Resonant characters for rational arrangements

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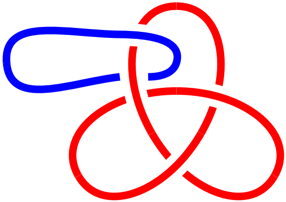
Linking number



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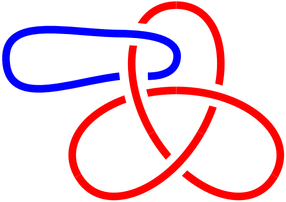

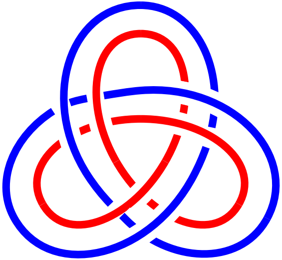

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- Projection of $K \Longrightarrow \tilde{K}$: $\mathcal{L}(K, \tilde{K})$ invariant of the projection.

Algebraic curves

Regular neighbourhood

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- $\sigma^{-1}(\mathcal{C}) = \overbrace{\bigcup_{j=1}^{r} \hat{\mathcal{C}}_i}^{\text{strict transforms}} \cup \overbrace{\bigcup_{k=1}^{s} E_k}^{\text{exceptional components}} = \bigcup_{j=1}^{r+s} A_j$; Γ dual graph weighted by $(g_j, e_j) = (g(A_j), A_j^2)$.

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- $\mathcal{T}(\mathcal{C})$ regular neighbourhood of $\sigma^{-1}(\mathcal{C}) \subset X$, plumbed union of tubular neighbourhoods $\pi_j : \mathcal{T}(A_j) \to A_j$ (Euler number A_j^2).

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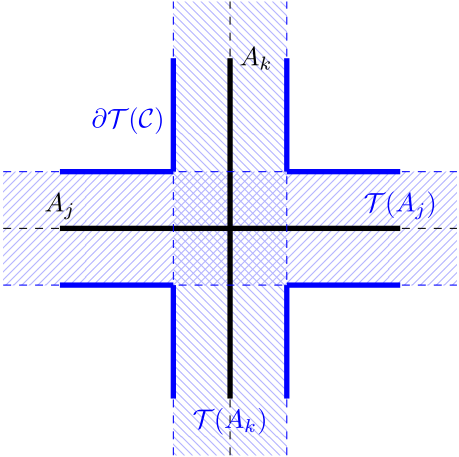

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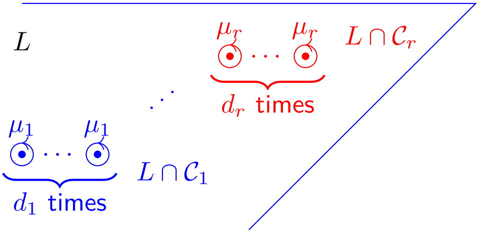

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- Goal: Define a linking number for cycles in Γ (see also Guerville-Meilhan).

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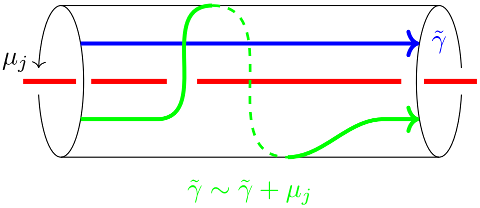

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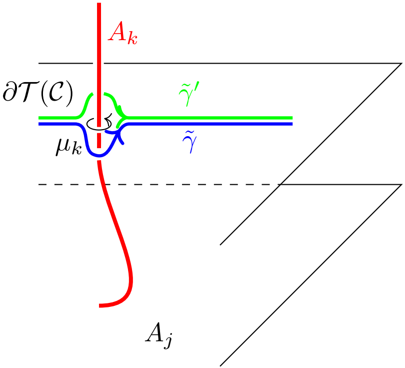

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Theorem (AFG)

The linking number of (ξ, γ) is a topological invariant of $(\mathbb{P}^2, \mathcal{C}, \xi)$ (in the case of line arrangements).

Nodal cubic and inflexion tangent lines

- $\mathcal{C} = \{\overbrace{(xyz + x^3 - y^3)}^{\mathcal{C}_1}\overbrace{(z^2 - 3(x-y)z + 9(x^2 - xy + y^2))}^{\mathcal{C}_2 \cup \mathcal{C}_3} = 0\}$: nodal cubic and two tangent lines at inflexion points.

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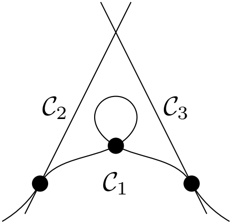

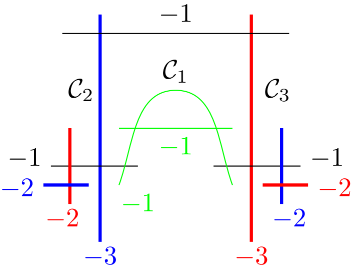

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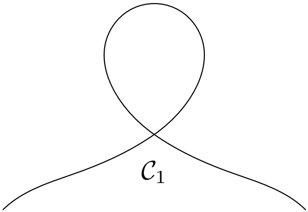

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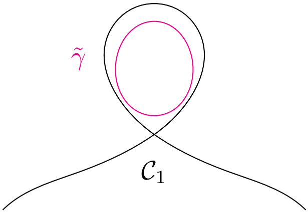

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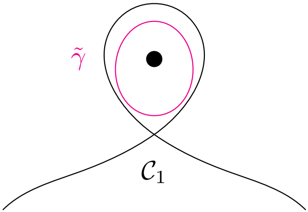

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  The preimage of $\mathcal{C}_1$: three lines.

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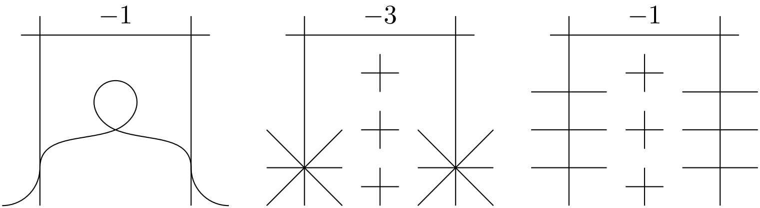
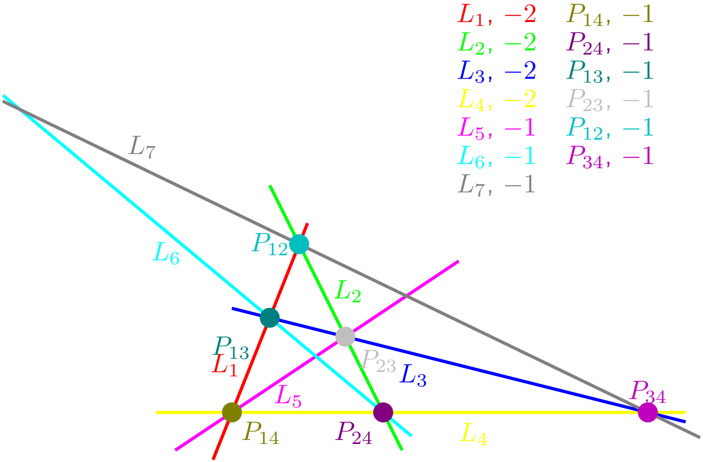


Extended Ceva arrangement



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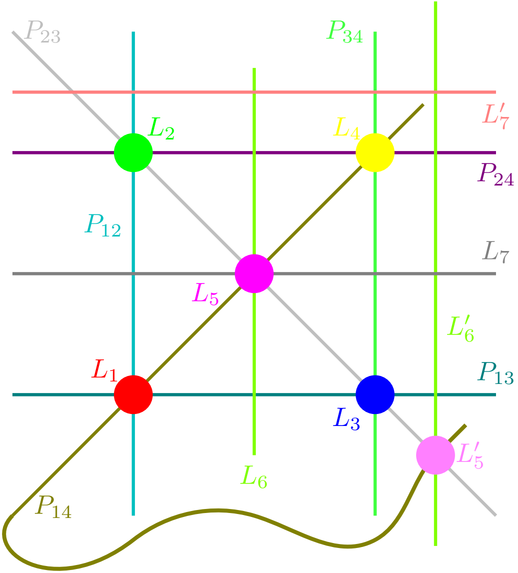
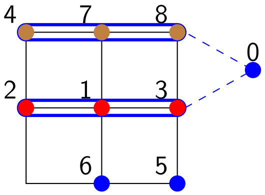


$$\left[xy^2 : x(y-z)^2 : y(y-z)(x-z)\right]$$

Zariski pairs of arrangements via linking numbers

Extended McLane arrangements



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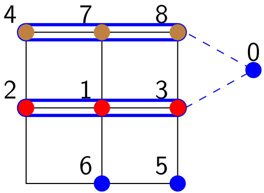

$\xi : (\mu_0, \ldots, \mu_8) \mapsto (1, \zeta, \zeta, \zeta, \overline{\zeta}, 1, 1, \overline{\zeta}, \overline{\zeta})$, cycle L_0, L_5, L_6.

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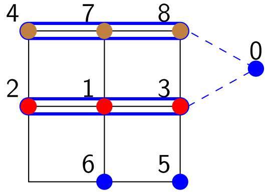

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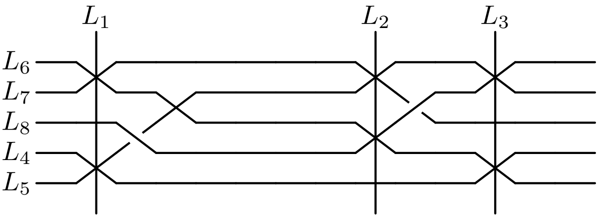

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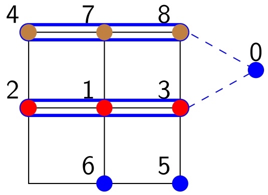

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Guerville's arrangement

Linking number oriented Zariski 4-tuple with equations in $\mathbb{Q}(\exp(\frac{2i\pi}{5}))$.

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Definition

The characteristic variety $V_k'(G) \equiv V_k'(Z) \subset \mathbb{T}_Z$ is the zero locus of $J_k(G)$.

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Remark

These two definitions may differ only in $\mathbf{1} \in \mathbb{T}_Z$.

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Corollary

Torsion points are dense in $V_k(Z)$.

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Hodge Theory

Hypersurface complement. Projective part.

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Geometric interpretation

$\ker \sigma_\xi$: equations of curves of degree $\ell_\xi - 3$ passing through the points $P \in \mathcal{P}$ with multiplicity at least $\left\lfloor \sum_{P \in L_j} q_j \right\rfloor - 1$.

More Libgober's results

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Problem

Compute $\dim_{\mathbb{C}} QP_\xi$ for resonant torsion characters.

Quasi-projective part

Theorem

S smooth projective surface, $\mathcal{D} \subset S$ normal crossing divisor, $\mathcal{D} = D_1 \cup \cdots \cup D_s$. Then,

$$\dim_{\mathbb{C}} H_1(S \setminus \mathcal{D}; \mathbb{C}) - \dim_{\mathbb{C}} H_1(S; \mathbb{C}) = \dim_{\mathbb{C}} \ker \left(\bigoplus_{j=1}^{s} H_2(D_j; \mathbb{C}) \to H_2(S; \mathbb{C}) \right).$$

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- $\mathbb{P}^2 \setminus \mathcal{C} \equiv X \setminus \left(\bigcup_{j=0}^{r} \hat{L}_j \cup \bigcup_{P \in \mathcal{P}} E_P \right) = X \setminus \left(\bigcup_{j=0}^{t} A_j \right)$

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$$\dim_{\mathbb{C}} H_1(S \setminus \mathcal{D};\mathbb{C}) - \dim_{\mathbb{C}} H_1(S;\mathbb{C}) = \dim_{\mathbb{C}} \ker\left(\bigoplus_{j=1}^{s} H_2(D_j;\mathbb{C}) \to H_2(S;\mathbb{C})\right).$$

- $\mathbb{P}^2 \setminus \mathcal{C} \equiv X \setminus \left(\bigcup_{j=0}^{r} \hat{L}_j \cup \bigcup_{P \in \mathcal{P}} E_P\right) = X \setminus \left(\bigcup_{j=0}^{t} A_j\right)$
- Smooth model $\rho_\xi : \bar{X}^\xi \to X$, such that $\bar{X}^\xi \setminus X^\xi = \mathcal{D}^\xi$ normal crossing divisor.

Quasi-projective part

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- Smooth model $\rho_\xi : \bar{X}^\xi \to X$, such that $\bar{X}^\xi \setminus X^\xi = \mathcal{D}^\xi$ normal crossing divisor.
- $\mathcal{D}_\xi = \bigcup_{j=0}^{t} \underbrace{\hat{\rho}_\xi^*(A_j)}_{\text{strict transform}} \cup \bigcup_{k=1}^{t'} \underbrace{B_j}_{\text{collapsed by } \rho_\xi}$.

Computing QP_ξ

$$\bigoplus_{j=0}^{t'} H_2(B_j;\mathbb{C}) \;\oplus\; \bigoplus_{j=0}^{t} H_2(\hat{\rho}_\xi^*(A_j);\mathbb{C}) \longrightarrow H_2(\bar{X}_\xi;\mathbb{C})$$

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- $H_2(\hat{\rho}_\xi^*(A_j);\mathbb{C})^\xi \neq 0 \iff \dim_{\mathbb{C}} H_2(\hat{\rho}_\xi^*(A_j);\mathbb{C}) = N \iff \rho_\xi$ is an unbranched covering over A_j

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$$\bigoplus_{A_j \subset \mathcal{C}_\xi} H_2(\hat\rho_\xi^*(A_j);\mathbb{C})^\xi \longrightarrow H_2(\bar X_\xi;\mathbb{C})^\xi$$

- $A_j \subset \mathcal{C}_\xi$, $\tilde A_j$ irreducible component of $\hat\rho_\xi^*(A_j)$, $\zeta = \exp\frac{2i\pi}{N}$, $H_2(\hat\rho_\xi^*(A_j);\mathbb{C})^\xi = \mathbb{C}\langle \hat A_j\rangle$,

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Theorem

$\dim_{\mathbb{C}} QP_\xi = \operatorname{corank} \mathbf{I}_\xi$ *(Hodge index theorem).*

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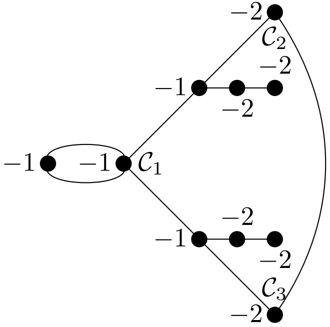
- If $A_j \cap A_k = \{P_1, \dots, P_m\} \to \{e_1, \dots, e_m\}$,

$$\hat{A}_j^\xi \cdot \hat{A}_k^\xi = \xi(\tilde{\gamma}_{e_1}) + \cdots + \xi(\tilde{\gamma}_{e_m}).$$

Examples

Example

Nodal cubic and inflexion tangent lines



Examples

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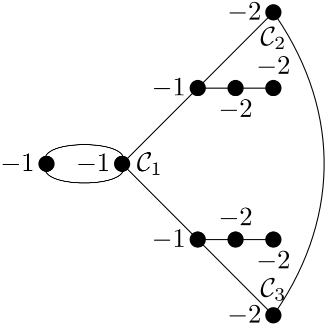

$$\begin{pmatrix} -1 & 1+\zeta \\ 1+\bar{\zeta} & -1 \end{pmatrix} = \begin{pmatrix} -1 & -\bar{\zeta} \\ -\zeta & -1 \end{pmatrix}$$

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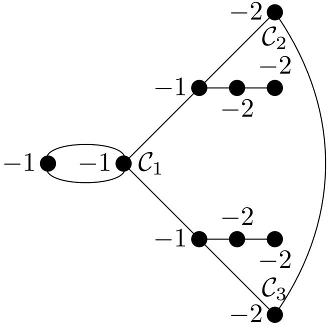

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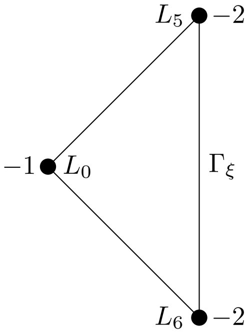
Examples

Example

Extended McLane arrangement



Examples

Example

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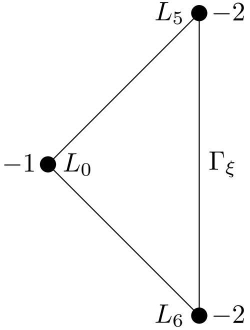
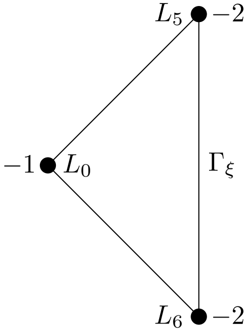


$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & \zeta \\ 1 & \bar{\zeta} & -2 \end{pmatrix}$$

Examples

Example

Extended McLane arrangement



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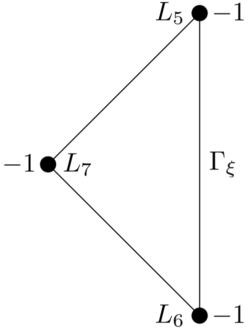
Examples

Example

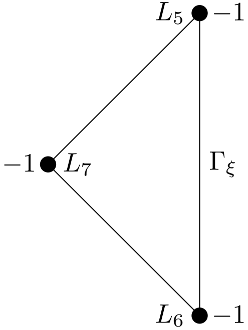
Extended Ceva arrangement



Examples

Example

Extended Ceva arrangement



$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

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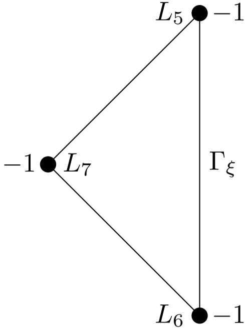

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\dim_{\mathbb{C}} QP_\xi = 2$$