On algebraic description of the Goldman-Turaev Lie bialgebra

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Contents

The Goldman-Turaev Lie bialgebra

Σ: a compact oriented surface
$\hat{\pi} = \hat{\pi}(\Sigma) := \pi_1(\Sigma)/\text{conjugacy} \cong \mathrm{Map}(S^1, \Sigma)/\text{homotopy}$

Two operations to loops on Σ

1. Goldman bracket ('86)

$$[\ ,\ ]\colon (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \to \mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}, \quad \alpha \otimes \beta \mapsto [\alpha, \beta]$$

   $\mathbf{1} \in \hat{\pi}$: the class of a constant loop

2. Turaev cobracket ('91)

$$\delta\colon \mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1} \to (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1})$$

Theorem (Goldman (bracket) +Turaev (cobracket, Lie bialgebra)+Chas (involutivity))

The triple $(\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}, [\ ,\ ], \delta)$ *is an involutive Lie bialgebra.*

Lie bialgebra

The operation $[\ ,\ ]$ is defined by using the intersection of two loops, while the operation δ by using the self-intersection of a loop.

Theorem (bis)

The triple $(\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}, [\ ,\ ], \delta)$ *is an involutive Lie bialgebra.*

Definition

A triple $(\mathfrak{g}, [\ ,\ ], \delta)$ is a Lie bialgebra if

1. the pair $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra,
2. the pair $(\mathfrak{g}, \delta)$ is a Lie coalgebra, and
3. the maps $[\ ,\ ]$ and δ satisfy a comatibility condition:

$$\forall \alpha, \beta \in \mathfrak{g}, \quad \delta[\alpha, \beta] = \alpha \cdot \delta(\beta) - \beta \cdot \delta(\alpha).$$

Moreover, if $[\ ,\ ] \circ \delta = 0$ then $(\mathfrak{g}, [\ ,\ ], \delta)$ is called involutive.

Fundamental group and tensor algebra

We have a binary operation $[\ ,\ ]$ and a unary operation δ on $\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}$.
The goal is to express them algebraically, i.e., by using tensors.

Assume $\partial\Sigma \neq \emptyset$ (e.g., $\Sigma = \Sigma_{g,1}$, $\Sigma = \Sigma_{0,n+1}$). Then any "group-like" Magnus expansion θ gives an isomorphism (of complete Hopf algebras)

$$\theta \colon \widehat{\mathbb{Q}\pi_1(\Sigma)} \xrightarrow{\cong} \widehat{T}(H)$$

onto the complete tensor algebra generated by $H := H_1(\Sigma;\mathbb{Q})$.
Moreover, we have an isomorphism (of $\mathbb{Q}$-vector spaces)

$$\theta \colon \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} \widehat{T}(H)^{cyc}.$$

Here,

1. the source $\widehat{\mathbb{Q}\hat{\pi}}$ is a certain completion of $\mathbb{Q}\hat{\pi}$,
2. ^{cyc} means taking the space of cyclic invariant tensors.

Algebraic description of the Goldman bracket

We can define $[\ ,\ ]^\theta$ by the commutativity of the following diagram.

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\hat{\pi} & \xrightarrow{[\ ,\ ]} & \mathbb{Q}\hat{\pi} \\ {\scriptstyle \theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \widehat{T}(H)^{\mathrm{cyc}} \widehat{\otimes} \widehat{T}(H)^{\mathrm{cyc}} & \xrightarrow{[\ ,\ ]^\theta} & \widehat{T}(H)^{\mathrm{cyc}} \end{array}$$

Theorem(Kawazumi-K., Massuyeau-Turaev), stated roughly

For some choice of θ, $[\ ,\ ]^\theta$ has a simple, θ-independent expression.

1. For $\Sigma = \Sigma_{g,1}$, it equals the associative version of the Lie algebra of symplectic derivations introduced by Kontsevich.
2. For $\Sigma = \Sigma_{0,n+1}$, it equals the Lie algebra of special derivations in the sense of Alekseev-Torossian (c.f. the work of Ihara).

Algebraic description of the Turaev cobracket

Similarly, we can define δ^θ by the commutativity of the following diagram.

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1} & \xrightarrow{\delta} & (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \\ \theta \downarrow & & \downarrow \theta\otimes\theta \\ \widehat{T}(H)^{\mathsf{cyc}} & \xrightarrow{\delta^\theta} & \widehat{T}(H)^{\mathsf{cyc}} \otimes \widehat{T}(H)^{\mathsf{cyc}} \end{array}$$

Question

Can we have a simple expression for δ^θ?

Our motivation: the Johnson homomorphism
$\mathcal{I}(\Sigma)$: the Torelli group of Σ
$\mathfrak{h}(\Sigma)$: Morita's Lie algebra (Kontsevich's "lie")

$$\mathcal{I}(\Sigma) \overset{\tau}{\hookrightarrow} \mathfrak{h}(\Sigma) \overset{\text{Kawazumi-K}}{\hookrightarrow} \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\delta} \widehat{\mathbb{Q}\hat{\pi}} \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}.$$

Then $\mathrm{Im}(\tau) \subset \mathrm{Ker}(\delta)$. For instance, the Morita trace factors through δ.

Definition of the Goldman bracket

Recall: $\hat{\pi} = \hat{\pi}(\Sigma) = \mathrm{Map}(S^1, \Sigma)/\text{homotopy}$.

Definition (Goldman)

$\alpha, \beta \in \hat{\pi}$: represented by free loops in general position

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \alpha_p \beta_p \in \mathbb{Q}\hat{\pi}.$$

Here, $\varepsilon_p(\alpha, \beta) = \pm 1$ is the local intersection number of α and β at p, and α_p is the loop α based at p.

This formula induces a Lie bracket on $\mathbb{Q}\hat{\pi}$, and $\mathbf{1} \in \hat{\pi}$ is centeral.

Background

Study of the Poisson structures on $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

The action σ

For $*_0, *_1 \in \partial\Sigma$, $\Pi\Sigma(*_0, *_1) := \mathrm{Map}(([0,1],0,1),(\Sigma,*_0,*_1))/\text{homotopy}$.

Definition (Kawazumi-K.)

For $\alpha \in \hat{\pi}$ and $\beta \in \Pi\Sigma(*_0, *_1)$,

$$\sigma(\alpha)\beta := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\beta_{*_0 p}\alpha_p \beta_{p *_1} \in \mathbb{Q}\Pi\Sigma(*_0, *_1).$$

This formula induces a $\mathbb{Q}$-linear map

$$\sigma = \sigma_{*_0, *_1} \colon \mathbb{Q}\hat{\pi} \to \mathrm{End}(\mathbb{Q}\Pi\Sigma(*_0, *_1)).$$

The Leibniz rule holds: for $\beta_1 \in \Pi\Sigma(*_0, *_1)$ and $\beta_2 \in \Pi\Sigma(*_1, *_2)$,

$$\sigma(\alpha)(\beta_1\beta_2) = (\sigma(\alpha)\beta_1)\beta_2 + \beta_1(\sigma(\alpha)\beta_2).$$

The action σ (continued)

Write $\partial\Sigma = \coprod_i \partial_i\Sigma$ with $\partial_i\Sigma \cong S^1$. For each i, choose $*_i \in \partial_i\Sigma$.

The small category $\mathbb{Q}\Pi\Sigma$

- Objects: $\{*_i\}_i$
- Morphisms: $\mathbb{Q}\Pi\Sigma(*_i, *_j)$

Consider the Lie algebra

$$\begin{aligned}&\mathrm{Der}(\mathbb{Q}\Pi\Sigma)\\ :=&\{(D_{i,j})_{i,j} \mid D_{i,j} \in \mathrm{End}(\mathbb{Q}\Pi\Sigma(*_i, *_j)), D_{i,j} \text{ satisfy the Leibniz rule.}\}\end{aligned}$$

Then the collection $(\sigma_{*_i,*_j})_{i,j}$ defines a Lie algebra homomorphism

$$\sigma\colon \mathbb{Q}\hat{\pi} \to \mathrm{Der}_{\partial}(\mathbb{Q}\Pi\Sigma).$$

Example

If $\partial\Sigma = S^1$, we have $\sigma\colon \mathbb{Q}\hat{\pi} \to \mathrm{Der}_\partial(\mathbb{Q}\pi_1(\Sigma))$.

Completions

We have a Lie algebra homomorphism

$$\sigma\colon \mathbb{Q}\hat{\pi} \to \mathrm{Der}_{\partial}(\mathbb{Q}\Pi\Sigma).$$

The augumentation ideal $I \subset \mathbb{Q}\pi_1(\Sigma)$ defines a filtration $\{I^m\}$ of $\mathbb{Q}\pi_1(\Sigma)$. We set

$$\widehat{\mathbb{Q}\pi_1(\Sigma)} := \varprojlim_m \mathbb{Q}\pi_1(\Sigma)/I^m.$$

Likewise, we can consider the completions of $\mathbb{Q}\hat{\pi}$ and $\mathbb{Q}\Pi\Sigma$.
For example,

1. the Goldman bracket induces a complete Lie bracket $[\ ,\ ]\colon \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\hat{\pi}}$,
2. we get a Lie algebra homomorphism

$$\sigma\colon \widehat{\mathbb{Q}\hat{\pi}} \to \mathrm{Der}_{\partial}(\widehat{\mathbb{Q}\Pi\Sigma}).$$

Magnus expansion

Let π be a free group of finite rank.
Set $H := \pi^{\mathrm{abel}} \otimes \mathbb{Q} \cong H_1(\pi; \mathbb{Q})$ and $\widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$.

Definition (Kawazumi)

A map $\theta\colon \pi \to \widehat{T}(H)$ is called a (generalized) Magnus expansion if

1. $\theta(x) = 1 + [x] + (\text{terms with deg} \geq 2)$,
2. $\theta(xy) = \theta(x)\theta(y)$.

Definition (Massuyeau)

A Magnus expansion θ is called group-like if $\theta(\pi) \subset \mathrm{Gr}(\widehat{T}(H))$.

If θ is a group-like Magnus expansion, then we have an isomorphism

$$\theta\colon \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}(H)$$

of complete Hopf algebras.

The case of $\Sigma = \Sigma_{g,1}$

Definition (Massuyeau)

A group-like expansion $\theta\colon \pi_1(\Sigma) \to \widehat{T}(H)$ is called symplectic if $\theta(\partial\Sigma) = \exp(\omega)$, where $\omega \in H^{\otimes 2}$ corresponds to $1_H \in \mathrm{Hom}(H,H) = H^* \otimes H \underset{\text{P.d.}}{\cong} H \otimes H$.

Fact: symplectic expansions do exist.

The Lie algebra of symplectic derivations (Kontsevich):

$$\mathrm{Der}_\omega(\widehat{T}(H)) := \{D \in \mathrm{End}(\widehat{T}(H)) \mid D \text{ is a derivation and } D(\omega) = 0\}.$$

The restriction map

$$\mathrm{Der}_\omega(\widehat{T}(H)) \to \mathrm{Hom}(H, \widehat{T}(H)) \underset{\text{P.d.}}{\cong} H \otimes \widehat{T}(H) \subset \widehat{T}(H), \quad D \mapsto D|_H$$

induces a $\mathbb{Q}$-linear isomorphism $\mathrm{Der}_\omega(\widehat{T}(H)) \cong \widehat{T}(H)^{\mathrm{cyc}}$.

The case of $\Sigma = \Sigma_{g,1}$: the Goldman bracket

Consider the diagram

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\hat{\pi} & \xrightarrow{[\ ,\ ]} & \mathbb{Q}\hat{\pi} \\ {\scriptstyle \theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \widehat{T}(H)^{cyc} \widehat{\otimes} \widehat{T}(H)^{cyc} & \xrightarrow{[\ ,\ ]^{\theta}} & \widehat{T}(H)^{cyc} \end{array}$$

where the vertical map θ is induced by $\pi \ni x \mapsto -(\theta(x) - 1) \in \widehat{T}(H)$.

Theorem (Kawazumi-K., Massuyeau-Turaev)

If θ is symplectic, $[\ ,\ ]^{\theta}$ equals the Lie bracket in $\widehat{T}(H)^{cyc} = \mathrm{Der}_{\omega}(\widehat{T}(H))$.

Explicit formula: for $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in H$,

$$[X_1 \cdots X_m, Y_1 \cdots Y_n]^{\theta} = \sum_{i,j} (X_i \cdot Y_j) X_{i+1} \cdots X_m X_1 \cdots X_{i-1} Y_{j+1} \cdots Y_n Y_1 \cdots Y_{j-1}.$$

The case of $\Sigma = \Sigma_{g,1}$: the action σ

Consider the diagram

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\pi_1(\Sigma) & \xrightarrow{\sigma} & \mathbb{Q}\pi_1(\Sigma) \\ {\scriptstyle \theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \widehat{T}(H)^{\mathrm{cyc}} \widehat{\otimes} \widehat{T}(H) & \longrightarrow & \widehat{T}(H) \end{array}$$

Here, the bottom horizontal arrow is the action of $\widehat{T}(H)^{\mathrm{cyc}} = \mathrm{Der}_\omega(\widehat{T}(H))$ by derivations.

Theorem (Kawazumi-K., Massuyeau-Turaev)

If θ is symplectic, this diagram is commutative.

- Kawazumi-K.: use (co)homology theory of Hopf algebras
- Massuyeau-Turaev: use the notion of Fox paring (see the next page)

The case of $\Sigma = \Sigma_{g,1}$: a refinement

Homotopy intersection form (Turaev, Papakyriakopoulos)

For $\alpha, \beta \in \pi_1(\Sigma)$, set $\eta(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\alpha_{*p}\beta_{p*} \in \mathbb{Q}\pi_1(\Sigma)$.

Theorem (Massuyeau-Turaev)

If θ is symplectic, then the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{Q}\pi_1(\Sigma) \times \mathbb{Q}\pi_1(\Sigma) & \xrightarrow{\eta} & \mathbb{Q}\pi_1(\Sigma) \\ \downarrow{\scriptstyle \theta\otimes\theta} & & \downarrow{\scriptstyle \theta} \\ \widehat{T}(H)\widehat{\otimes}\widehat{T}(H) & \xrightarrow{(\ \overset{\bullet}{\leadsto}\ )+\rho_s} & \widehat{T}(H). \end{array}$$

Here, $X_1 \cdots X_m \overset{\bullet}{\leadsto} Y_1 \cdots Y_n = (X_m \cdot Y_1)X_1 \cdots X_{m-1}Y_2 \cdots Y_n$ *and*
$\rho_s(a, b) = (a - \varepsilon(a))s(\omega)(b - \varepsilon(b))$, *where*
$s(\omega) = \frac{1}{\omega} + \frac{1}{(e^{-\omega}-1)} = -\frac{1}{2} - \frac{\omega}{12} + \frac{\omega^3}{720} - \frac{\omega^5}{30240} + \cdots$. *(Bernoulli numbers appear!)*

The case of $\Sigma = \Sigma_{0,n+1}$

We regard $\Sigma_{0,n+1} = D^2 \setminus \coprod_{i=1}^n \mathrm{Int}(D_i)$. Then $H \cong \bigoplus_{i=1}^n \mathbb{Q}[\partial D_i]$.

Definition (Massuyeau (implicit in the work of Alekseev-Enriquez-Torossian))

A Magnus expansion θ is called special if

1. $\exists g_i \in \mathrm{Gr}(\widehat{T}(H))$ such that $\theta(\partial D_i) = g_i \exp([\partial D_i]) g_i^{-1}$,
2. $\theta(\partial D^2) = \exp([\partial D^2])$.

The Lie algebra of special derivations (in the sense of Alekeev-Torossian):

$$\begin{aligned} &\mathrm{sder}(\widehat{T}(H)) \\ :=&\{D \in \mathrm{Der}(\widehat{T}(H)) \mid D([\partial D_i]) = [[\partial D_i], \exists u_i], D([\partial D^2]) = 0\}. \end{aligned}$$

We can naturally identify $\mathrm{sder}(\widehat{T}(H))$ with $\widehat{T}(H)^{\mathrm{cyc}}$.

Theorem (Kawazumi-K., Massuyeau-Turaev)

If θ is special, then $[\ ,\ ]^\theta$ equals the Lie bracket in $\mathrm{sder}(\widehat{T}(H))$.

General case ($\partial\Sigma \neq \emptyset$)

Write $\Sigma = \Sigma_{g,n+1}$ and $\partial\Sigma = \coprod_{i=0}^{n} \partial_i\Sigma$.
Put $\overline{\Sigma} := \Sigma \cup (\coprod_{i=0}^{n} D^2) \cong \Sigma_g$.
Choose a section s of $i_*\colon H_1(\Sigma) \to H_1(\overline{\Sigma})$.
We need

1. a notion of Magnus expansion for the small category $\mathbb{Q}\Pi\Sigma$,
2. a (s-dependent) boundary condition for such an expansion θ.

Then, we have a simple (s-dependent) expression for $[\ ,\ ]^\theta$ and σ^θ.

An application:

Theorem (Kawazumi-K., the infinitesimal Dehn-Nielsen theorem)

For any Σ with $\partial\Sigma \neq \emptyset$, the map $\sigma\colon \widehat{\mathbb{Q}\hat{\pi}} \to \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi\Sigma})$ is a Lie algebra isomorphism.

Definition of the Turaev cobracket

Definition (Turaev)

$\alpha \in \hat{\pi}$: represented by a generic immersion

$$\delta(\alpha) := \sum_{p \in \Gamma_\alpha} \alpha_p^1 \otimes \alpha_p^2 - \alpha_p^2 \otimes \alpha_p^1 \in (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}).$$

Here:

- Γ_α is the set of double points of α,
- α_p^1, α_p^2 are two branches of α created by p. They are arranged so that (α_p^1, α_p^2) gives a positive frame of $T_p(\Sigma)$.

This formula induces a Lie cobracket on $\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}$.

Background

A skein quantization of Poisson algebras on surfaces.

Self-intersection μ

Definition (essentially introduced by Turaev)

$\alpha \in \pi_1(\Sigma)$: represented by a generic immersion

$$\mu(\alpha) := -\sum_{p \in \Gamma_\alpha} \varepsilon_p(\alpha)\ \alpha_{*p}\alpha_{p*} \otimes \alpha_p \in \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}).$$

This formula induces a $\mathbb{Q}$-linear map

$$\mu \colon \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}).$$

1. μ is a refinement of δ; we have
$$\delta(|\alpha|) = \mathrm{Alt}(|\ | \otimes \mathrm{id})\mu(\alpha),$$
where $|\ | \colon \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}$ is the natural projection.
2. The operations μ and δ extends naturally to completions.
3. There is a framed version of δ, related to the Enomoto-Satoh trace and Alekseev-Torossian's divergence cocycle (Kawazumi).

Algebraic description of μ at the graded level

Define μ^θ by the commutativity of the following diagram.

$$\begin{array}{ccc} \mathbb{Q}\pi_1(\Sigma) & \xrightarrow{\mu} & \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}) \\ \theta \downarrow & & \downarrow \theta\otimes\theta \\ \widehat{T}(H) & \xrightarrow{\mu^\theta} & \widehat{T}(H) \otimes \widehat{T}(H)^{\mathrm{cyc}} \end{array}$$

Theorem (Kawazumi-K., Massuyeau-Turaev)

For $\Sigma = \Sigma_{g,1}$ and for any symplectic expansion θ,

$$\mu^\theta = \mu^{alg} + \mu^\theta_{(0)} + \mu^\theta_{(1)} + \cdots ,$$

where $\mu^\theta_{(i)}$ is a map of degree i and μ^{alg} a map of degree -2. For $i \geq 1$, $\mu^\theta_{(i)}$ depend on the choice of θ, but μ^{alg} does not.

Algebraic description of δ at the graded level

Corollary

For $\Sigma = \Sigma_{g,1}$ and for any symplectic expansion θ,

$$\delta^\theta = \delta^{alg} + \delta^\theta_{(1)} + \cdots .$$

Explicit formula: for $X_1, \ldots, X_m \in H$,

$$\begin{aligned} &\delta^{\mathrm{alg}}(X_1 \cdots X_m) \\ &= -\sum_{i<j} (X_i \cdot X_j) \mathrm{Alt}\left(X_{i+1} \cdots X_{j-1} \otimes X_{j+1} \cdots X_m X_1 \cdots X_{i-1}\right). \end{aligned}$$

Open question

Is there a symplectic expansion θ such that $\delta^\theta = \delta^{\mathrm{alg}}$?

Note: for $g = 1$, there is a θ such that $\delta^\theta \neq \delta^{\mathrm{alg}}$. Namely,

$$\{\text{symplectic expansions}\} \supsetneq \{\theta \mid \delta^\theta = \delta^{\mathrm{alg}}\}.$$

Algebraic description of δ: the case of $\Sigma_{0,n+1}$

For $\Sigma = \Sigma_{0,n+1}$ and for any special expansion θ,

$$\delta^{\theta} = \delta^{\mathrm{alg}} + \delta^{\theta}_{(1)} + \cdots ,$$

where δ^{alg} is a map of degree -1.

Explicit formula: for $X_1, \ldots, X_m \in H$,

$$\begin{aligned} &\delta^{\mathrm{alg}}(X_1 \cdots X_m) \\ =&\sum_{i<j} \delta_{X_i, X_j} \operatorname{Alt} \begin{pmatrix} X_i \cdots X_{j-1} \otimes X_{j+1} \cdots X_m X_1 \cdots X_{i-1} \\ +X_j \cdots X_m X_1 \cdots X_{i-1} \otimes X_{i+1} \cdots X_{j-1} \end{pmatrix} \end{aligned}$$

The proof uses a capping argument: consider the embedding

$$\Sigma_{0,n+1} \hookrightarrow \Sigma_{0,n+1} \cup \left(\coprod_{i=1}^{n} \Sigma_{1,1} \right) = \Sigma_{n,1}.$$

Recent development

Why is δ^θ more difficult than $[\ ,\ ]^\theta$? The main reason is that

$$\mathrm{Self}(\alpha\beta) = \mathrm{Self}(\alpha) \sqcup \mathrm{Self}(\beta) \sqcup (\alpha \cap \beta).$$

Partial results

1. For $\Sigma = \Sigma_{0,n+1}$, Kawazumi obtained a description of δ^θ with respect to the exponential Magnus expansion ($\theta(x_i) = \exp([x_i])$).
2. For $\Sigma = \Sigma_{1,1}$, there is a symplectic expansion θ such that $\delta^\theta = \delta^{\mathrm{alg}}$ modulo terms of degree ≥ 9. (K., using computer)

Theorem (Massuyeau '15)

Let $\Sigma = \Sigma_{0,n+1}$. For a special expansion θ arising from the Kontsevich integral, δ^θ equals δ^{alg}. (Actually a description for μ^θ is obtained.)

Summary

Two operations to loops on Σ

$$[\ ,\ ]\colon \widehat{\mathbb{Q}\hat{\pi}}\hat{\otimes}\widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\hat{\pi}}, \quad \overset{\text{refinement}}{\rightsquigarrow} \quad \eta\colon \widehat{\mathbb{Q}\pi_1(\Sigma)}\hat{\otimes}\widehat{\mathbb{Q}\pi_1(\Sigma)} \to \widehat{\mathbb{Q}\pi_1(\Sigma)}$$

$$\delta\colon \widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\hat{\pi}}\hat{\otimes}\widehat{\mathbb{Q}\hat{\pi}}, \quad \overset{\text{refinement}}{\rightsquigarrow} \quad \mu\colon \widehat{\mathbb{Q}\pi_1(\Sigma)} \to \widehat{\mathbb{Q}\pi_1(\Sigma)}\hat{\otimes}\widehat{\mathbb{Q}\hat{\pi}}$$

Current status of finding a simple expression for $[\ ,\ ]^\theta$ and δ^θ:

	Magnus expansion	$[\ ,\ ]^\theta$	δ^θ
$\Sigma_{g,1}$	symplectic	OK	?
$\Sigma_{0,n+1}$	special	OK	OK (Massuyeau)
general case	a ∂-condition	OK	?

1. We know that $\mathrm{gr}(\delta^\theta) = \delta^{\mathsf{alg}}$.
2. To get "?", we need a refinement of symplectic/special condition.