Degeneration of Fermat hypersurfaces in positive characteristic

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The context

1. Introduction
2. The main theorems and corollaries
   - Theorem 2.2 and corollaries
   - Theorem 2.5
3. The proof of the main theorems
   - The proof of the Theorem 2.2
4. The case of plane curves

1 Introduction

2 The main theorems and corollaries
- Theorem 2.2 and corollaries
- Theorem 2.5

3 The proof of the main theorems
- The proof of the Theorem 2.2

4 The case of plane curves

Introduction

We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. We denote by $M_{n+1}(k)$ the set of square matrices of size $n+1$ with coefficients in k. For a nonzero matrix $A = (a_{ij})_{0 \leq i,j \leq n} \in M_{n+1}(k)$, we denote by X_A the hypersurface of degree $q+1$ defined by the equation

$$\sum a_{ij} x_i x_j^q = 0$$

in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$.

The well-known proposition

Proposition 1.1 (Lang 1956, Beauville 1986, Shimada 2001)

Let $A = (a_{ij})_{0 \leq i,j \leq n} \in M_{n+1}(k)$ *and* $X_A \subset \mathbb{P}^n$ *be as above. Then the following conditions are equivalent:*

(i) $\mathrm{rank}(A) = n + 1$,

(ii) X_A *is smooth,*

(iii) X_A *is isomorphic to the Fermat hypersurface of degree* $q + 1$*, and*

(iv) *there exists a linear transformation of coordinates* $T \in GL_{n+1}(k)$ *such that* ${}^tTAT^{(q)} = I_{n+1}$*, where* ${}^tT$ *is the transpose of* T*,* $T^{(q)}$ *is the matrix obtained from* T *by raising each coefficient to its q-th power, and* I_{n+1} *is the identity matrix.*

The Fermat hypersurfaces

The Fermat hypersurface of degree $q+1$ defined over an algebraically closed field of positive characteristic p has been a subject of numerous papers. It has many interesting properties :

- Supersingularity (Tate 1965, Shioda 1974, Shioda and Katsura 1979)
- Unirationality (Shioda 1974,Shioda and Katsura 1979, Shimada 1992), etc....

Moreover, the hypersurface X_A associated with the matrix A with coefficients a_{ij} in the finite field $\mathbb{F}_{q^2}$, which is called a Hermitian variety, has also been studied for many applications, such as coding theory (Høholdt, van Lint and Pellikaan 1998).

The quadratic form

In the case where characteristic $p \neq 2$, the hypersurface defined by the quadratic form $\sum a_{ij}x_i x_j = 0$ is projectively isomorphic to the hypersurface defined by

$$x_0^2 + \cdots + x_{r-1}^2 = 0,$$

where r is the rank of $A = (a_{ij})$. Recently, the case where characteristic 2 has been extended by Dolgachev and Duncan.

The Hermitian form

Question :

What is the normal form of the hypersurfaces defined by a form

$$\sum a_{ij} x_i x_j^q = 0.$$

When A satisfies ${}^t A = A^{(q)}$ and hence this form is the Hermitian form over $\mathbb{F}_{q^2}$, the hypersurface X_A is projectively isomorphic over $\mathbb{F}_{q^2}$ to

$$x_0^{q+1} + \cdots + x_{r-1}^{q+1} = 0,$$

where r is the rank of A (Hirschfeld 1991).

The purposes

We classify the hypersurfaces X_A associated with the matrices A of rank *n over an algebraically closed field* and determine their projective isomorphism classes.

1 Introduction

2 The main theorems and corollaries
- Theorem 2.2 and corollaries
- Theorem 2.5

3 The proof of the main theorems
- The proof of the Theorem 2.2

4 The case of plane curves

Some definitions and notions

Definition 2.1

Two hypersurfaces $X_A,\ X_{A'}$ associated with the matrices A, A' are projectively isomorphic if and only if there exists a linear transformation $T \in GL_{n+1}(k)$ such that $A' = {}^tTAT^{(q)}$. In this case, we denote $A \sim A'$.

We define I_s to be the $s \times s$ identity matrix, and E_r to be the $r \times r$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In particular, $E_1 = (0)$ and E_0 is the 0×0 matrix.

Theorem 2.2

Theorem 2.2

Let $A = (a_{ij})_{0\leq i,j\leq n}$ be a nonzero matrix in $M_{n+1}(k)$, and let X_A be the hypersurface of degree $q+1$ defined by $\sum a_{ij}x_ix_j^q = 0$ in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Suppose that the rank of A is n. Then the hypersurface X_A is projectively isomorphic to one of the hypersurfaces X_s associated with the matrices

$$W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array}\right),$$

where $0 \leq s \leq n$. Moreover, if $s \neq s'$, then X_s and $X_{s'}$ are not projectively isomorphic.

The corollaries

Corollary 2.3

If A is a general point of $\{A \in M_{n+1}(k) |\ rank(A) = n\}$, then $A \sim W_{n-1}$.

Corollary 2.4

Suppose that $n \geq 2, s < n$ and $(n, s) \neq (2, 0)$. Then X_s is rational.

Theorem 2.5 (1)

For $M \in GL_{n+1}(k)$, we denote by $[M] \in PGL_{n+1}(k)$ the image of M by the natural projection.

Theorem 2.5

Let X_s be the hypersurface associated with the matrix W_s in the projective space $\mathbb{P}^n$. The projective automorphism group $\mathrm{Aut}(X_s)$ with $s \leq n-2$ is the group consisting of $[M]$, with

$$M = \left(\begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline 0 & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array}\right),$$

where $T \in GL_{n-1}(k)$, $\mathbf{a}, \mathbf{c}$ are row vectors of dimension $n-1$, and $d, e \in k$, and they satisfy the following conditions:

Theorem 2.5 (2)

(i) $[T] \in \mathrm{Aut}(X_s^{n-2})$, ${}^tTW_s'T^{(q)} = \delta W_s'$, $\delta = \delta^q \neq 0$, *where* X_s^{n-2} *is the hypersurface defined in* $\mathbb{P}^{n-2}$ *by the matrix*

$$W_s' = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array}\right)$$

(ii) $d = \delta$,

(iii) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot T^{(q)} = \delta(0, \cdots, 0, 1)$,

(iv) ${}^tTW_s' \cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0$,

(v) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot {}^t\mathbf{a}^{(q)} + ed^q = 0$.

Theorem 2.5 (3)

Moreover, we have

$$\mathrm{Aut}(X_n) = \left\{ \left[\begin{array}{c|c} T_n & \\ \hline \mathbf{u} & 1 \end{array}\right] \;\middle|\; \begin{array}{l} {}^tT_n T_n^{(q)} = \lambda I_n,\ T_n \in GL_n(k), \\ \lambda \neq 0, \\ \mathbf{u} \text{ is a row vector of dimension } n \end{array} \right\},$$

and

$$\mathrm{Aut}(X_{n-1}) = \left\{ \left[\begin{array}{c|c|c} T_{n-1} & & \\ \hline & \beta & \\ \hline & & 1 \end{array}\right] \;\middle|\; \begin{array}{l} {}^tT_{n-1} T_{n-1}^{(q)} = \beta^q I_{n-1}, \\ T_{n-1} \in GL_{n-1}(k), \\ 0 \neq \beta \in k \end{array} \right\}.$$

1 Introduction

2 The main theorems and corollaries
- Theorem 2.2 and corollaries
- Theorem 2.5

3 The proof of the main theorems
- The proof of the Theorem 2.2

4 The case of plane curves

The lemma (1)

Lemma 3.1

Put

$$B_s = \left(\begin{array}{c|c} D_s & \\ \hline \mathbf{b}_s & \\ 0 & \\ \vdots & E_{n-s+1} \\ 0 & \end{array}\right),$$

where $s \geq 1$, $n - s + 1 \geq 1$, $D_s \in M_s(k)$, *and* $\mathbf{b}_s$ *is a row vector of dimension* s. *Suppose that the rank of* B_s *is* n. *Then*

The lemma (2)

$$B_s \sim W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array}\right),$$

or

$$B_s \sim B_{s-1} = \left(\begin{array}{c|c} D_{s-1} & \\ \hline \mathbf{b}_{s-1} & \\ 0 & \\ \vdots & E_{n-s+2} \\ 0 & \end{array}\right),$$

where $D_{s-1} \in M_{s-1}(k)$, *and* $\mathbf{b}_{s-1}$ *is a row vector of dimension* $s-1$.

Remark 3.2

When $s = 1$, *we have* $B_{s-1} = B_0 \sim E_{n+1} = W_0$.

The proof of the Theorem 2.2 (1)

Because the rank of the matrix A is n, Proposition 1.1 implies that X_A is singular. By using a linear transformation of coordinates if nessesary, we can assume that X_A has a singular point $(0, \cdots, 0, 1)$. Then we have $a_{in} = 0$ for any $0 \leq i \leq n$. The matrix A is now of the form

$$A = \left(\begin{array}{c|c} D_n & \\ \hline \mathbf{b}_n & \end{array} \right) = B_n,$$

where $D_n \in M_n(k)$, and $\mathbf{b}_n$ is a row vector of dimension n. Using Lemma 3.1 repeatedly and Remark 3.2, we have that X_A is isomorphic to one of the hypersurfaces defined by W_s with $0 \leq s \leq n$.

The proof of the Theorem 2.2 (2)

Next we prove that $s \neq s'$ implies $W_s \not\sim W_{s'}$. For this, we introduce some notions. Let X_s^n be the hypersurface defined by the matrix W_s in the projective space $\mathbb{P}^n$. The defining equation of X_s^n can be written as

$$F_q x_n + F_{q+1} = 0,$$

where

$$F_q = \begin{cases} 0 & \text{if } s = n \\ x_{n-1}^q & \text{if } s < n, \end{cases}$$

and

$$F_{q+1} = \begin{cases} x_0^{q+1} + \cdots + x_{n-1}^{q+1} & \text{if } s = n \\ x_0^{q+1} + \cdots + x_{s-1}^{q+1} + x_s^q x_{s+1} + \cdots + x_{n-2}^q x_{n-1} & \text{if } s < n. \end{cases}$$

It is easy to see that X_s^n has only one singular point $P_0 = (0, \cdots, 0, 1)$.

The proof of the Theorem 2.2 (3)

Let φ be the map defined by

$$\begin{array}{rcl} \varphi : \mathbb{P}^n \setminus \{P_0\} & \longrightarrow & \mathbb{P}^{n-1} \cong \{\text{the lines passing throught } P_0\} \\ P & \longmapsto & \overline{PP_0}. \end{array}$$

Let $\overline{X_s^n} = \varphi(X_s^n \setminus \{P_0\})$. For any line $l \in \overline{X_s^n}$, then

$$\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\}) = \begin{cases} \emptyset & \text{if } F_q = 0 \text{ and } F_{q+1} \neq 0, \\ \{\text{a single point}\} & \text{if } F_q \neq 0, \\ l \setminus \{P_0\} & \text{if } F_q = 0 \text{ and } F_{q+1} = 0. \end{cases}$$

The proof of the Theorem 2.2 (4)

Putting $V_s = \{F_q = 0,\ F_{q+1} = 0\} \subset \mathbb{P}^{n-1}$, and $H_s = \{F_q = 0\} \subset \mathbb{P}^{n-1}$, we have

$$V_s = \begin{cases} X_s^{n-2} & \text{if } s \leq n-2, \\ \text{nonsingular Fermat hypersurface in } \mathbb{P}^{n-1} & \text{if } s = n, \\ \text{nonsingular Fermat hypersurface in } \mathbb{P}^{n-2} & \text{if } s = n-1, \end{cases}$$

where X_s^{n-2} is the hypersurface in $\mathbb{P}^{n-2}$ associated with the matrix

$$\left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array} \right).$$

The proof of the Theorem 2.2 (5)

For any $s \neq s'$, suppose that X_s^n and $X_{s'}^n$ are isomorphic and let $\psi : X_s^n \longrightarrow X_{s'}^n$ be an isomorphism. Because each of X_s^n and $X_{s'}^n$ has only one singular point P_0, we have $\psi(P_0) = P_0$, and hence ψ induces an isomorphism $\overline{\psi}$ from $\overline{X_s^n}$ to $\overline{X_{s'}^n}$. For any line $l \in \overline{X_s^n}$ and $l' \in \overline{X_{s'}^n}$ such that $\overline{\psi}(l) = l'$, we have

$$\sharp(\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\})) = \sharp(\varphi^{-1}(l') \cap (X_{s'}^n \setminus \{P_0\})).$$

Thus $V_s \cong V_{s'}$ and $H_s \cong H_{s'}$. Hence for any $s \neq s'$, if $V_s \ncong V_{s'}$ or $H_s \ncong H_{s'}$ then $X_s^n \ncong X_{s'}^n$.

In the case $n = 1$, we have that X_0^1 consists of two points, and X_1^1 consists of a single point. In the case $n = 2$, we have that X_0^2 consists of two irreducible components, X_1^2 is irreducible, and X_2^2 consists of $(q+1)$ lines. Hence, in the case $n = 1$ and $n = 2$, we see that $s \neq s'$ implies $W_s \nsim W_{s'}$. By induction on n, we have the proof. □

1. Introduction
2. The main theorems and corollaries
   - Theorem 2.2 and corollaries
   - Theorem 2.5
3. The proof of the main theorems
   - The proof of the Theorem 2.2
4. The case of plane curves

The case of plane curves (1)

Theorem 4.1

Let $A = (a_{ij})_{0 \leq i,j \leq 2} \in M_3(k)$ *be a nonzero matrix and let* X_A *be the curve defined by* $\sum a_{ij} x_i x_j^q = 0$ *in* $\mathbb{P}^2$*. Suppose that the rank of* A *is smaller than 3.*

(i) *When the rank of* A *is 1, the curve* X_A *is projectively isomorphic to one of the following curves*

$$Z_0 : x_0^{q+1} = 0, \text{ or } Z_1 : x_0^q x_1 = 0.$$

(ii) *When the rank of* A *is 2, the curve* X_A *is projectively isomorphic to one of the following curves*

$$X_0 : x_0^q x_1 + x_1^q x_2 = 0, \; X_1 : x_0^{q+1} + x_1^q x_2 = 0, \text{or } X_2 : x_0^{q+1} + x_1^{q+1} = 0.$$

The case of plane curves (2)

Remark 4.2

In fact, the case when the plane curve X_A of degree $p+1$ has been proved by Homma.

Note that the plane curve X_1 is strange. Moreover this curve is irreducible and nonreflexive. Ballico and Hefez (1991) proved that a reduced irreducible nonreflexive plane curve of degree $q+1$ is isomorphic to one of the following curves:

(1) $X_I \; : \; x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0$,

(2) a nodal curve whose defining equation is given by Fukasawa (2013), Hoang and Shimada (2015),

(3) strange curves.