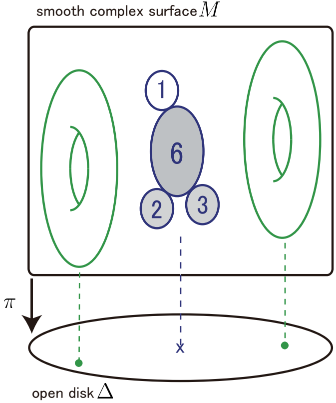
X_0 is **normally minimal** $\overset{\text{df}}{\Longleftrightarrow}$

- X_0 has at most normal crossings.
- An irreducible component of X_0, if it is a (-1)-curve, intersects other components at at least 3 points.

Degeneration of Riemann surfaces

M : a smooth complex surface Δ : an open disk in $\mathbb{C}$
$\pi : M \to \Delta$: a proper surjective holomorphic map
i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**
$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.



X_0 is **normally minimal** $\overset{\text{df}}{\Longleftrightarrow}$

- X_0 has at most normal crossings.
- An irreducible component of X_0, if it is a (-1)-**curve**, intersects other components at at least 3 points.

Degeneration of Riemann surfaces

M : a smooth complex surface Δ : an open disk in $\mathbb{C}$

$\pi : M \to \Delta$: a proper surjective holomorphic map

i.e. a family of (compact) complex curves over Δ

$\pi : M \to \Delta$ is called a **degeneration of Riemann surfaces**

$\overset{\text{df}}{\Longleftrightarrow}$ it has a **unique singular value 0** $\in \Delta$.

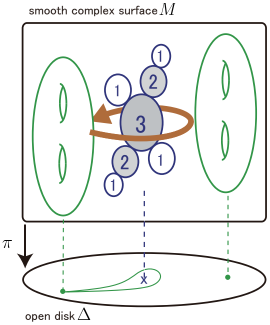
1. Genus $1 \rightsquigarrow$ "8" types of min degenerations
   (Kodaira, 63)
2. Genus $2 \rightsquigarrow$ about "120" types of min degenerations
   (Namikawa-Ueno, 73)
3. Genus $3 \rightsquigarrow$ about "1600" types of min degenerations
   (Ashikaga-Ishizaka, 02,
   via **Matsumoto-Montesinos' theorem**)

Theorem (Matsumoto-Montesinos, 91/92)

{ top. equiv. classes of minimal degenerations of Riemann surfs. of genus g } $\overset{1:1}{\longleftrightarrow}$ { conj. classes in MCG_g of **pseudo-periodic mapp. classes** of negative twist }

via **topological monodromy**, for $g \geq 2$.

Given degeneration



Take a base point $s \in \Delta \setminus \{0\}$
and consider a reference fiber X_s.

$\rightsquigarrow$ a monodromy homeom. $f : X_s \to X_s$

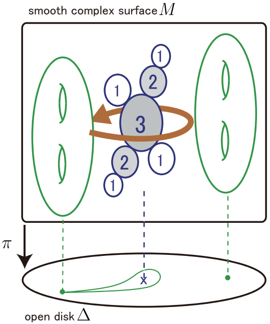
$\rightsquigarrow$ an isotopy class $[f] \in \mathrm{MCG}_g$
(**topological monodromy**)

Theorem (Matsumoto-Montesinos, 91/92)

$$\left\{\begin{array}{l}\text{top. equiv. classes of}\\ \text{minimal degenerations of}\\ \text{Riemann surfs. of genus } g\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{conj. classes in } \mathrm{MCG}_g \text{ of}\\ \textbf{pseudo-periodic mapp. classes}\\ \text{of negative twist}\end{array}\right\}$$

via **topological monodromy**, for $g \geq 2$.

Given degeneration



Take a base point $s \in \Delta \setminus \{0\}$ and consider a reference fiber X_s.

$\rightsquigarrow$ a monodromy homeom. $f : X_s \to X_s$

$\rightsquigarrow$ an isotopy class $[f] \in \mathrm{MCG}_g$ (**topological monodromy**)

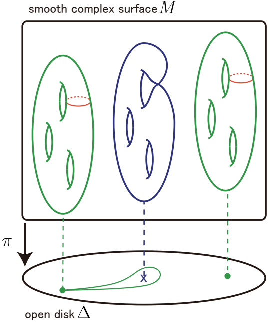
Matsumoto-Montesinos' Theorem

Theorem (Matsumoto-Montesinos, 91/92)

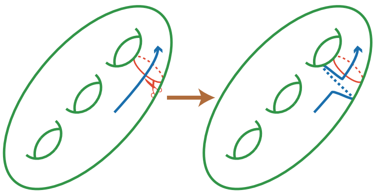
$$\left\{\begin{array}{l}\text{top. equiv. classes of}\\ \text{minimal degenerations of}\\ \text{Riemann surfs. of genus } g\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{conj. classes in } \mathrm{MCG}_g \text{ of}\\ \textbf{pseudo-periodic mapp. classes}\\ \text{of negative twist}\end{array}\right\}$$

via **topological monodromy**, for $g \geq 2$.

Lefschetz fiber



Right-handed Dehn twist



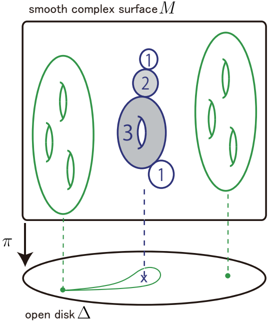
Matsumoto-Montesinos' Theorem

Theorem (Matsumoto-Montesinos, 91/92)

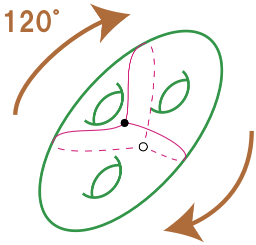
$$\left\{\begin{array}{l}\text{top. equiv. classes of}\\ \text{minimal degenerations of}\\ \text{Riemann surfs. of genus } g\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{conj. classes in } \mathrm{MCG}_g \text{ of}\\ \textbf{pseudo-periodic mapp. classes}\\ \text{of negative twist}\end{array}\right\}$$

via **topological monodromy**, for $g \geq 2$.

Stellar fiber



Periodic mapping class

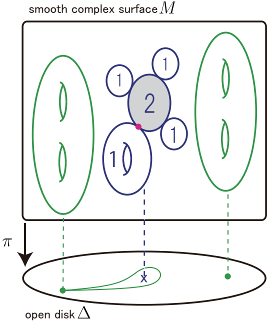


Theorem (Matsumoto-Montesinos, 91/92)

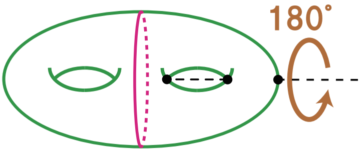
$\left\{\begin{array}{l}\text{top. equiv. classes of}\\ \text{minimal degenerations of}\\ \text{Riemann surfs. of genus } g\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{conj. classes in } \mathrm{MCG}_g \text{ of}\\ \textbf{pseudo-periodic mapp. classes}\\ \text{of negative twist}\end{array}\right\}$

via **topological monodromy**, for $g \geq 2$.

Constellar fiber

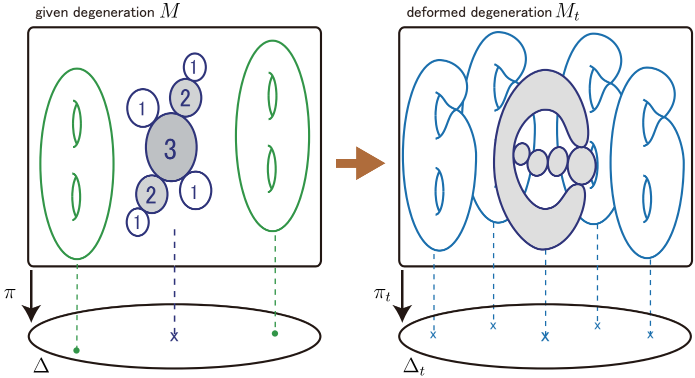


Pseudo-periodic mapping class



Splitting of Singular fibers

$\{\pi_t : M_t \to \Delta_t\}_t$: a "family of families of complex curves"
s.t. $\pi : M \to \Delta$ coincides with $\pi_0 : M_0 \to \Delta_0$



If $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has **k singular values $s_1, s_2, \ldots, s_k$**,
i.e. **k singular fibers $X_{t,s_1}, X_{t,s_2}, \ldots, X_{t,s_k}$** appear,
$\Longrightarrow$ we say **X_0 splits into $X_{t,s_1}, X_{t,s_2}, \ldots, X_{t,s_k}$**.

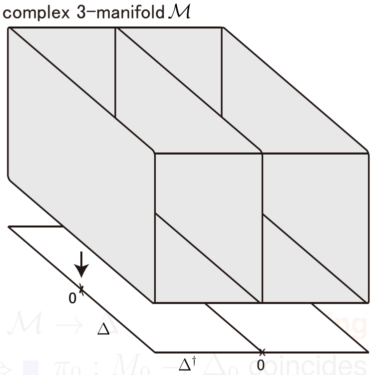
Splitting families for degenerations

$\mathcal{M}$: a complex 3-manifold $\Delta^\dagger$: a sufficiently small open disk

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a proper flat surjective holomorphic map

i.e. a family of (compact) complex curves over $\Delta \times \Delta^\dagger$

Assume $\mathrm{proj}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion.



For each $t \in \Delta^\dagger$, set

$\Delta_t := \Delta \times \{t\}$,

$M_t := \Psi^{-1}(\Delta_t)$

$\left(= (\mathrm{proj}_2 \circ \Psi)^{-1}(t)\right)$,

$\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t$.

$\Psi : \mathcal{M} \to$ … **ing family** for $\pi : M \to \Delta$

$\overset{\text{df}}{\iff}$ ■ $\pi_0 : M_0 \to \Delta_0$ coincides with $\pi : M \to \Delta$, and

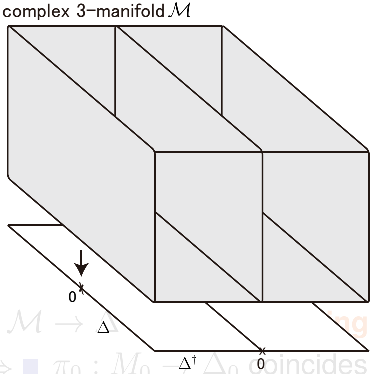
■ $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has **at least two singular fibers**.

$\mathcal{M}$: a complex 3-manifold $\Delta^\dagger$: a sufficiently small open disk

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a proper flat surjective holomorphic map

i.e. a family of (compact) complex curves over $\Delta \times \Delta^\dagger$

Assume $\mathrm{proj}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion.



For each $t \in \Delta^\dagger$, set

$$\Delta_t := \Delta \times \{t\},$$
$$M_t := \Psi^{-1}(\Delta_t)$$
$$(= (\mathrm{proj}_2 \circ \Psi)^{-1}(t)),$$
$$\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t.$$

$\Psi : \mathcal{M} \to \Delta$ ... ng family for $\pi : M \to \Delta$

$\overset{\mathrm{df}}{\iff}$ ■ $\pi_0 : M_0 \to \Delta_0$ coincides with $\pi : M \to \Delta$, and

■ $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has at least two singular fibers.

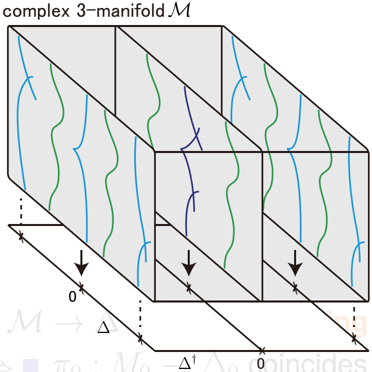
Splitting families for degenerations

$\mathcal{M}$: a complex 3-manifold $\quad \Delta^\dagger$: a sufficiently small open disk

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a proper flat surjective holomorphic map

i.e. a family of (compact) complex curves over $\Delta \times \Delta^\dagger$

Assume $\mathrm{proj}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion.



For each $t \in \Delta^\dagger$, set

$$\Delta_t := \Delta \times \{t\},$$
$$M_t := \Psi^{-1}(\Delta_t)$$
$$(= (\mathrm{proj}_2 \circ \Psi)^{-1}(t)),$$
$$\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t.$$

$\Psi : \mathcal{M} \to$ ... ing family for $\pi : M \to \Delta$

$\overset{\mathrm{df}}{\iff}$ ■ $\pi_0 : M_0 \to \Delta_0$ coincides with $\pi : M \to \Delta$, and

■ $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has **at least two singular fibers**.

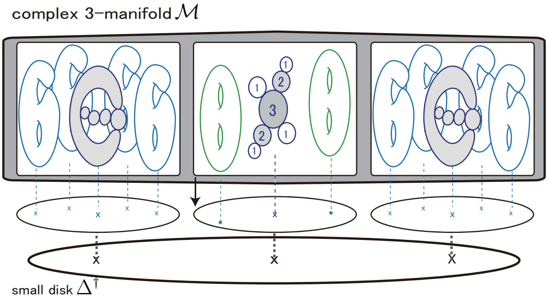
Splitting families for degenerations

$\mathcal{M}$: a complex 3-manifold $\Delta^\dagger$: a sufficiently small open disk

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a proper flat surjective holomorphic map

i.e. a family of (compact) complex curves over $\Delta \times \Delta^\dagger$

Assume $\mathrm{proj}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion.



For each $t \in \Delta^\dagger$, set

$$\Delta_t := \Delta \times \{t\},$$
$$M_t := \Psi^{-1}(\Delta_t)$$
$$(= (\mathrm{proj}_2 \circ \Psi)^{-1}(t)),$$
$$\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t.$$

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a **splitting family** for $\pi : M \to \Delta$

$\overset{\text{df}}{\Longleftrightarrow}$
- $\pi_0 : M_0 \to \Delta_0$ coincides with $\pi : M \to \Delta$, and
- $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has **at least two singular fibers**.

A singular fiber (or precisely, its degeneration) is **atomic**

$\overset{\mathrm{df}}{\Longleftrightarrow}$ it does NOT admit any splitting families.

Fact

Lefschetz fibers and **multiple smooth curves** are atomic.

How to construct splitting families

- **Double covering method**
  1. for degenerations of genus 1 (Moishezon)
  2. for degenerations of genus 2 (Horikawa)
  3. for hyperelliptic degenerations (Arakawa-Ashikaga)
- **Barking deformation**
  for linear degenerations
  whose singular fiber has a simple crust (Takamura)

A singular fiber (or precisely, its degeneration) is **atomic**

$\overset{\text{df}}{\Longleftrightarrow}$ it does NOT admit any splitting families.

Fact

Lefschetz fibers and **multiple smooth curves** are atomic.

How to construct splitting families

- **Double covering method**
  1. for degenerations of genus 1 (Moishezon)
  2. for degenerations of genus 2 (Horikawa)
  3. for hyperelliptic degenerations (Arakawa-Ashikaga)
- **Barking deformation**

  for **linear degenerations**

  whose singular fiber has a **simple crust** (Takamura)

A singular fiber (or precisely, its degeneration) is **atomic**
$\overset{\text{df}}{\Longleftrightarrow}$ it does NOT admit any splitting families.

Fact
Lefschetz fibers and **multiple smooth curves** are atomic.

How to construct splitting families

- **Double covering method**
  1. for degenerations of genus 1 (Moishezon)
  2. for degenerations of genus 2 (Horikawa)
  3. for hyperelliptic degenerations (Arakawa-Ashikaga)
- **Barking deformation**
  for **linear degenerations**
  whose singular fiber has a **simple crust** (Takamura)

Splittability of singular fibers

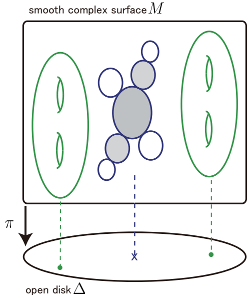
A singular fiber (or precisely, its degeneration) is **atomic**
$\overset{\text{df}}{\Longleftrightarrow}$ it does NOT admit any splitting families.

Fact
Lefschetz fibers and **multiple smooth curves** are atomic.

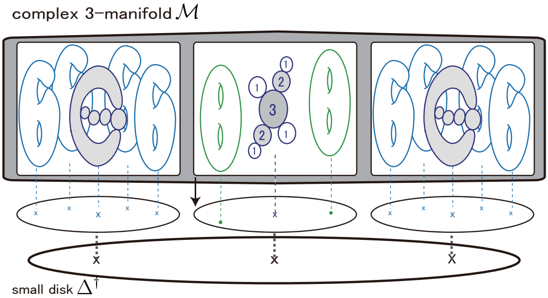
How to construct splitting families

- **Double covering method**
  1. for degenerations of genus 1 (Moishezon)
  2. for degenerations of genus 2 (Horikawa)
  3. for hyperelliptic degenerations (Arakawa-Ashikaga)
- **Barking deformation**
  for **linear degenerations**
  whose singular fiber has a **simple crust** (Takamura)

Degeneration
of Rieman surfaces



Splitting family
for degeneration of Riemann surfaces

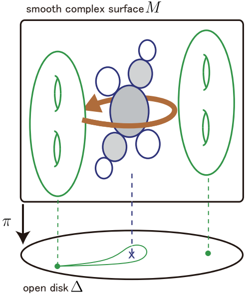


The central fiber is **singular**.
General fibers are **smooth**.

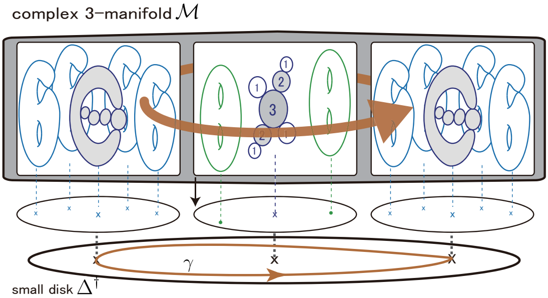
The central family has **one singular fiber**.
General families have k **singular fibers**.

Degeneration VS Splitting Family

Degeneration
of Rieman surfaces



Splitting family
for degeneration of Riemann surfaces

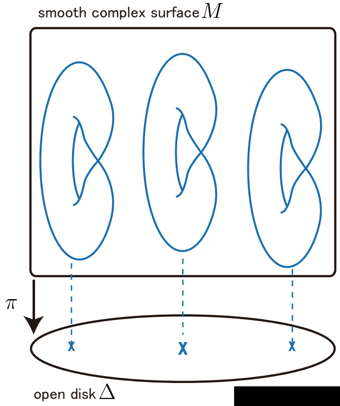


The central fiber is **singular**.
General fibers are **smooth**.

The central family has **one singular fiber**.
General families have k **singular fibers**.

Mapping class grps of degenerations

In what follows, we allow
degenerations to have **finitely many singular fibers**.



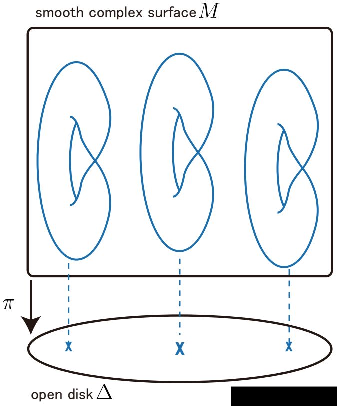
Mapping class grps of degenerations

In what follows, we allow
degenerations to have **finitely many singular fibers**.

$\pi : M \to \Delta$: a degeneration of Riemann surfaces.
with **k singular fibers $X_{s_1}, X_{s_2}, \ldots, X_{s_k}$**

Consider a pair of orientation preserving self-homeomorphisms
$F : M \to M$ and $\phi : \Delta \to \Delta$
satisfying $\pi \circ F = \phi \circ \pi$.
(note: F preserves fibers of π.)
Such pairs (F, ϕ) are called
topological automorphisms
of the degeneration $\pi :\to \Delta$.

$\mathrm{Aut}(\pi) := \{(F, \phi) \ : \ \pi \circ F = \phi \circ \pi\}$
: the set of topological automorphisms
We call $\mathrm{MCG}(\pi) := \pi_0(\mathrm{Aut}(\pi))$
the **mapping class group** of $\pi :\to \Delta$.

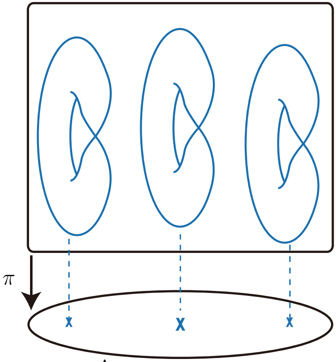


Mapping class grps of degenerations

In what follows, we allow
degenerations to have **finitely many singular fibers**.

$\pi : M \to \Delta$: a degeneration of Riemann surfaces.
with k **singular fibers** $X_{s_1}, X_{s_2}, \ldots, X_{s_k}$

Consider a pair of orientation preserving self-homeomorphisms
$F : M \to M$ and $\phi : \Delta \to \Delta$
satisfying $\pi \circ F = \phi \circ \pi$.
(note: F preserves fibers of π.)



Such pairs (F, ϕ) are called
topological automorphisms
of the degeneration $\pi :\to \Delta$.

$\mathrm{Aut}(\pi) := \{(F, \phi)\ :\ \pi \circ F = \phi \circ \pi\}$
: the set of topological automorphisms

We call $\mathrm{MCG}(\pi) := \pi_0(\mathrm{Aut}(\pi))$
the **mapping class group** of $\pi :\to \Delta$.

Mapping class grps of degenerations

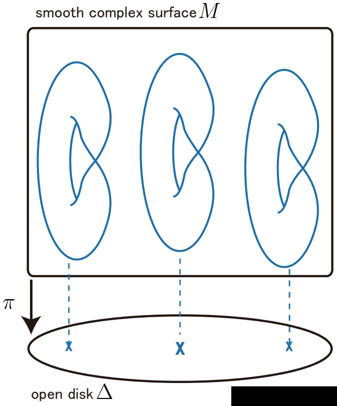
In what follows, we allow
degenerations to have **finitely many singular fibers**.

$\pi : M \to \Delta$: a degeneration of Riemann surfaces.
with **k singular fibers $X_{s_1}, X_{s_2}, \ldots, X_{s_k}$**

Consider a pair of orientation preserving self-homeomorphisms
$F : M \to M$ and $\phi : \Delta \to \Delta$
satisfying $\pi \circ F = \phi \circ \pi$.
(note: F preserves fibers of π.)
Such pairs (F, ϕ) are called
topological automorphisms
of the degeneration $\pi :\to \Delta$.

$\mathrm{Aut}(\pi) := \{(F, \phi) \ : \ \pi \circ F = \phi \circ \pi\}$
: the set of topological automorphisms

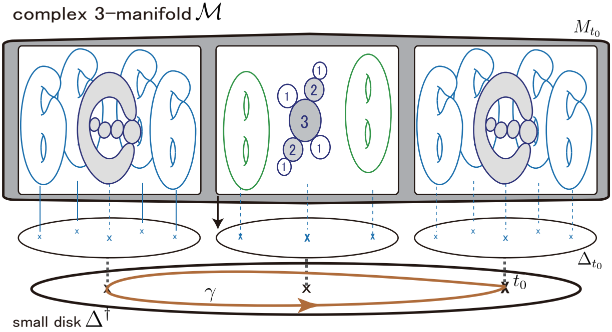
We call $\mathrm{MCG}(\pi) := \pi_0(\mathrm{Aut}(\pi))$
the **mapping class group** of $\pi :\to \Delta$.



$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a splitting family

Take a base point $t_0 \in \Delta^\dagger \setminus \{0\}$, and
a loop γ in $\Delta^\dagger \setminus \{0\}$ with base point t_0
that goes once around 0 in the counterclockwise direction.

We regard $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ as a "**reference degeneration**".

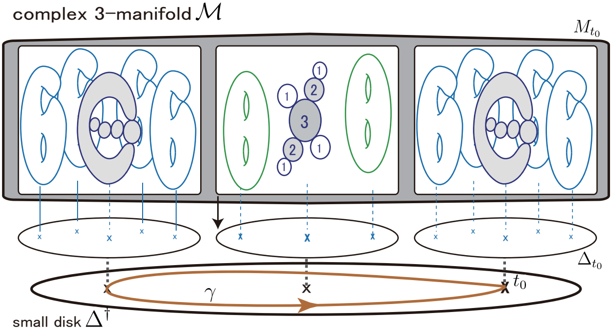


Monodromy of splitting families

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a splitting family

Take a base point $t_0 \in \Delta^\dagger \setminus \{0\}$, and
a loop γ in $\Delta^\dagger \setminus \{0\}$ with base point t_0
that goes once around 0 in the counterclockwise direction.

We regard $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ as a "**reference degeneration**".



Monodromy of splitting families

Consider the diagram

$$W := \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma,$$

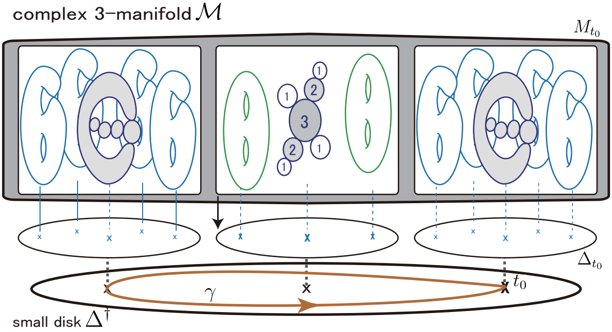
(!) $W = \bigcup_{t\in\gamma} M_t$ is a real 5-manifold.

$\exists$ a **Thom stratification** for $\Psi|_{\Delta\times\gamma} : W \to \Delta \times \gamma$ s.t.

proj_2 maps each stratum in $\Delta \times \gamma$ onto γ submersively.

$\Rightarrow$ **Thom's second isotopy lemma** ensures that

$\Psi|_{\Delta\times\gamma} : W \to \Delta \times \gamma$ is (topologically) locally trivial over γ.



Monodromy of splitting families

Consider the diagram

$$W := \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma,$$

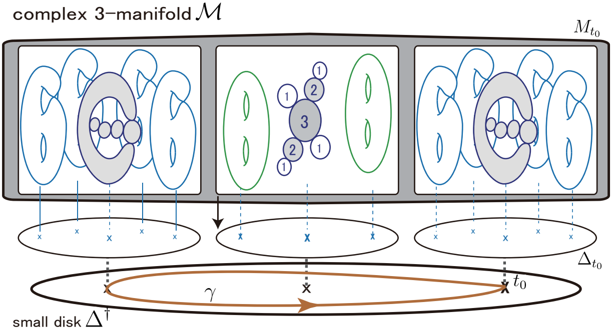
(!) $W = \bigcup_{t\in\gamma} M_t$ is a real 5-manifold.

$\exists$ a **Thom stratification** for $\Psi|_{\Delta\times\gamma} : W \to \Delta \times \gamma$ s.t.

proj_2 maps each stratum in $\Delta \times \gamma$ onto γ submersively.

$\Rightarrow$ **Thom's second isotopy lemma** ensures that

$\Psi|_{\Delta\times\gamma} : W \to \Delta \times \gamma$ is (topologically) locally trivial over γ.

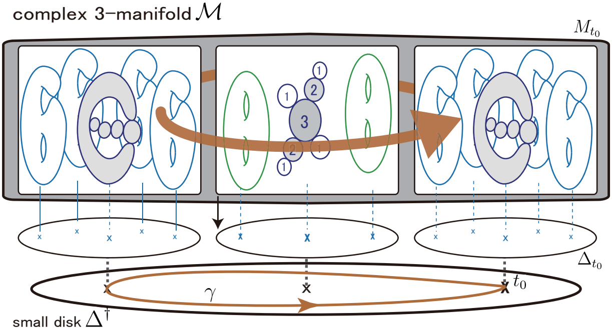


$$W = \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma.$$

Pasting these trivializations of $\Psi|_{\Delta\times\gamma}$ along γ gives us
a topological automorphism (F, ϕ) of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$.

(F, ϕ) is called the **monodromy automorphism**.

The **topological monodromy** of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$
is defined as its homotopy class $[F, \phi] \in MCG(\pi_{t_0})$



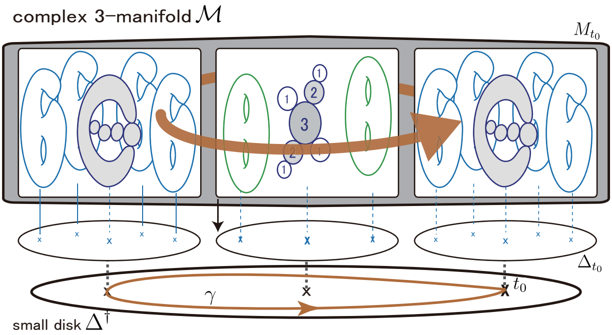
Monodromy of splitting families

$$W = \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma.$$

Pasting these trivializations of $\Psi|_{\Delta\times\gamma}$ along γ gives us
a topological automorphism (F, ϕ) of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$.

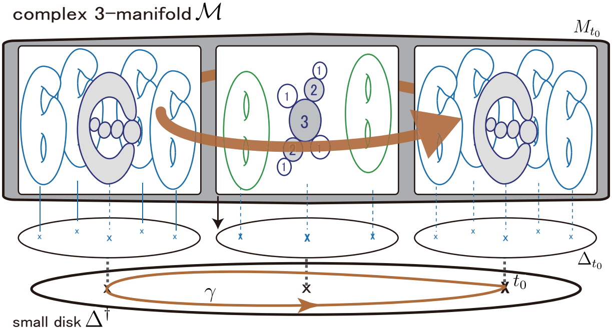
(F, ϕ) is called the **monodromy automorphism**.

The **topological monodromy** of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$
is defined as its homotopy class $[F, \phi] \in MCG(\pi_{t_0})$



Monodromy of splitting families

In this talk, we focus on
the restrictions of monodromy automorphisms
to the union of singular fibers.



Discriminant of splitting families

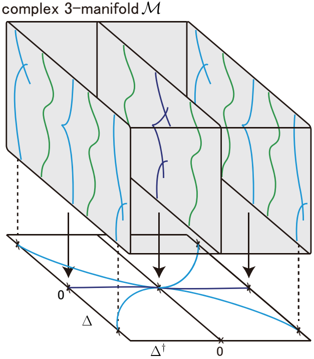
$\mathcal{D} := \Psi(\mathrm{Sing}(\Psi))$: the **discriminant** of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$

(!) It is a plane curve in $\Delta \times \Delta^\dagger$ with at most one singularity at $\underline{0}$.

Suppose $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ has k singular values $s_1, s_2, \ldots, s_k$

(i.e. it has **k singular fibers $X_{s_1,t_0}, X_{s_2,t_0}, \ldots, X_{s_k,t_0}$**).

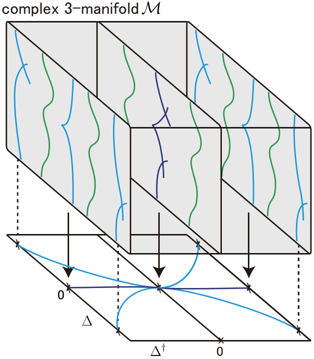
$\Rightarrow$ The second projection $\mathrm{proj}_2 : \Delta \times \Delta^\dagger \to \Delta^\dagger$ induces an unramified k-fold covering $\mathcal{D} \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$.



Discriminant of splitting families

$\mathcal{D} := \Psi(\mathrm{Sing}(\Psi))$: the **discriminant** of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$
(!) It is a plane curve in $\Delta \times \Delta^\dagger$ with at most one singularity at $\underline{0}$.

Suppose $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ has k singular values $s_1, s_2, \ldots, s_k$
(i.e. it has ***k* singular fibers $\boldsymbol{X_{s_1,t_0}, X_{s_2,t_0}, \ldots, X_{s_k,t_0}}$**).
$\Rightarrow$ The second projection $\mathrm{proj}_2 : \Delta \times \Delta^\dagger \to \Delta^\dagger$ induces
an unramified k-fold covering $\mathcal{D} \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$.

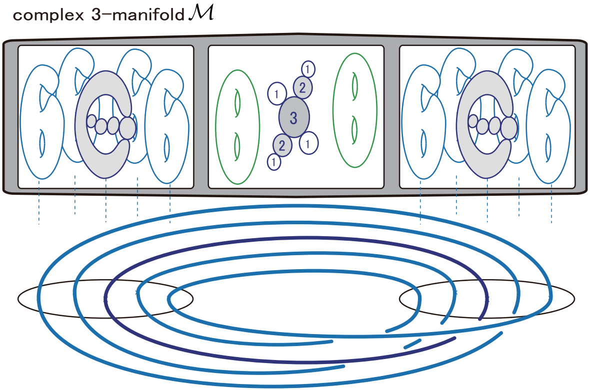


Discriminant of splitting families

$$W = \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma.$$

In the solid torus $\Delta \times \gamma$,
$L := \mathcal{D} \cap (\Delta \times \gamma)$ forms a quasi-positive (k-string) closed braid.

$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_\ell$: the irreducible decomposition
$\Rightarrow K_i := \mathcal{D}_i \cap L$, $i = 1, 2, \ldots, \ell$, are the knot components of L.



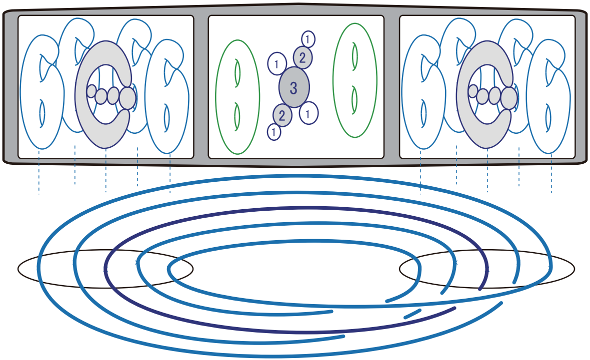
Discriminant of splitting families

$$W = \Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi|_{\Delta\times\gamma}} \Delta \times \gamma \xrightarrow{\mathrm{proj}_2} \gamma.$$

In the solid torus $\Delta \times \gamma$,
$L := \mathcal{D} \cap (\Delta \times \gamma)$ forms a quasi-positive (k-string) closed braid.

$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_\ell$: the irreducible decomposition
$\Rightarrow K_i := \mathcal{D}_i \cap L$, $i = 1, 2, \ldots, \ell$, are the knot components of L.

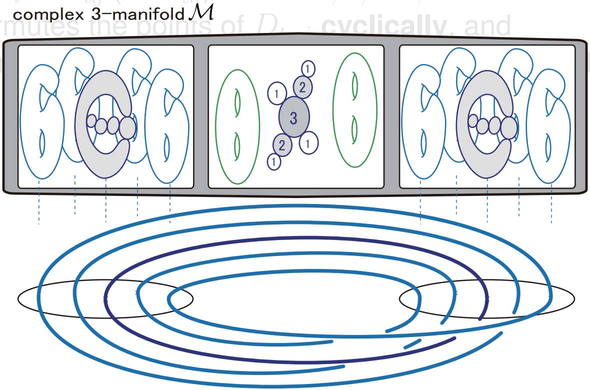
complex 3−manifold $\mathcal{M}$



Polydromy of splitting families

(F, ϕ) : the **monodromy automorphism** of Ψ (w.r.t. π_{t_0})

D_{t_0} $(= L \cap \Delta_{t_0})$: the discriminant of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$
$\Longrightarrow$ ϕ permutes the points of D_{t_0}, and
F permutes **the singular fibers** over D_{t_0}.



Polydromy of splitting families

(F, ϕ) : the **monodromy automorphism** of Ψ (w.r.t. π_{t_0})

D_{t_0} $(= L \cap \Delta_{t_0})$: the discriminant of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$
$\Longrightarrow$ ϕ permutes the points of D_{t_0}, and
F permutes **the singular fibers** over D_{t_0}.

$D_{t_0,i} := K_i \cap \Delta_{t_0}$ $(\subset D_{t_0})$ for $i = 1, 2, \ldots, \ell$
$\Longrightarrow$ ϕ permutes the points of $D_{t_0,i}$ **cyclically**, and
F permutes **the singular fibers** over $D_{t_0,i}$ **cyclically**.

Polydromy of splitting families

Set $\mathbb{X}_i := \coprod_{s \in D_{t_0,i}} X_s$, and $f_i := F\big|_{\mathbb{X}_i} : \mathbb{X}_i \to \mathbb{X}_i$.

For $c_i := \# D_{t_0,i}$, $f_i^{c_i}$ maps each singular fiber to itself.

We say f_i is a **polydromy** of a **tassel** $\mathbb{X}_i$ of **order** c_i.

The projection $\mathcal{D}_i \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$ induced by proj_2
is an unramified 1-fold covering (i.e. an automorphism)
$\iff f_i$ is a polydromy of order $c_i = 1$.

$\mathbb{X}_i = X_{s_i}$ (so $f_i : X_{s_i} \to X_{s_i}$).

Theorem (O)

f_i : a **polydromy** of X_{s_i} of **order** $c_i = 1$
Θ : an irreducible component of X_{s_i}
b : a positive integer satisfying $f_i^b(\Theta) = \Theta$.
Then $f_i^b\big|_\Theta$ is a **pseudo-periodic map of negative twist**.

Polydromy of splitting families

Set $\mathbb{X}_i := \coprod_{s \in D_{t_0,i}} X_s$, and $f_i := F\big|_{\mathbb{X}_i} : \mathbb{X}_i \to \mathbb{X}_i$.
For $c_i := \# D_{t_0,i}$, $f_i^{c_i}$ maps each singular fiber to itself.

We say f_i is a **polydromy** of a **tassel** $\mathbb{X}_i$ of **order** c_i.

The projection $\mathcal{D}_i \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$ induced by proj_2
is an unramified 1-fold covering (i.e. an automorphism)
$\iff f_i$ is a polydromy of order $c_i = 1$.

$\mathbb{X}_i = X_{s_i}$ (so $f_i : X_{s_i} \to X_{s_i}$).

Theorem (O)

f_i : a **polydromy** of X_{s_i} of **order** $c_i = 1$
Θ : an irreducible component of X_{s_i}
b : a positive integer satisfying $f_i^b(\Theta) = \Theta$.
Then $f_i^b\big|_\Theta$ is a **pseudo-periodic map of negative twist**.

Polydromy of splitting families

Set $\mathbb{X}_i := \coprod_{s \in D_{t_0,i}} X_s$, and $f_i := F|_{\mathbb{X}_i} : \mathbb{X}_i \to \mathbb{X}_i$.
For $c_i := \# D_{t_0,i}$, $f_i^{c_i}$ maps each singular fiber to itself.

We say f_i is a **polydromy** of a **tassel** $\mathbb{X}_i$ of **order** c_i.

The projection $\mathcal{D}_i \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$ induced by proj_2
is an unramified 1-fold covering (i.e. an automorphism)
$\iff f_i$ is a polydromy of order $c_i = 1$.

$\mathbb{X}_i = X_{s_i}$ (so $f_i : X_{s_i} \to X_{s_i}$).

Theorem (O)

f_i : a **polydromy** of X_{s_i} of **order** $c_i = 1$
Θ : an irreducible component of X_{s_i}
b : a positive integer satisfying $f_i^b(\Theta) = \Theta$.
Then $f_i^b|_\Theta$ is a **pseudo-periodic map of negative twist**.

Polydromy of splitting families

Set $\mathbb{X}_i := \coprod_{s \in D_{t_0,i}} X_s$, and $f_i := F|_{\mathbb{X}_i} : \mathbb{X}_i \to \mathbb{X}_i$.
For $c_i := \# D_{t_0,i}$, $f_i^{c_i}$ maps each singular fiber to itself.

We say f_i is a **polydromy** of a **tassel** $\mathbb{X}_i$ of **order** c_i.

The projection $\mathcal{D}_i \setminus \{\underline{0}\} \to \Delta^\dagger \setminus \{0\}$ induced by proj_2
is an unramified 1-fold covering (i.e. an automorphism)
$\iff f_i$ is a polydromy of order $c_i = 1$.

$\mathbb{X}_i = X_{s_i}$ (so $f_i : X_{s_i} \to X_{s_i}$).

Theorem (O)

f_i : a **polydromy** of X_{s_i} of **order** $c_i = 1$
Θ : an irreducible component of X_{s_i}
b : a positive integer satisfying $f_i^b(\Theta) = \Theta$.
Then $f_i^b|_\Theta$ is a **pseudo-periodic map of negative twist**.

How to construct splitting families

- **Double covering method**
  1. for degenerations of genus 1 (Moishezon)
  2. for degenerations of genus 2 (Horikawa)
  3. for hyperelliptic degenerations (Arakawa-Ashikaga)
- **Barking deformation**

  for **linear degenerations**

  whose singular fiber has a **simple crust** (Takamura)

— Toy model —

$\pi(z,\zeta) = z^6\zeta^5$: a holomorphic function on $\mathbb{C}^2$

Consider a family of holomorphic functions on $\mathbb{C}^2$ given by

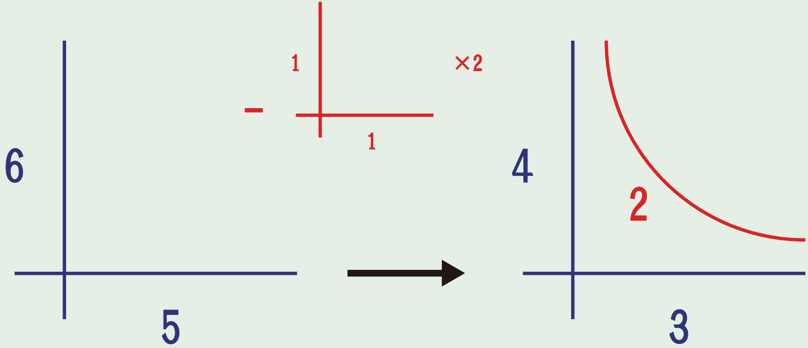
$$\pi_t(z,\zeta) = z^4\zeta^3(z\zeta + t)^2 \quad (t \in \Delta^\dagger).$$

— Toy model —

$\pi(z,\zeta) = z^6\zeta^5$: a holomorphic function on $\mathbb{C}^2$

Consider a family of holomorphic functions on $\mathbb{C}^2$ given by

$$\pi_t(z,\zeta) = z^4\zeta^3(z\zeta + t)^2 \quad (t \in \Delta^\dagger).$$

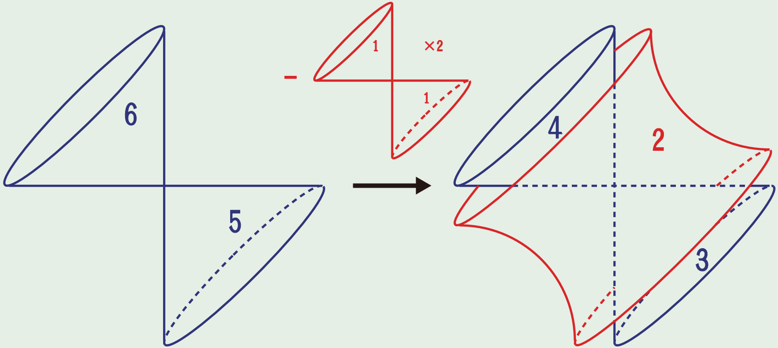


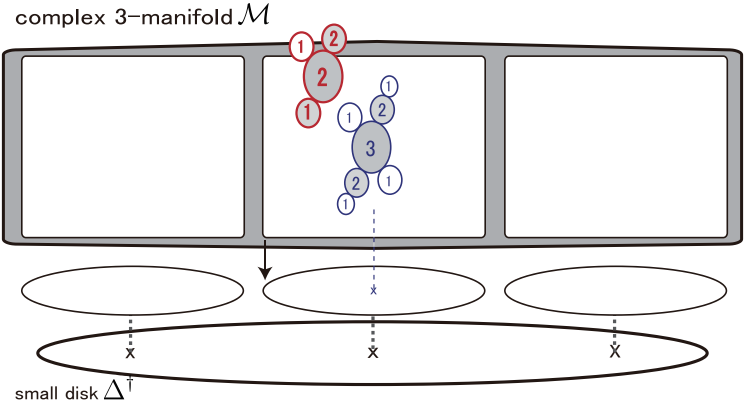
— Toy model —

$\pi(z,\zeta) = z^6\zeta^5$: a holomorphic function on $\mathbb{C}^2$

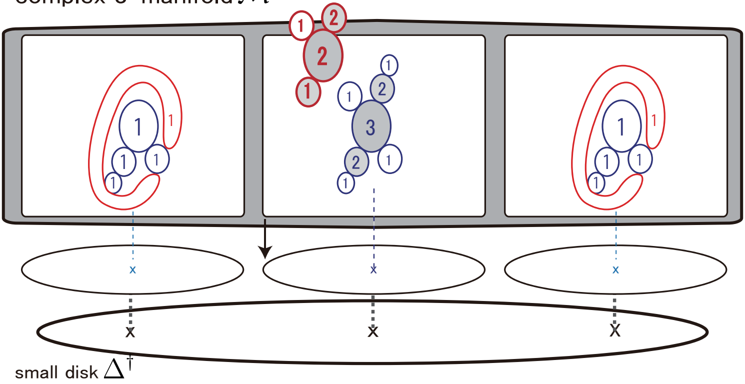
Consider a family of holomorphic functions on $\mathbb{C}^2$ given by

$$\pi_t(z,\zeta) = z^4\zeta^3(z\zeta + t)^2 \quad (t \in \Delta^\dagger).$$



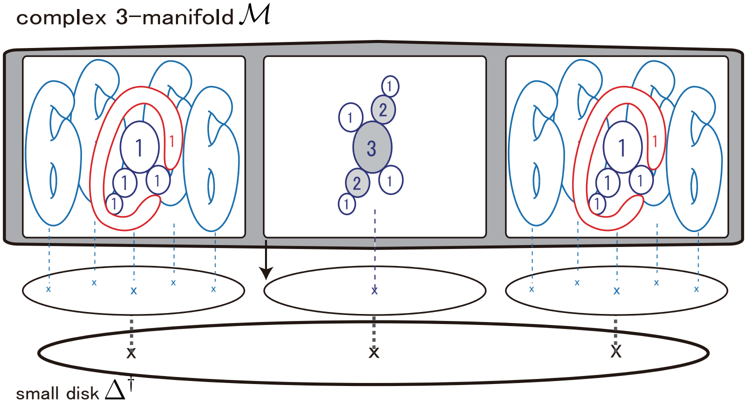


- If X_0 contains **"simple crust"** Y as a subdivisor,
  $\Longrightarrow$ $\exists$ a deformation family for the given degeneration associated with Y.



- X_0 is deformed to **the central fiber** $X_{t,0}$ of $\pi_t : M_t \to \Delta_t$ in such a way that Y **looks "barked" off** from X_0.
- There exist other singular fibers than $X_{t,0}$, which are called **subordinate fibers**.

What is **barking**?



- $\boldsymbol{X_0}$ is deformed to **the central fiber $X_{t,0}$** of $\pi_t : M_t \to \Delta_t$ in such a way that $\boldsymbol{Y}$ **looks "barked" off** from $\boldsymbol{X_0}$.
- There exist other singular fibers than $\boldsymbol{X_{t,0}}$, which are called **subordinate fibers**.

The central fiber $X_{t,0}$ forms a **tassel** by itself.
i.e. $X_{t,0}$ has a **polydromy** $f : X_{t,0} \to X_{t,0}$ of order 1.

Theorem (O)

$\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$: a barking family for a linear degeneration associated with **a simple crust** (Y, ℓ)

f_0 **: a polydromy** of **The central fiber** $X_{t,0}$ (so of order 1)

1. If Θ is a **stable component**, $\Longrightarrow f(\Theta) = \Theta$, and $f\big|_\Theta$ is isotopic to the **identity map**.
2. If Θ is a **bark component**, $\Longrightarrow f(\Theta), f^2(\Theta), \ldots, f^b(\Theta)(= \Theta)$ are **bark components**, and $f^b\big|_\Theta$ is **a monodromy homeomorphism** of a degeneration with "**the modification of** $\frac{1}{b}Y$" (in particular, a pseudo-periodic map of negative twist).

The central fiber $X_{t,0}$ forms a **tassel** by itself.
i.e. $X_{t,0}$ has a **polydromy $f : X_{t,0} \to X_{t,0}$** of order 1.

Theorem (O)

$\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$: a barking family for a linear degeneration associated with **a simple crust (Y, ℓ)**

f_0 : a polydromy of **The central fiber $X_{t,0}$** (so of order 1)

1. If Θ is a **stable component**,
$\Longrightarrow f(\Theta) = \Theta$, and $f\big|_{\Theta}$ is isotopic to the **identity map**.
2. If Θ is a **bark component**,
$\Longrightarrow f(\Theta), f^2(\Theta), \ldots, f^b(\Theta)(= \Theta)$ are **bark components**, and $f^b\big|_{\Theta}$ is **a monodromy homeomorphism** of a degeneration with "**the modification of $\frac{1}{b}Y$**" (in particular, a pseudo-periodic map of negative twist).

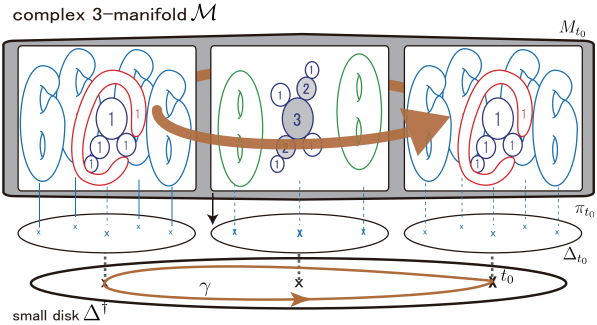
The central fiber $X_{t,0}$ forms a **tassel** by itself.
i.e. $X_{t,0}$ has a **polydromy $f : X_{t,0} \to X_{t,0}$** of order 1.

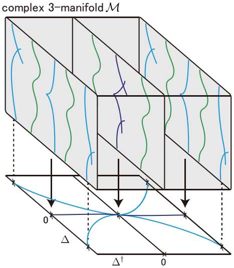
Theorem (O)

$\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$: a barking family for a linear degeneration associated with **a simple crust (Y, ℓ)**

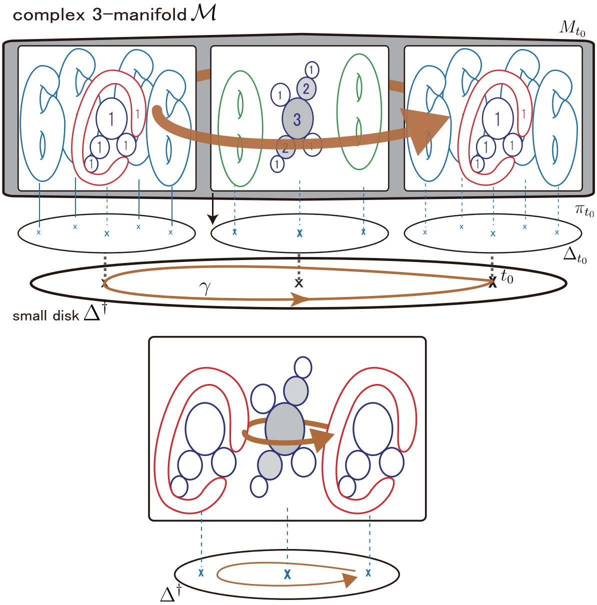
f_0 : a polydromy of **The central fiber $X_{t,0}$** (so of order 1)

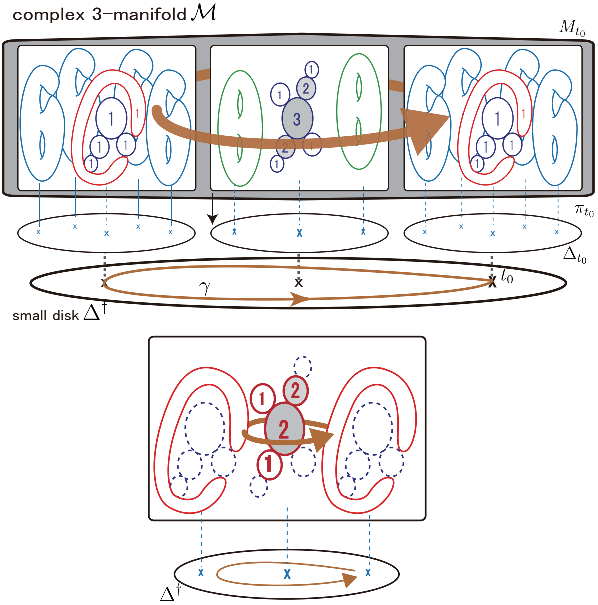
1. If Θ is a **stable component**,
$\Longrightarrow f(\Theta) = \Theta$, and $f\big|_{\Theta}$ is isotopic to the **identity map**.
2. If Θ is a **bark component**,
$\Longrightarrow f(\Theta), f^2(\Theta), \ldots, f^b(\Theta)(= \Theta)$ are **bark components**, and $f^b\big|_{\Theta}$ is **a monodromy homeomorphism** of a degeneration with "**the modification of $\frac{1}{b}Y$**" (in particular, a pseudo-periodic map of negative twist).

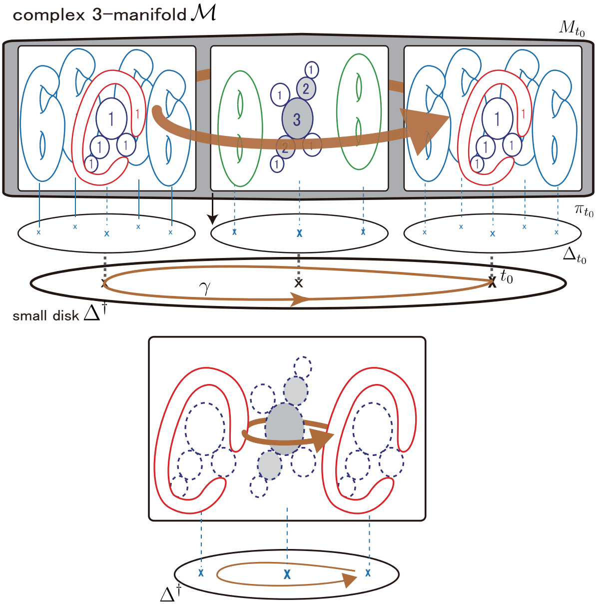

complex 3−manifold $\mathcal{M}$
M_{t_0}
π_{t_0}
Δ_{t_0}
t_0
γ
small disk $\Delta^\dagger$

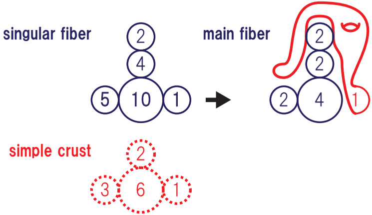

complex 3−manifold $\mathcal{M}$
0
Δ
$\Delta^\dagger$
0

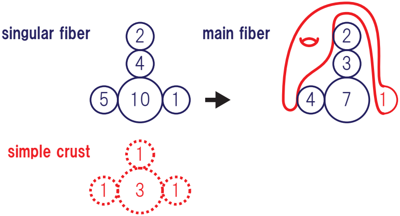
Idea of proof

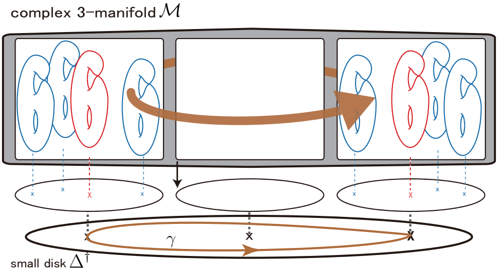



complex 3−manifold $\mathcal{M}$
M_{t_0}
π_{t_0}
Δ_{t_0}
t_0
γ
small disk $\Delta^\dagger$
$\Delta^\dagger$


complex 3–manifold $\mathcal{M}$
M_{t_0}
π_{t_0}
Δ_{t_0}
t_0
γ
small disk $\Delta^\dagger$
$\Delta^\dagger$


singular fiber
2
4
5
10
1
main fiber
2
2
2
4
1
simple crust
2
3
6
1


singular fiber
2
4
5
10
1
main fiber
2
3
4
7
1
simple crust
1
1
3
1


complex 3−manifold $\mathcal{M}$
γ
small disk $\Delta^\dagger$

Thank you for listening!