Non-Kähler complex structures on $\mathbb{R}^4$

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2016.3.8

Our problem

A complex mfd (M, J) is said to be Kähler if there exists a symplectic form ω compatible with J, i.e.,
(1) $\omega(u, Ju) > 0$ for any $u \neq 0 \in TM$,
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- If $n = 1$, the answer is "No".
- If $n \geq 3$, "Yes" (Calabi-Eckmann).
- Then, what about if $n = 2$?

Calabi-Eckmann's construction

$H_1 : S^{2p+1} \to \mathbb{C}P^p$, $H_2 : S^{2q+1} \to \mathbb{C}P^q$: the Hopf fibrations.
$H_1 \times H_2 : S^{2p+1} \times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q$ is a T^2-bundle.
The Calabi-Eckmann manifold $M_{p,q}(\tau)$ is a complex mfd diffeo to $S^{2p+1} \times S^{2q+1}$ s.t. $H_1 \times H_2$ is a holomorphic torus bundle (τ is the modulus of a fiber torus).
$E_{p,q}(\tau)$: the top dim cell of the natural cell decomposition.
If $p > 0$ and $q > 0$, then it contains holomorphic tori.

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- This argument doesn't work if $p = 0$ or $q = 0$.

Non-Kählerness and holomorphic curves

Lemma (1)

If a complex manifold $(\mathbb{R}^{2n}, J)$ contains a compact holomorphic curve C, then it is non-Kähler.

Proof.

Suppose it is Kähler. Then, there is a symp form ω compatible with J. Then, $\int_C \omega > 0$. On the other hand, ω is exact. By Stokes' theorem, $\int_C \omega = \int_C d\alpha = 0$. This is a contradiction. $\square$

Main Theorem

Let $P = \{0 < \rho_1 < 1, 1 < \rho_2 < \rho_1^{-1}\} \subset \mathbb{R}^2$.

Theorem

For any $(\rho_1, \rho_2) \in P$, there are a complex manifold $E(\rho_1, \rho_2)$ diffeomorphic to $\mathbb{R}^4$ and a surjective holomorphic map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$ such that the only singular fiber $f^{-1}(0)$ is an immersed holomorphic sphere with one node, and the other fiber is either a holomorphic torus or annulus.

The Matsumoto-Fukaya fibration

$f_{MF} : S^4 \to \mathbb{C}P^1$ is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.
F_1: the fiber with the positive singularity $\big((z_1, z_2) \mapsto z_1 z_2\big)$
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- $N_1 \cup (N_2 \backslash X) \cong \mathbb{R}^4$ (X is a nbd of $-$ sing),
- In the smooth sense, f is a restriction of f_{MF}.

The Matsumoto-Fukaya fibration 2

Originally, it is constructed by taking the composition of the Hopf fibration $H : S^3 \to \mathbb{C}P^1$ and its suspension $\Sigma H : S^4 \to S^3$. $f_{MF} = H \circ \Sigma H$.

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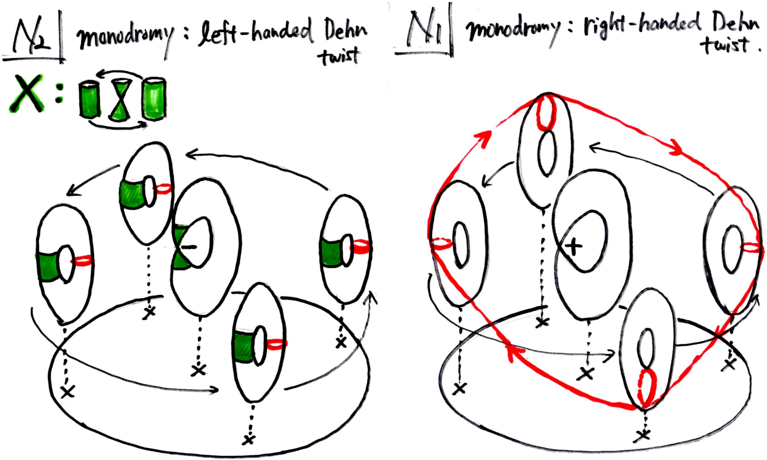
- **How to glue ∂N_2 to ∂N_1 is as the following pictures (in the next page).**

Gluing N_1 and N_2



Kirby diagrams

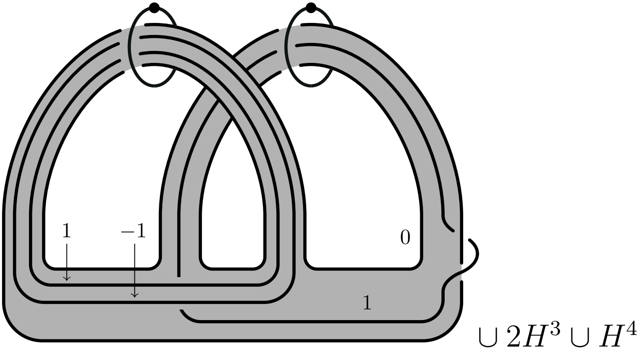


Figure: The Matsumoto-Fukaya fibration on S^4.

Kirby diagrams 2

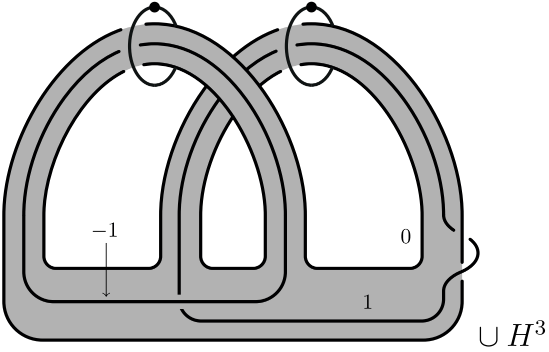


Figure: The map f on $S^4 \setminus X \cong \mathbb{R}^4$.

Key Lemma

Lemma (2)

Let us glue $A \times D^2$ *to* N_1 *so that for each* $t \in \partial D^2 = S^1$, *the annulus* $A \times \{t\}$ *embeds in the fiber torus* $f^{-1}(t)$ *as a thickned meridian, and that it rotates in the longitude direction once as* $t \in S^1$ *rotates once. Then, the interior of the resultant manifold is diffeomorphic to* $\mathbb{R}^4$.

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- **We will realize this gluing by complex manifolds!**

Kodaira's holomorphic model

$\Delta(r) = \{z \in \mathbb{C} \mid |z| < r\}$,
$\Delta(r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$.
Consider an elliptic fibration

$$\pi : \mathbb{C}^* \times \Delta(0, \rho_1)/\mathbb{Z} \to \Delta(0, \rho_1),$$

where the action is $n \cdot (z, w) = (zw^n, w)$.
It naturally extends to $f_1 : W \to \Delta(\rho_1)$.
W is a tubular neighborhood of a singular elliptic fiber of type I_1. It is a holomorphic model of N_1.

Gluing domains in the two pieces

The model for $N_2 \backslash X$ is $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.
The gluing domain is
$V_2 := \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

$$Y := \{(z\varphi(w), w) \in \mathbb{C}^* \times \Delta(\rho_0, \rho_1) \mid z \in \Delta(1, \rho_2)\},$$

where $\varphi(w) = \exp\left(\frac{1}{4\pi i}(\log w)^2 - \frac{1}{2}\log w\right)$.

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- $\varphi(re^{i(\theta+2\pi)}) = re^{i\theta}\varphi(re^{i\theta}) = w\varphi(w)$.

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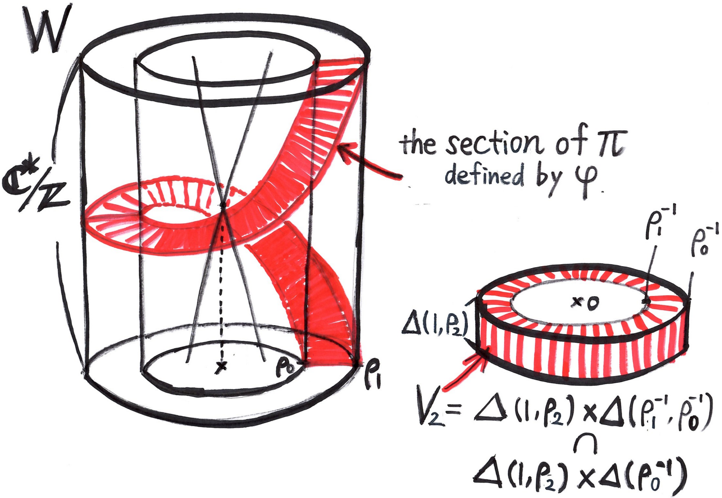
Gluing domains in the two pieces 2



Gluing the two pieces

By the biholomorphism between the gluing domains

$$\Phi : V_2 \to V_1;\ (z, w^{-1}) \mapsto [(z\phi(w), w)],$$

we obtain a complex manifold

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- $\Delta(\rho_1)$ and $\Delta(\rho_0^{-1})$ are glued to become $\mathbb{C}P^1$.
- f is defined to be $f_1 : W \to \Delta(\rho_1)$ on W, and the second projection on $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

Classification of holomorphic curves

Lemma (3)

Any compact holomorphic curve in $E(\rho_1, \rho_2)$ is a compact fiber of the map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$.

Proof.

Let $i : C \to E(\rho_1, \rho_2)$ be a compact holomorphic curve. The composition $f \circ i : C \to \mathbb{C}P^1$ is a holomorphic map between compact Riemann surfaces. It is either a brached covering or a constant map. Since it is homotopic to a constant map, it is a constant map. $\square$

Properties of $E(\rho_1, \rho_2)$

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- Any meromorphic function is the pullback of that on $\mathbb{C}P^1$ by f.

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- It cannot be holomorphically embedded in any compact complex surface.

Thank you for your attention!