Autonomous limit of 4-dimensional Painlevé-type equations and degeneration of curves of genus two

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Goal

Geometrically understand the higher-dimensional analogues of the 2-dimensional Painlevé equations.

Idea

Characterize integrable systems studying the degeneration of the Liouville tori and the spectral curves.

Contents

1. A review of Painlevé-type equations
2. Classification of 4-dimensional Painlevé-type equations
3. Autonomouos (isospectral) limit of Painlevé-type equations
4. Degeneration of the spectral curves and the Liouville tori

The Painlevé equations

- The Painlevé equations are **8 types of nonlinear second order ordinary differential equations with the Painlevé property** (its general solution has no critical singularities that depend on initial values) other than linear equations, differential equations satisfied by elliptic functions, equations solvable by quadratures.
- The Painlevé equations govern the **isomonodromic deformation** of certain linear equations.
- The Painlevé equations can be expressed as **Hamiltonian systems**.
- These 8 equations are linked by certain limiting processes (**degeneration**).

$$
\begin{aligned}
\mathrm{P_I} \quad \frac{dy^2}{dt^2} &= 6y^2 + t,\\
\mathrm{P_{II}} \quad \frac{dy^2}{dt^2} &= 2y^3 + ty + \alpha,\\
\mathrm{P_{IV}} \quad \frac{dy^2}{dt^2} &= \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y},\\
\mathrm{P_{III}^{D_8}} \quad \frac{d^2y}{dt^2} &= \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{y^2}{t^2} - \frac{1}{t},\\
\mathrm{P_{III}^{D_7}} \quad \frac{d^2y}{dt^2} &= \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} - \frac{2y^2}{t^2} + \frac{\beta}{4t} - \frac{1}{y},\\
\mathrm{P_{III}^{D_6}} \quad \frac{d^2y}{dt^2} &= \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{\alpha y^1}{4t^2} + \frac{\beta}{4t} + \frac{\gamma y^3}{4t^2} + \frac{\delta}{4y}, \quad \gamma\delta \neq 0\\
\mathrm{P_V} \quad \frac{dy^2}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1},\\
\mathrm{P_{VI}} \quad \frac{dy^2}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}\\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right).
\end{aligned}
$$

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The Painlevé equations and Isomonodromic Deformation

The Painlevé equations has another important aspect; they govern isomonodromic deformation of certain linear equations.
Let us consider m coupled linear ODE of first order:

$$\frac{d}{dx}y(x) = A(x)y(x).$$

Here $y(x)$ is an m-component vector, and $A(x)$ is an $m \times m$ matrix, rational in x and has poles at t_ν $(\nu = 1, \ldots, n)$ and at $t_\infty = \infty$. Let us consider a fundamental matrix solution $Y(x)$:

$$\frac{d}{dx}Y(x) = A(x)Y(x). \tag{1}$$

Let $\mathbb{P}^1_t = \mathbb{P}^1 \setminus \{t_1, \ldots, t_n, t_\infty\}$, and $\pi\colon \overline{\mathbb{P}^1_t} \to \mathbb{P}^1_t$ the universal covering. Let γ be a path in $\overline{\mathbb{P}^1_t}$, starting at the point x, and ending at x_γ such that $\pi(x) = \pi(x_\gamma)$. There exists a nonsingular constant matrix M_γ such that

$$Y(x_\gamma) = Y(x)M_\gamma.$$

The mapping $[\gamma] \mapsto M_\gamma$ defines a representation of the fundamental group of $\mathbb{P}^1_t$, the **monodromy representation** associated with the differential system.

$$\frac{\partial}{\partial x}Y(x,t) = A(x,t)Y(x,t), \tag{1}$$

Given a differential system (1), *is it possible to deform it while preserving its monodromy representation?* The answer is that to ensure the isomonodromy of the deformation, $Y(x)$**, as a function of deformation parameters, has to satisfy a set of linear partial differential equations**.

$$\frac{\partial}{\partial t_i}Y(x,t) = B_i(x,t)Y(x,t).$$

Linear equation

$$\frac{\partial}{\partial x}Y(x,t) = A(x,t)Y(x,t),$$

admit isomonodromic deformation.

$$\downarrow$$

$Y(x,t)$ satisfies

$$\begin{cases} \frac{\partial}{\partial x}Y(x,t) = & A(x,t)Y(x,t) \\ \frac{\partial}{\partial t_i}Y(x,t) = & B_i(x,t)Y(x,t), \end{cases}$$

$$\Big\downarrow \quad \boxed{\text{Frobenius integrability}}$$

$$\frac{\partial A(x,t)}{\partial t_i} - \frac{\partial B_i(x,t)}{\partial x} + [A(x,t), B_i(x,t)] \tag{2}$$

The Painlevé equations are the solutions of (2) for certain $A(x,t)$'s.

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Hamiltonians of the Painlevé equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

$$\begin{aligned}
H_{\mathrm{I}}\,(t;q,p) &= p^2 - q^3 - tq,\\
H_{\mathrm{II}}\,(\alpha;t;q,p) &= p^2 - (q^2+t)p - \alpha q,\\
H_{\mathrm{IV}}\,(\alpha,\beta;t;q,p) &= pq(p-q-t) + \beta p + \alpha q,\\
tH_{\mathrm{III}}(D_8)\,(t;q,p) &= p^2q^2 + qp - q - \frac{t}{q},\\
tH_{\mathrm{III}}(D_7)\,(\alpha;t;q,p) &= p^2q^2 + \alpha qp + tp + q,\\
tH_{\mathrm{III}}(D_6)\,(\alpha,\beta;t;q,p) &= p^2q^2 - (q^2 - \beta q - t)p - \alpha q,\\
tH_{\mathrm{V}}\,(\alpha,\beta,\gamma;t;q,p) &= p(p+t)q(q-1) + \beta pq + \gamma p - (\alpha+\gamma)tq,\\
t(t-1)H_{\mathrm{VI}}\,(\alpha,\beta,\gamma,\epsilon;t;q,p) &= q(q-1)(q-t)p^2\\
&\quad + \{\epsilon q(q-1) - (2\alpha+\beta+\gamma+\epsilon)q(q-t)\\
&\quad + \gamma(q-1)(q-t)\}p + \alpha(\alpha+\beta)(q-t).
\end{aligned}$$

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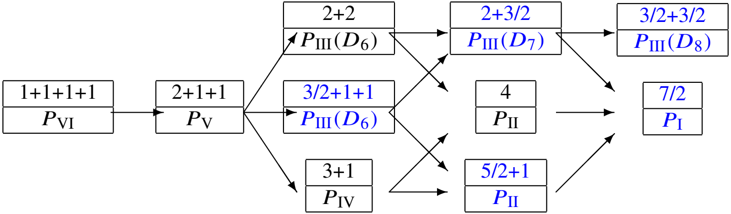
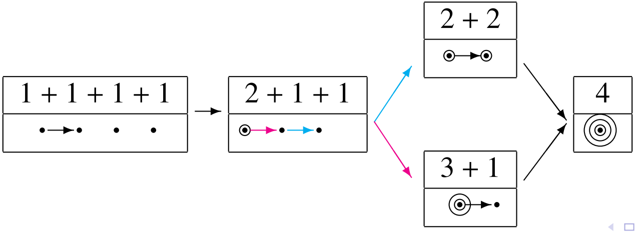
The first to fifth Painlevé equations can e derived from the sisth Painlevé equation by degeneration process.



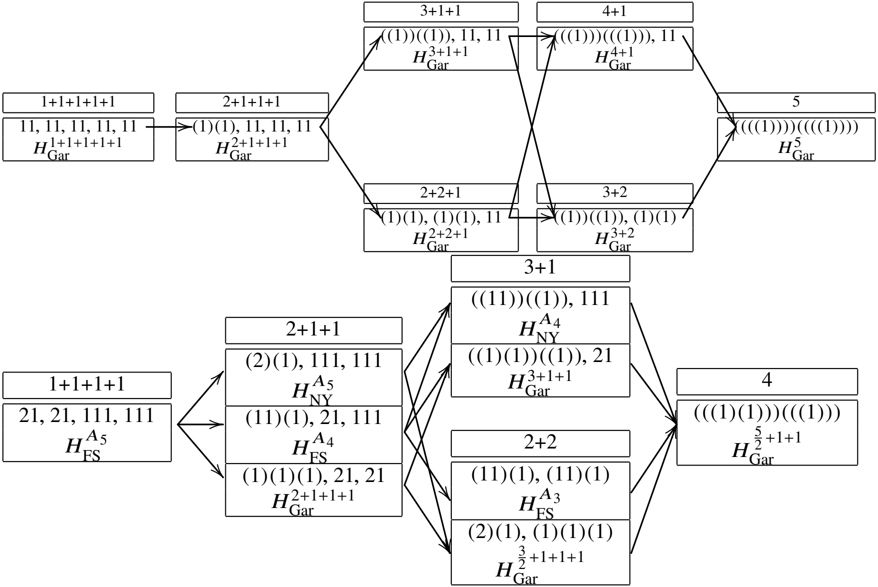
These degeneration processes correspond to the confluence of the singularities of corresponding linear equations. (cf. confluence of hypergeometric functions:Gauss to Kummer, Bessel, Hermite, Airy)

Confluence of Singular Points of linear equations

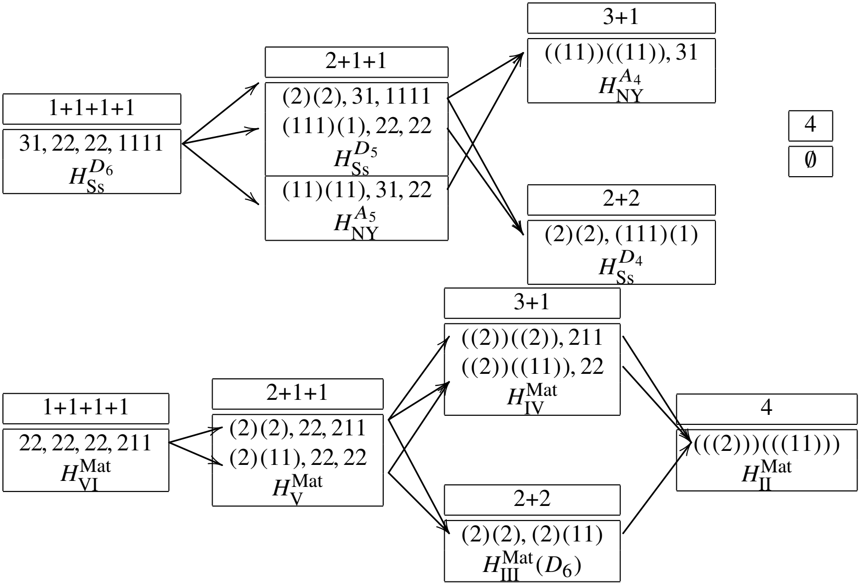


- Various extensions and analogues of the 2-dimensional Painlevé equations are known.
- The relations among the equations from different origins are not obvious clear and the overall picture is not clear
- Recently, (roughly speaking,) **a “classification" of the 4-dimensional Painlevé-type equations** was accomplished (Sakai [9], Kawakami-N.-Sakai [3], Kawakami [4]).
- The classification is based on the classification of the Fuchsian linear equations with 4 accessory parameters up to Katz’ operations and the degeneration processes.
- There are **40 types** of 4-dimensional Painlevé-type equations in their list.

Degeneration Scheme of the 4-dimensional Painlevé-type equations (KNS)



Degeneration Scheme of the 4-dimensional Painlevé-type equations (KNS)



- Some of these 40 equations look similar to each other.
  eg. H^{4+1}_{Gar} and $H^{\mathrm{Mat}}_{\mathrm{II}}$ pointed out by H. Chiba:

$$H^{4+1}_{\mathrm{Gar},t_1} = p_1^2 - \left(q_1^2 + t_1\right) p_1 + \kappa_1 q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2 + t_2\right) + \theta_0 q_2,$$

$$\tilde{H}^{\mathrm{Mat}}_{\mathrm{II}} = p_1^2 - \left(\frac{q_1^2}{4} + t\right) p_1 - \left(\theta_0 + \frac{\kappa_2}{2}\right) q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2\right) + \theta_0 q_2.$$

- Is there any way to distinguish these 40 types of equations?
- We need to study intrinsic nature (geometry) of these equations.
- Let us first consider easier cases; the autonomous cases(=integrable cases, the Hitchin systems).

Integrable system in Liouville's sense

Definition 1

A Hamiltonian system (M^{2n}, ω, H) is **(completely) integrable in Liouville's sense** if it possesses $n = \frac{1}{2} \dim M$ (i.e. the maximal number of) independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket; $\{f_i, f_j\} = 0$ for all i, j.

This definition of integrability is motivated by Liouville's theorem.

Theorem 1 (Arnold-Liouville)

Let (M, ω, H) be a completely integrable Hamiltonian system with integrals of motions $f_1 = H, \ldots, f_n$, and let M'_f denote the connected component of a regular level set that passes through $x \in M$. If M'_f is compact, then there exists a diffeomorphism from M'_f to the torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$, under which the vector fields $X_{f_1}, \ldots, X_{f_n}$ are mapped to linear (i.e. translation-invariant) vector fields.

Example (Harmonic oscillator)

The harmonic oscillator has a Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\alpha^2 q^2.$$

The phase space is fibered into ellipses $H = c$ except for the point $(0, 0)$ which is a stationary point. In the coordinates $p = \rho \cos\theta$, $q = \frac{\rho}{\alpha} \sin\theta$ the flow reads:

$$\rho = \sqrt{2H}, \quad \theta = \alpha t + \theta_0.$$

In the algebraic settings, we allow family of abelian varieties to have degenerate fibers.

Definition 2

An **algebraically completely integrable Hamiltonian system** consists of a proper fiat morphism $H: M \to B$ where M is a smooth Poisson variety and B is a smooth variety such that, over the complement $B \setminus \Delta$ of some proper closed subvariety $\Delta \subset B$, H is a Lagrangian fibration whose fibers are isomorphic to abelian varieties.

There is a correspondence between

- **Lax pair flows** of matricial polynomial:

$$\frac{d}{dt}A(x;t) = [P(A(x;t), x^{-1})_+, A(x,t)]$$

and

- **linear flows in the Jacobian** of the desingularization $\tilde{S}$ of the compactification of the **spectral curve**

$$S_0 = \left\{(x,y) \in \mathbb{C}^2 \mid \det(yI - A(x,t)) = 0\right\},$$

through the **eigenvector mappings**.

The Lax equation

Many integrable systems are known to have Lax pair expressions:

$$\frac{dA(x;t)}{dt} + [A(x;t), B(x;t)] = 0, \tag{3}$$

where $A(x;t)$ and $B(x;t)$ are m by m matrices and x is the spectral parameter. From this differential equation, $\mathrm{tr}\left(A(x;t)^k\right)$ are conserved quantities of the system:

$$\frac{d}{dt}\,\mathrm{tr}\left(A(x;t)^k\right) = \mathrm{tr}\left(k\,[A(x;t), B(x;t)]\,A(x;t)^{k-1}\right) = 0.$$

Therefore, all the eigenvalues and the coefficients of the characteristic polynomials are all conserved quantities.

Isomonodoromic and isospectral deformation

Isomonodormic deformation

$$\begin{cases} \dfrac{\partial Y(x,t)}{\partial x} = A(x,t)Y(x,t), \\ \dfrac{\partial Y(x,t)}{\partial t} = B(x,t)Y(x,t), \end{cases} \iff \frac{\partial A(x,t)}{\partial t} - \frac{\partial B(x,t)}{\partial x} + [A(x,t), B(x,t)] = 0.$$

Isospectral deformation

$$\begin{cases} A(x;t)Y(x;t) = Y(x;t)A_0(x;t_0), \\ \dfrac{dY(x;t)}{dt} = B(x;t)Y(x;t). \end{cases} \iff \frac{dA(x;t)}{dt} + [A(x;t), B(x;t)] = 0.$$

Remark

- The only difference is the existence of the term $\frac{\partial B}{\partial x}$ in isomonodromic deformation equation.
- In fact, we can consider isospectral problems as the special limit of isomonodromic problem with a parameter δ.

Isomonodromic deformation to isospectral deformation

To see its connection to the isospectral problem, we restate the isomonodromic problem as follows:

$$\begin{cases} \delta \dfrac{\partial Y}{dx} = A(x,\tilde{t})Y, \\ \dfrac{\partial Y}{\partial t} = B(x,\tilde{t})Y, \end{cases}$$

where $\tilde{t}$ is a variable which satisfies $\frac{d\tilde{t}}{dt} = \delta$. The integrability condition $\frac{\partial^2 Y}{\partial x \partial t} = \frac{\partial^2 Y}{\partial t \partial x}$ is equivalent to the following:

$$\frac{\partial A(x,\tilde{t})}{\partial t} - \delta \frac{\partial B(x,\tilde{t})}{\partial x} + [A(x,\tilde{t}), B(x,\tilde{t})] = 0. \tag{4}$$

$$\Longrightarrow (\delta = 1): \text{ (usual) Isomonodromic deformation}$$

$$\Longrightarrow (\delta = 0): \text{ Isospectral deformation}$$

Autonomous limit of Painlevé equation

Example (Autonomous limit of the second Painlevé equation)

We take the second Painlevé equation as an example.

$$\begin{cases} \delta \dfrac{\partial Y}{\partial x} = & A(x,\tilde{t})Y, \qquad A(x,\tilde{t}) = \left(A_{\infty}^{(-3)}(\tilde{t})x^2 + A_{\infty}^{(-2)}(\tilde{t})x + A_{\infty}^{(-1)}(\tilde{t})\right), \\ \dfrac{\partial Y}{\partial t} = & B(x,\tilde{t})Y \qquad B(x,\tilde{t}) = \left(A_{\infty}^{(-3)}(\tilde{t})x + B_1(\tilde{t})\right), \end{cases}$$

where

$$\hat{A}_{\infty}^{(-3)}(\tilde{t}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_{\infty}^{(-2)}(\tilde{t}) = \begin{pmatrix} 0 & 1 \\ -p + q^2 + \tilde{t} & 0 \end{pmatrix},$$

$$\hat{A}_{\infty}^{(-1)}(\tilde{t}) = \begin{pmatrix} -p + q^2 + \tilde{t} & q \\ (p - q^2 - \tilde{t})q - \kappa_2 & p - q^2 \end{pmatrix}, \quad \hat{B}_1(\tilde{t}) = \begin{pmatrix} q & 1 \\ p - q^2 - \tilde{t} & 0 \end{pmatrix},$$

$$A_{\infty}^{(-i)} = U^{-1}\hat{A}_{\infty}^{(-i)}U \quad \text{for } i = 1,2,3 \quad B_1 = U^{-1}\hat{B}_1 U, \quad U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Example (continued)

The deformation equation (4) is equivalent to the following differential equations.

$$\frac{dq}{dt} = 2p - q^2 - \tilde{t}, \qquad \frac{dp}{dt} = 2pq + \delta - \kappa_1, \qquad \frac{du}{dt} = 0.$$

The first two equations are equivalent to the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H_{\mathrm{II}}(\delta)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{II}}(\delta)}{\partial q},$$

with the Hamiltonian

$$H_{\mathrm{II}}(\delta) := p^2 - (q^2 + \tilde{t})p + (\kappa_1 - \delta)q.$$

When $\delta = 1$, it is the usual Hamiltonian of H_{II}.

Example (continued)

Taking the limit $\delta \to 0$, we obtain an integrable system with a Hamiltonian

$$h := H_{\mathrm{II}}(0) = p^2 - (q^2 + \tilde{t})p + \kappa_1 q,$$

and a Lax pair[a]

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0.$$

The spectral curve of the Lax pair is

$$\det(yI - A(x)) = y^2 - (x^2 + \tilde{t})y - \kappa_1 x - h = 0. \tag{5}$$

[a] We rewrite $A(x) = A(x, \tilde{t})$ and $B(x) = B(x, \tilde{t})$.

Autonomous limit of 4-dimensional Painlevé-type equation

One of the easiest examples in 4-dimensional Painlevé-type equation is the first matrix Painlevé equation [Kawakami].

Example (autonomous limit of the first matrix Painlevé equation)

The linear equation is given by

$$A(x) = \left(A_0 x^2 + A_1 x + A_2\right),\ B(x) = A_0 x + B_1,$$
$$A_0 = \begin{pmatrix} O_2 & I_2 \\ O_2 & O_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} O_2 & Q \\ I_2 & O_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -P & Q^2 + \tilde{t} I_2 \\ -Q & P \end{pmatrix},$$
$$B_1 = \begin{pmatrix} O_2 & 2Q \\ I_2 & O_2 \end{pmatrix}, Q = \begin{pmatrix} q_1 & u \\ -q_2/u & q_1 \end{pmatrix}, P = \begin{pmatrix} p_1/2 & -p_2 u \\ (p_2 q_2 - \kappa_2)/u & p_1/2 \end{pmatrix}.$$

The spectral curve is defined by

$$\det\left(y I_4 - A(x)\right) = y^4 - \left(2\,x^3 + 2\tilde{t} x + h\right) y^2 + x^6 + 2\tilde{t} x^4 + h x^3 + \tilde{t}^2 x^2 + \left(\tilde{t} h - \kappa_2^2\right) x + g = 0.$$

Example (continued)

The explicit forms of two functionally independent invariants are

$$
\begin{aligned}
h := H_{\mathrm{I}}^{\mathrm{Mat}} &= \operatorname{tr}\left(P^2 - Q^3 - \tilde{t}Q\right) \\
&= -2p_2\left(p_2 q_2 - \kappa_2\right) + \frac{p_1^2}{2} - 2q_1\tilde{t} - 2q_1\left(q_1^2 - q_2\right) + 4q_1 q_2, \\
g := G_{\mathrm{I}}^{\mathrm{Mat}} &= q_2\left(p_1 p_2 + 3q_1^2 - q_2 + \tilde{t}\right)^2 - \kappa_2 p_1 \\
&\qquad \left(p_1 p_2 + 3q_1^2 - q_2 + \tilde{t}\right) - 2\kappa_2^2 q_1 .
\end{aligned}
$$

From the similar direct computations, we obtain the following.

Theorem 2 (N.)

As the autonomous limits of 4-dimensional Painlevé-type equations, we obtain 40 types of integrable systems

Characterization of integrable systems

- The Fomenko school considered topological classification of 4-dimensional real integrable systems by studying bifurcation diagram (the set of images of critical points of the momentum mapping).
- **The degeneration of Liouville tori characterize integrable systems.**
- With this guiding principal, what can we conclude for our case, which are more natural to be considered as complex integrable systems?

Liouville tori fibration

Let us start from 2-dimensional integrable systems derived as the autonomous limits of the Painlevé equations. For the generic value $h \in \mathbb{C}$, the fiber of the momentum map

$$H_{\mathrm{J}} \colon M \to \mathbb{C}, \quad (p, q) \mapsto H_{\mathrm{J}}(p, q)$$

is an affine part of 1-dimensional complex torus, and can be completed into an elliptic curve. After extending the base curve to $\mathbb{P}^1$, an elliptic surface is naturally defined as the fibration of the Liouville tori.

Theorem 3

Each elliptic surface thus defined from the Liouville tori fibration of the autonomous 2-dimensional Painlevé equation has the following type of singular fiber over $\infty \in \mathbb{P}^1$.

	H_{VI}	H_{V}	$H_{\mathrm{III}(D_6)}$	$H_{\mathrm{III}(D_7)}$	$H_{\mathrm{III}(D_8)}$	H_{IV}	H_{II}	H_{I}
Kodaira	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Example (The autonomous first Painlevé equation)

The Hamiltonian of P_{I} is $H_{\mathrm{I}} = p^2 - (q^3 + \tilde{t}q)$. We view it as elliptic curve over $\mathbb{C}(h)$:

$$X_1 = \left\{(q, p, h) \in \mathbb{A}^2_{(q,p)} \times \mathbb{A}^1_h \mid p^2 = q^3 + \tilde{t}q + h\right\} \to \mathbb{A}^1_h.$$

Replacing $\bar{q} = q/h^2,\ \bar{p} = p/h^3,\ \bar{h} = 1/h$, we obtain the ∞-model:

$$X_2 = \left\{(\bar{q}, \bar{p}, \bar{h}) \in \mathbb{A}^2_{(\bar{q},\bar{p})} \times \mathbb{A}^1_{\bar{h}} \mid \bar{p}^2 = \bar{q}^3 + \tilde{t}\bar{h}^4\bar{q} + \bar{h}^5\right\} \to \mathbb{A}^1_{\bar{h}}.$$

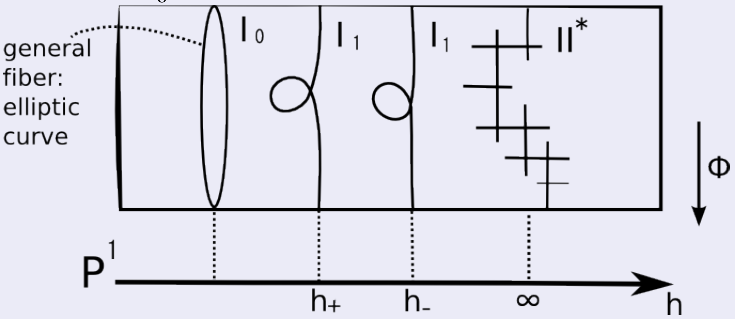
After minimal desingularization of the Weierstrass model $W = \bar{X}_1 \cup \bar{X}_2 \to \mathbb{P}^1$, we obtain the the desired elliptic surface.

Example

The discriminant and the j-invariant of $\bar{p}^2 = \bar{q}^3 + \tilde{t}\bar{h}^4\bar{q} + \bar{h}^5$ are

$$\Delta = \bar{h}^{10}(27 + 4\tilde{t}^3\bar{h}^2), \quad j = \bar{h}^2 \frac{\tilde{t}^3}{27 + 4\tilde{t}^3\bar{h}^2}$$

Thus, the singular fiber of $h = \infty$ of the Liouville tori fibration has Kodaira type II*, or $E_8^{(1)}$ in Dynkin's notation.



Liouville tori fibration (2-dimensional case)

- Consider a Hamiltonian of an autonomous 2-dimensional Painlevé equation in Weierstrass form

$$p^2 = q^3 + a(h)q + b(h), \qquad (a(h), b(h) \in \mathbb{C}[h]). \tag{6}$$

as an elliptic curve over $\mathbb{C}(h)$, where h is the Hamiltonian.

$\downarrow$ $\boxed{\text{compactification}}$

- Weierstrass model: $\phi\colon W \to \mathbb{P}^1$.

$\downarrow$ $\boxed{\text{minimal disingularization}}$

- Kodaira-Néron model: $\phi\colon S \to \mathbb{P}^1$.
- Possible singular fibers of elliptic surfaces are classified by Kodaira.
- Tate's algorithm tells the Kodaira type of singular fiber from the discriminant and the j-invariant of the equation (7).

Tate's algorithm

- Tate's algorithm: compute the type of fiber from

$$\Delta = 4a(h)^3 + 27b(h)^2 : \quad \text{discriminant of the cubic,}$$

$$j = \frac{4a(h)^3}{\Delta} : \quad j\text{-invariant}$$

of the equation $y^2 = x^3 + a(h)x + b(h)$.

- The Kodaira types of singular fibers are determined as in the table by the valuation $\mathrm{ord}_v(\Delta)$, $\mathrm{ord}_v(j)$ of Δ and j.

Kod.	Dynkin	$\mathrm{ord}_v(\Delta)$	$\mathrm{ord}_v(j)$	Kod.	Dynkin	$\mathrm{ord}_v(\Delta)$	$\mathrm{ord}_v(j)$
I_0	-	0	≥ 0	I_0^*	$D_4^{(1)}$	6	≥ 0
I_m	$A_{m-1}^{(1)}$	m	$-m$	I_m^*	$D_{4+m}^{(1)}$	$6+m$	$-m$
II	-	2	≥ 0	IV^*	$E_6^{(1)}$	8	≥ 0
III	$A_1^{(1)}$	3	≥ 0	III^*	$E_7^{(1)}$	9	≥ 0
IV	$A_2^{(1)}$	4	≥ 0	II^*	$E_8^{(1)}$	10	≥ 0

: Tate's algorithm and Kodaira types

For generic value $h \in \mathbb{C}$, the fiber of momentum map

$$H_{\mathrm{J}}\colon M \to \mathbb{C}, \quad (p, q) \mapsto H_{\mathrm{J}}(p, q),$$

the fiber is a genus 1 curve. After transforming the curve into the Weierstrass form, we can follow the same procedure as in the case of H_{I}, and the result follows.

Remark

The configurations of the components of the singular fibers are exactly the same as those of the anticanonical divisors of the corresponding Okamoto's space of initial conditions of the Painlevé equations.

	H_{VI}	H_{V}	$H_{\mathrm{III}(\mathrm{D}_6)}$	$H_{\mathrm{III}(\mathrm{D}_7)}$	$H_{\mathrm{III}(\mathrm{D}_8)}$	H_{IV}	H_{II}	H_{I}
Dyn	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

The intersection diagram of the aniticanonical divisors of Okamoto's space of the 2-dimensional Painlevé equations

Degenerations of spectral curves

- For 4-dimensional integrable systems, the Liouville tori are two dimensional (i.e. compactified to Abelian surfaces)

$$\mu\colon M \to \mathbb{C}^2 \quad (q_1, p_1, q_2, p_2) \mapsto (H_1, H_2).$$

  $\to$ not easy to study their degenerations.
- Jacobians of spectral curves can be identified with the Liouville tori via the eigenvector mapping.
- $\boxed{\text{bifurcation of Liouville tori}}$

  $\simeq \boxed{\text{discriminant locus of the spectral curve fibration}}$
- It is easier to study the degeneration of the spectral curves than the degeneration of the Liouville tori.

Theorem 4

The elliptic surface constructed as the spectral curve fibration of each autonomous 2-dimensional Painlevé equation has the singular fiber of the following type over ∞.

	H_{VI}	H_{V}	$H_{\mathrm{III}(D_6)}$	$H_{\mathrm{III}(D_7)}$	$H_{\mathrm{III}(D_8)}$	H_{IV}	H_{II}	H_{I}
Kod	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dyn	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

The singular fiber at $h = \infty$ of spectral curve fibrations of autonomous 2-dimensional Painlevé equations

Outline of the proof.

- Derive defining equations of spectral curves from Lax equations.
- Transform spectral curves into Weierstrass form (using Magma, Sage or Maple).
- Compactification and minimal desingularization.
- Compute discriminants and j-invariants and apply Tate's algorithm to find the types of singular fibers.

Spectral curve fibration

Example (spectral curve fibration of the first Painlevé equation)

$$\frac{d^2q}{dt^2} = 6q^2 + \tilde{t} \Longleftrightarrow \frac{\partial A}{\partial t} - \delta \frac{\partial B}{\partial x} + [A, B] = 0,$$

$$A(x) = \begin{pmatrix} -p & x^2 + qx + q^2 + \tilde{t} \\ x - q & p \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & x + 2q \\ 1 & 0 \end{pmatrix}.$$

The spectral curve associated with its autonomous equation is defined by

$$\det (yI_2 - A(x)) = 0.$$

This is equivalent to

$$y^2 = x^3 + \tilde{t}x + H_{\mathrm{I}},$$

where $h := H_{\mathrm{I}} = p^2 - q^3 + \tilde{t}q$. We view it as the defining equation of elliptic curve over $\mathbb{A}^1_h$.

Example (continued)

- Let X_1 be the affine surface defined by equation $y^2 = x^3 + \tilde{t}x + h$:

$$X_1 = \left\{(x, y, h) \in \mathbb{A}^2_{(x,y)} \times \mathbb{A}^1_h \mid y^2 = x^3 + \tilde{t}x + h\right\} \to \mathbb{A}^1_h.$$

- Upon replacing $\bar{h} = h^{-1},\ \bar{x} = h^{-2}x,\ \bar{y} = h^{-3}y$ we obtain "∞-model":

$$X_2 = \left\{(\bar{x}, \bar{y}, \bar{h}) \in \mathbb{A}^2_{(\bar{x},\bar{y})} \times \mathbb{A}^1_{\bar{h}} \mid \bar{y}^2 = \bar{x}^3 + \tilde{t}\bar{h}^4\bar{x} + \bar{h}^5\right\} \to \mathbb{A}^1_{\bar{h}}.$$

- Weierstrass model $\varphi\colon W = \overline{X_1} \cup \overline{X_2} \to \mathbb{P}^1 = \mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$.

$$\Big\downarrow\ \boxed{\text{minimal disingularization}}$$

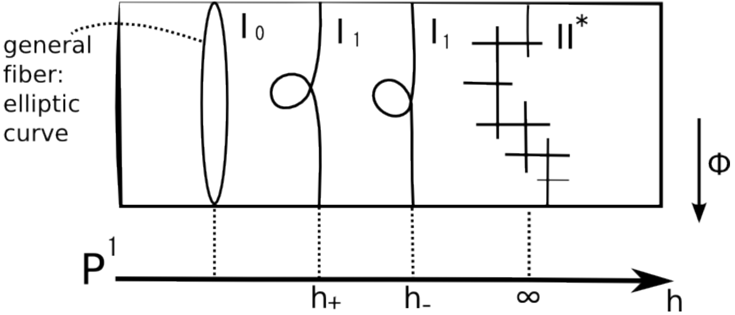
- Kodaira-Néron model $\phi\colon S \to \mathbb{P}^1 = \mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$.

The discriminant and the j-invariant are $\bar{x}^3 + \tilde{t}\bar{h}^4\bar{x} + \bar{h}^5$ is

$$\Delta = 4\left(\tilde{t}\bar{h}^4\right)^3 + 27\left(\bar{h}^5\right)^2 = \bar{h}^{10}\left(27 + 4\tilde{t}^3\bar{h}^2\right), \ \mathrm{ord}_\infty(\Delta) = 10$$

$$j = \frac{4}{\Delta}\left(\tilde{t}\bar{h}^4\right)^3 = \frac{4\tilde{t}^3\bar{h}^{12}}{\bar{h}^{10}(27 + 4\tilde{t}^3\bar{h}^2)} = \frac{4\tilde{t}^3\bar{h}^2}{27 + 4\tilde{t}^3\bar{h}^2}, \ \mathrm{ord}_\infty(j) = 2.$$

The surface $S \to \mathbb{P}^1$ has singular fiber of type II^* (or $E_8^{(1)}$ in Dynkin's notation) at $h = \infty$.



Spectral curve fibration (2-dimensional case)

- Consider a spectral curve of an autonomous 2-dimensional Painlevé equation in Weierstrass form

$$y^2 = x^3 + a(h)x + b(h), \qquad (a(h), b(h) \in \mathbb{C}[h]). \tag{7}$$

as an elliptic curve over $\mathbb{C}(h)$, where h is the Hamiltonian.

$$\Big\downarrow \ \boxed{\text{compactification}}$$

- Weierstrass model: $\phi \colon W \to \mathbb{P}^1$.

$$\Big\downarrow \ \boxed{\text{minimal disingularization}}$$

- Kodaira-Néron model: $\phi \colon S \to \mathbb{P}^1$.
- Possible singular fibers of elliptic surfaces are classified by Kodaira.
- Tate's algorithm tells the Kodaira type of singular fiber from the discriminant and the j-invariant of the equation (7).

For generic value $h \in \mathbb{C}$, the spectral curve of H_{J} ($\mathrm{J} = \mathrm{I}, \mathrm{II}, \mathrm{III}(D_6), \mathrm{III}(D_7), \mathrm{III}(D_8), \mathrm{IV}, \mathrm{V}, \mathrm{IV}$) is a genus 1 curve. After transforming the curve into the Weierstrass form, we can follow the same procedure as in the case of H_{I}, and the result follows.

Remark

- The agreements of singular fibers at $h = \infty$ of the spectral curve fibrations and the Liouville tori fibrations are not coincidences.
- The Liouville tori are related to the Jacobians of spectral curves through eigenvector mapping, and taking Jacobian is isomorphism in genus 1 cases.

Genus 2 fibration and Liu's algorithm

- We apply a similar method to our 40 autonomous 4-dimensional Painlevé-type equations.
- The number of independent conserved quantities is 2.
- The diminsion of the Louvielle tori is 2 and the genus of the spectral curves is 2.
- Let h be one of the independent conserved quantity of the system. (We fix the other conserved quantity to the generic value.)
- We construct spectral curve fibrations from explicit forms of spectral curves in the Weierstrass form.

$$y^2 = \sum_{i=0}^{6} a_i(h) x^{6-i}$$

- Attach another affine model given by $\bar{x} = x/h,\ \bar{y} = y/h^3,\ \bar{h} = 1/h$.

- Classification of fibers in pencils of genus 2 curves are given by Ogg [8], Iitaka [2] and Namikawa-Ueno [7].
- There are 120 types in Namikawa-Ueno's classification.
- Liu [6] gives the genus two counterpart of Tate's algorithm.

	genus of spectral curve	types of singular fibers in pencils	algorithm to determine types of fibers
2-dim. Painlevé	1	Kodaira	Tate's algorithm
4-dim. Painlevé	2	Namikawa-Ueno	Liu's algorithm

List of singular fibers of spectral curve fibrations

Let us denote $H_1 = h$, $H_2 = g$.

Theorem 5 (N.)

40 types of autonomous 4-dimensional Painlevé type equation define two rational surfaces with relatively minimal fibrations $\phi_i : X_i \to \mathbb{P}^1$ $(i = 1, 2)$ of spectral curves of genus 2 as spectral curve fibrations. Their singular fiber at $H_1 = \infty$ and $H_2 = \infty$ are as in the following tables.

Outline of the proof.

- Derive defining equations of spectral curves.
- Convert spectral curves into Weierstrass form (using Maple).
- Compactify and consider the minimal desingularization.
- Compute the discriminant, Igusa invariants and other invariants of the sextic.
- Apply Liu's algorithm to determine the Namikawa-Ueno type of fiber at $H_i = \infty$ $(i = 1, 2)$.

Example

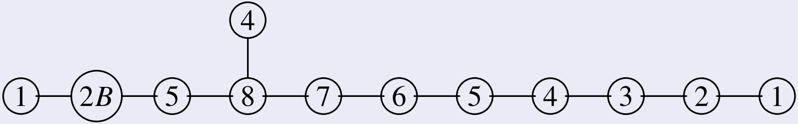
- The characteristic polynomial of a Lax equation of the Garnier system of type $\frac{9}{2}$ is expressed as

$$y^2 = 9x^5 + 9\tilde{t}_1 x^3 + 3\tilde{t}_2 x^2 - hx + g,\ h = H^{9/2}_{\mathrm{Gar},\tilde{t}_1},\ g = H^{9/2}_{\mathrm{Gar},\tilde{t}_2}.$$

- Upon replacements $\bar{x} = x/h,\ \bar{y} = y/h^3,\ \bar{h} = 1/h$, we have "$\infty$"-model:

$$\bar{y}^2 = 9\bar{h}^6 x^5 + 9\tilde{t}_1 \bar{h}^6 x^3 + 3\bar{h}^6 \tilde{t}_2 x^2 - \bar{h}^5 x + g\bar{h}^6.$$

- Compute Igusa invariants and other invariants of this quintic.
- Namikawa-Ueno type of the fiber at $h = \infty$ is VII*, from Liu's algorithm. (Type 22 in Ogg's notation.)



The numbers in circles: multiplicities of components in the reducible fibers. "B":(-3)-curve, the rest: (-2)-curves.

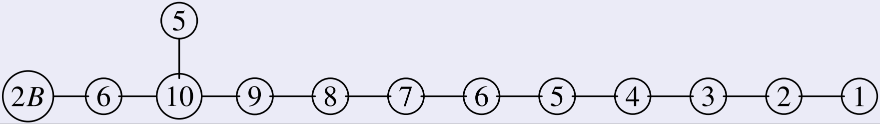
Example (continued)

Similarly, we can associate another surface to each system: spectral curve fibration with respect to another conserved quantity "g". After replacing $\bar{x} = x/g$, $\bar{y} = y/g^3$, $\bar{g} = 1/g$ in the above example, we obtain an affine equation around $g = \infty$;

$$\bar{y}^2 = 9\bar{g}x^5 + 9\tilde{t}_1\bar{g}^3x^3 + 3\bar{g}^4\tilde{t}_2x^2 - \bar{g}^5hx + \bar{g}^6.$$

From Liu's algorithm, the fiber at $g = \infty$ is type VIII − 4 in Namikawa-Ueno's notation.

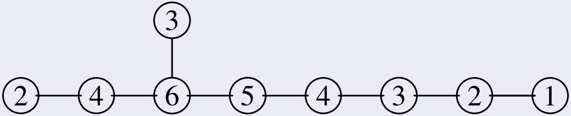
VIII − 4: $H^{\frac{9}{2}}_{\mathrm{Gar},\mathrm{t}_2}$



Remark

- Its dual graph contains, as a subgraph, the extended Dynkin diagram of the unimodular integral lattice D_{12}^{+}.[a]
- Mordell-Weil group of $f \colon X \to \mathbb{P}^1$ is trivial (Kitagawa [5, Thm 3.1]).
- It can be thought as a generalization of the fact (Thm4) that the spectral curve fibration defined by the autonomous $\mathrm{H_I}$ (the most degenerated 2-dimensional Painlevé equation) has the singular fiber of type $E_8 = D_8^{+}$.

Kodaira type II^*(Dynkin type E_8) : H_{I}



[a]The notation is as in Conway-Sloane [1]. In some literatures, this lattice is expressed as Γ_{12}.

Spectral curve fibration with respect to $H_{\mathrm{Gar},\tilde{t}_1}$

Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$H^{1+1+1+1+1}_{\mathrm{Gar},t_1}$	11,11,11,11,11	I^*_{1-0-0}	-	$(2)^2\times(4)$	33
$H^{2+1+1+1}_{\mathrm{Gar},t_1}$	(1)(1),11,11,11	I^*_{1-1-0}	-	$(4)\times(4)$	33
$H^{3/2+1+1+1}_{\mathrm{Gar},t_1}$	(2)(1),(1)(1)(1)	I^*_{1-2-0}	-	$(4)\times(2)^2$	33
$H^{3+1+1}_{\mathrm{Gar},t_1}$	((1))((1)),11,11	IV^*-I^*_1-(-1)	E_6-D_5-(-1)	$(3)\times(4)$	29a
$H^{2+2+1}_{\mathrm{Gar},t_1}$	(1)(1),(1)(1),11	IV^*-I^*_1-(-1)	E_6-D_5-(-1)	$(3)\times(4)$	29a
$H^{5/2+1+1}_{\mathrm{Gar},t_1}$	(((1)(1)))(((1)))	III^*-I^*_1-(-1)	E_7-D_5-(-1)	$(2)\times(4)$	29a
$H^{3/2+2+1}_{\mathrm{Gar},t_1}$	$(1)_2$, (1)(1), 11	IV^*-I^*_2-(-1)	E_6-D_6-(-1)	$(3)\times(2)^2$	29a
$H^{3/2+3/2+1}_{\mathrm{Gar},t_1}$	$(1)_2$, $(1)_2$, 11	III^*-I^*_2-(-1)	E_7-D_6-(-1)	$(2)\times(2)^2$	29a
$H^{4+1}_{\mathrm{Gar},t_1}$	(((1)))(((1))),11	III^*-II^*_1	E_7-II^*_1	(8)	23
$H^{3+2}_{\mathrm{Gar},t_1}$	((1))((1)),(1)(1)	III^*-II^*_1	E_7-II^*_1	(8)	23
$H^{5/2+2}_{\mathrm{Gar},t_1}$	$(((1)))_2$, (1)(1)	II^*-II^*_1	E_8-II^*_1	$(2)^2$	25
$H^{7/2+1}_{\mathrm{Gar},t_1}$	$(((((1)))))_2$, 11	II^*-II^*_1	E_8-II^*_1	$(2)^2$	25
$H^{3/2+3}_{\mathrm{Gar},t_1}$	$(1)_2$, ((1))((1))	IV^*-III^*-(-1)	E_6-E_7-(-1)	(6)	29
$H^{5/2+3/2}_{\mathrm{Gar},t_1}$	$(((1)))_2$, $(1)_2$	III^*-III^*-(-1)	E_7-E_7-(-1)	$(2)^2$	29
H^{5}_{Gar,t_1}	((((1))))((((1))))	IX-3	-	(5)	21
$H^{9/2}_{\mathrm{Gar},t_1}$	$(((((((1)))))))_2$	VII^*	-	(2)	22

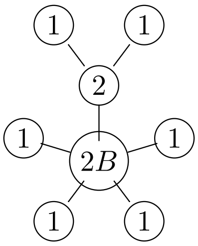
The singular fibers at $H_{\mathrm{Gar},t_1}=\infty$ of spectral curve fibrations

Spectral curve fibration with respect to $H_{\mathrm{Gar},\tilde{t}_2}$

Hamiltonian	spectral type	N-U type	Dynkin	Φ	Ogg
$H^{1+1+1+1+1}_{\mathrm{Gar},t_2}$	11,11,11,11,11	I^*_{1-0-0}	-	$(2)^2\times(4)$	33
$H^{2+1+1+1}_{\mathrm{Gar},t_2}$	(1)(1),11,11,11	I^*_{1-1-0}	-	$(1)\times(1)$	33
$H^{3/2+1+1+1}_{\mathrm{Gar},t_2}$	(2)(1),(1)(1)(1)	I^*_{1-2-0}	-	$(1)\times(2)$	33
$H^{2+2+1}_{\mathrm{Gar},t_2}$	(1)(1),(1)(1),11	I^*_{1-1-1}	-	$(4)\times(4)$	33
$H^{3/2+2+1}_{\mathrm{Gar},t_2}$	$(1)_2$, (1)(1), 11	I^*_{1-1-2}	-	$(3)\times(2)^2$	33
$H^{3/2+3/2+1}_{\mathrm{Gar},t_2}$	$(1)_2$, $(1)_2$, 11	I^*_{1-2-2}	-	$(2)\times(2)^1$	33
$H^{3+1+1}_{\mathrm{Gar},t_2}$	((1))((1)),11,11	IV^*-I^*_1-(-1)	E_6-D_5-(-1)	$(3)\times(4)$	29a
$H^{5/2+1+1}_{\mathrm{Gar},t_2}$	(((1)(1)))(((1)))	III^*-I^*_1-(-1)	E_7-D_5-(-1)	$(2)\times(4)$	29a
$H^{3+2}_{\mathrm{Gar},t_2}$	((1))((1)),(1)(1)	IV^*-IV^*-(-1)	E_6-E_6-(-1)	$(3)^2$	29
$H^{5/2+2}_{\mathrm{Gar},t_2}$	$(((1)))_2$, (1)(1)	IV^*-III^*-(-1)	E_6-E_7-(-1)	(6)	29
$H^{3/2+3}_{\mathrm{Gar},t_2}$	$(1)_2$, ((1))((1))	III^*-II^*_2	E_7-II^*_2	(8)	23
$H^{5/2+3/2}_{\mathrm{Gar},t_2}$	$(((1)))_2$, $(1)_2$	II^*-II^*_2	E_8-II^*_2	$(2)^2$	25
$H^{4+1}_{\mathrm{Gar},t_2}$	(((1)))(((1))),11	IX-3	-	(5)	21
$H^{7/2+1}_{\mathrm{Gar},t_2}$	$(((((1)))))_2$, 11	VII^*	-	(2)	22
H^{5}_{Gar,t_2}	((((1))))((((1))))	V^*	-	(3)	19
$H^{9/2}_{\mathrm{Gar},t_2}$	$((((((((1))))))))_2$	VIII-4	-	(0)	20

The singular fibers at $H_{\mathrm{Gar},\tilde{t}_2} = \infty$ of spectral curve fibrations

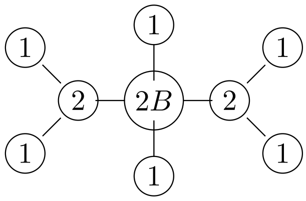
I^*_{1-0-0}: $H^{1+1+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{1+1+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$



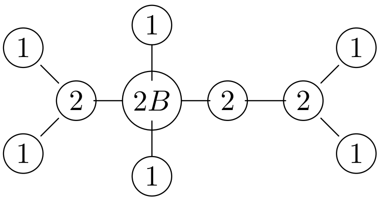
$$\begin{pmatrix} -1 & 0 & -p-q & q \\ 0 & -1 & q & -n-q \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{I}^*_{\mathrm{n-p-q}}$

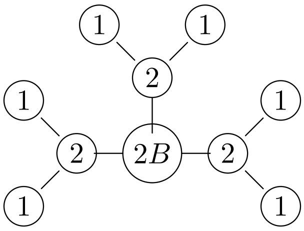
I^*_{1-1-0}: $H^{2+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{2+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$



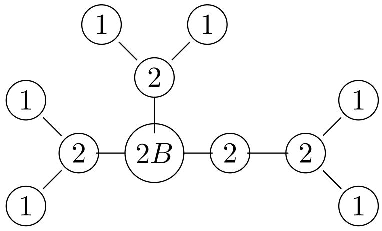
I^*_{1-2-0}: $H^{\frac{3}{2}+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{\frac{3}{2}+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$



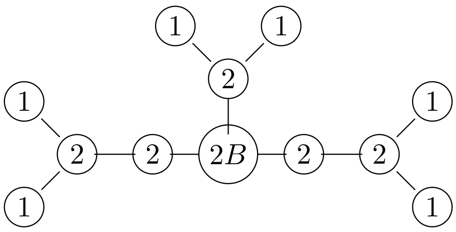
I^*_{1-1-1}: $H^{2+2+1}_{\mathrm{Gar},\tilde{t_2}}$



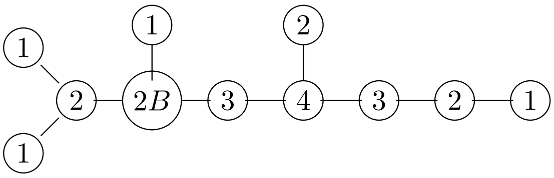
I^*_{1-1-2}: $H^{\frac{3}{2}+2+1}_{\mathrm{Gar},\tilde{t_1}}$



I^*_{1-2-2}: $H^{\frac{3}{2}+\frac{3}{2}+1}_{\mathrm{Gar},\tilde{t_1}}$



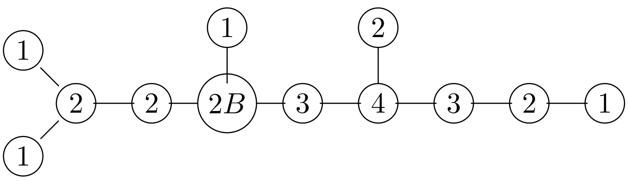
$\mathrm{III}^*-\mathrm{I}^*_1-(-1)$: $H^{\frac{5}{2}+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{\frac{5}{2}+1+1}_{\mathrm{Gar},\tilde{t_2}}$



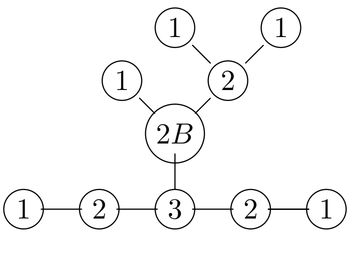
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{III}-\mathrm{I_n}-\mathrm{m}$

$\mathrm{III}^*-\mathrm{I}^*_2-(-1)$: $H^{\frac{3}{2}+\frac{3}{2}+1}_{\mathrm{Gar},\tilde{t_1}}$



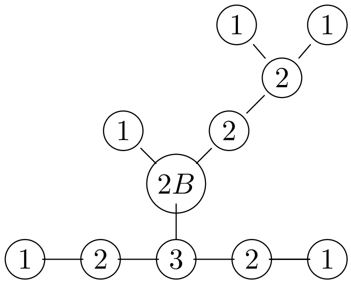
$\mathrm{IV}^*-\mathrm{I}^*_1-(-1)$: $H^{3+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{3+1+1}_{\mathrm{Gar},\tilde{t_2}}$, $H^{2+2+1}_{\mathrm{Gar},\tilde{t_1}}$



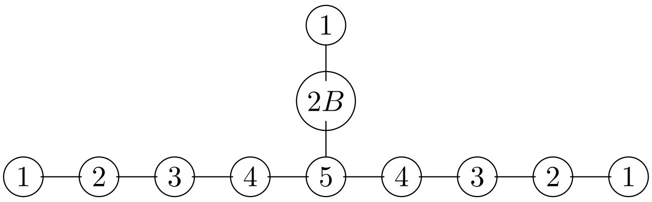
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{IV}^*-\mathrm{I}^*_\mathrm{n}-\mathrm{m}$

$\mathrm{IV}^*-\mathrm{I}^*_2-(-1)$: $H^{\frac{3}{2}+2+1}_{\mathrm{Gar},\tilde{t_2}}$

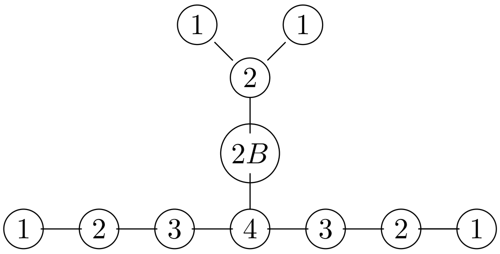


$\mathrm{IX}-3\colon H^{4+1}_{\mathrm{Gar},\tilde{t_2}}, H^{5}_{\mathrm{Gar},\tilde{t_1}}$



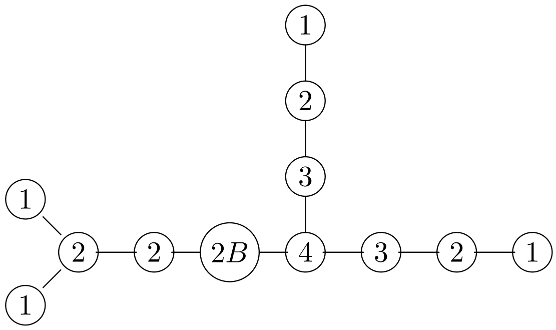
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{III}^*-\mathrm{II}^*_1\colon H^{4+1}_{\mathrm{Gar},\tilde{t_1}}, H^{3+2}_{\mathrm{Gar},\tilde{t_1}}$

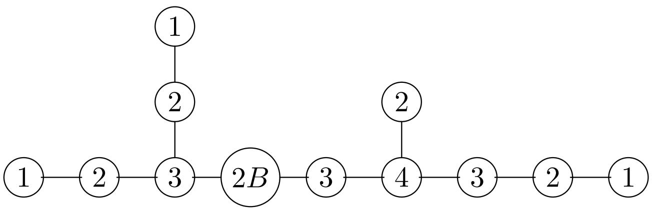


$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -n-1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{III}^*-\mathrm{II}^*_2\colon H^{\frac{3}{2}+3}_{\mathrm{Gar},\tilde{t_2}}$

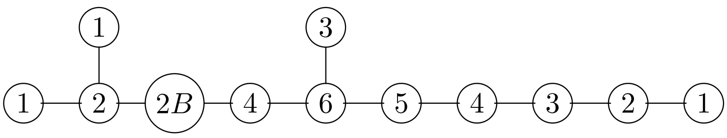


$\mathrm{IV}^*-\mathrm{III}^*-(-1)\colon H^{\frac{5}{2}+2}_{\mathrm{Gar},\tilde{t_1}}, H^{\frac{3}{2}+3}_{\mathrm{Gar},\tilde{t_1}}$



$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

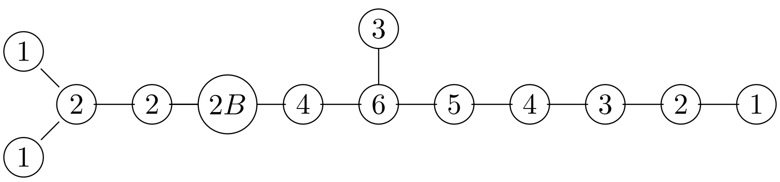
$\mathrm{II}^*-\mathrm{II}^*_1\colon H^{\frac{5}{2}+2}_{\mathrm{Gar},\tilde{t_2}}, H^{\frac{7}{2}+1}_{\mathrm{Gar},\tilde{t_1}}$



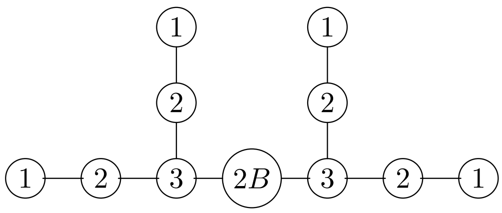
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -n \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{II}^*-\mathrm{II}^*_{\mathrm{n}}$

$\mathrm{II}^*-\mathrm{II}^*_2\colon H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar},\tilde{t_2}}$

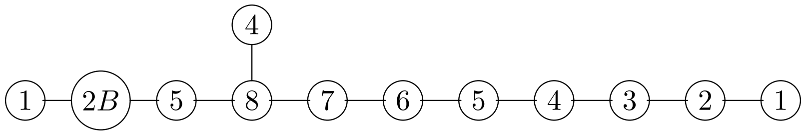


$\mathrm{IV}^* - \mathrm{IV}^* - (-1)\colon H^{3+2}_{\mathrm{Gar},\tilde{t_2}}$



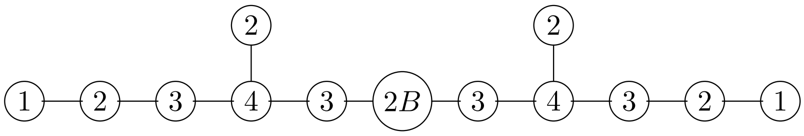
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{VII}^*\colon H^{\frac{7}{2}+1}_{\mathrm{Gar},\tilde{t_2}}, H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_1}}$



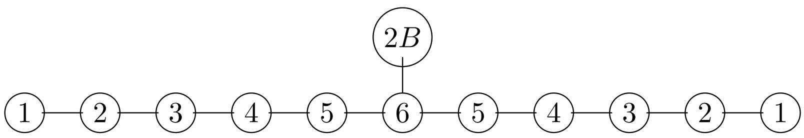
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{III}^* - (-1)\colon H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar},\tilde{t_1}}$



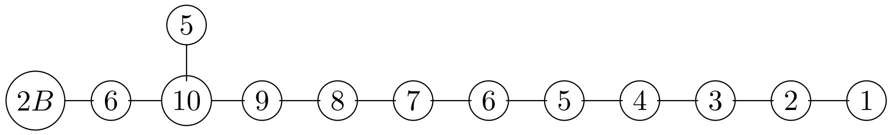
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{V}^*\colon H^{5}_{\mathrm{Gar},\tilde{t_2}}$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{VIII} - 4\colon H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_2}}$



$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

Spectral curve fibration of Fuji-Suzuki equations with respect to H

Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$H_{\mathrm{FS}}^{A_5}$	21,21,111,111	II_{4-0}	-	(16)	41
$H_{\mathrm{FS}}^{A_4}$	(11)(1),21,111	II_{4-1}	-	(17)	41
$H_{\mathrm{FS}}^{A_3}$	$(1)_2$, 21, 111	II_{4-2}	-	(17)	41
$H_{\mathrm{Suz}}^{\frac{3}{2}+2}$	(11)(1), $(1)_2 1$	II_{4-3}	-	(19)	41
$H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$	$(1)_3$, (11)(1)	II_{4-4}	-	(20)	41
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$	$(1)_3$, $(1)_2 1$	II_{4-5}	-	(21)	41
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$	$(1)_3$, $(1)_3$	II_{3-6}	-	(18)	41
$H_{\mathrm{NY}}^{A_5}$	(2)(1),111,111	II_{5-1}	-	$(2) \times (2)$	41a
$H_{\mathrm{NY}}^{A_4}$	((11))((1)),111	$\mathrm{IV}^* - \mathrm{II}_4$	$E_6 - \mathrm{II}_4$	(13)	41b

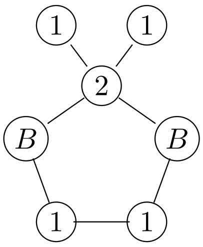
The singular fibers at $H = \infty$ of spectral curve fibrations
of autonomous 4-dimensional (degenerate) Fuji-Suzuki equations

Spectral curve fibration of Fuji-Suzuki equations with respect to G

Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$G_{\mathrm{FS}}^{A_5}$	21,21,111,111	III	-	$(3)^2$	42
$G_{\mathrm{FS}}^{A_4}$	(11)(1),21,111	III_1 (par[5])	-	(9)	43
G_{Suz}^{2+2}	(11)(1),(11)(1)	III_2 (par[5])	-	(9)	43
$G_{\mathrm{Suz}}^{\frac{3}{2}+2}$	$(11)(1), (1)_2 1$	III_3 (par[5])	-	$(3)^2$	43
$G_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$	$(1)_3, (11)(1)$	III_4 (par[5])	-	(9)	43
$G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$	$(1)_3, (1)_2 1$	III_5 (par[5])	-	(9)	43
$G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$	$(1)_3, (1)_3$	III_6 (par[5])	-	$(3)^2$	43
$G_{\mathrm{NY}}^{A_5}$	(2)(1),111,111	IV-III^*-(-1)	A_2-E_7-(-1)	(6)	42
$G_{\mathrm{NY}}^{A_4}$	((11))((1)),111	$\mathrm{IX}-4$	-	(5)	44

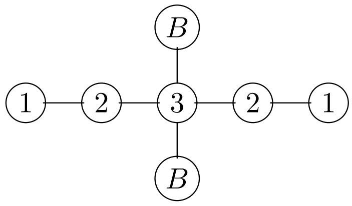
The singular fibers at $G = \infty$ of spectral curve fibrations
of autonomous 4-dimensional (degenerate) Fuji-Suzuki equations

II_{4-0}: $H_{\mathrm{FS}}^{A_5}$



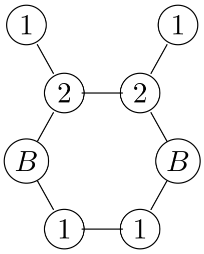
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

III: $G_{\mathrm{FS}}^{A_5}$



$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

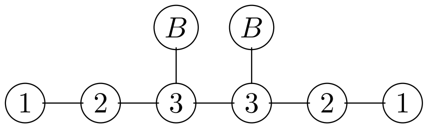
II_{4-1}: $H_{\mathrm{FS}}^{A_4}$



$$\begin{pmatrix} -1 & 0 & -p & 0 \\ 1 & 1 & p & n \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $\mathrm{II_{n-p}}(\mathrm{p} \geq 1)$

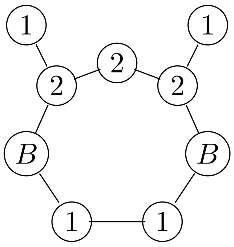
$5 - \mathrm{III}_1$: $G_{\mathrm{FS}}^{A_4}$



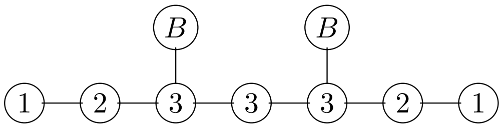
$$\begin{pmatrix} 0 & 1 & -n & n \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

for $\mathrm{III_n}$

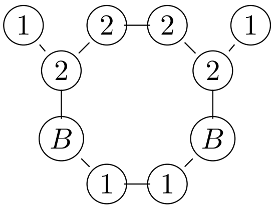
II_{4-2}: $H_{\mathrm{FS}}^{A_3}$



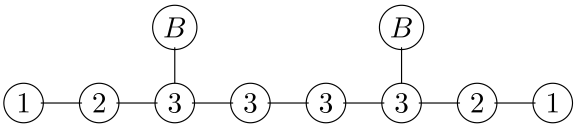
$5 - \mathrm{III}_2$: $G_{\mathrm{FS}}^{A_3}$



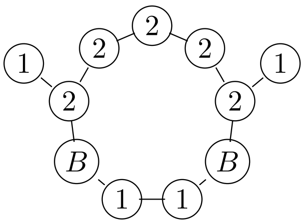
II_{4-3}: $H_{\mathrm{KFS}}^{\frac{3}{2}+2}$



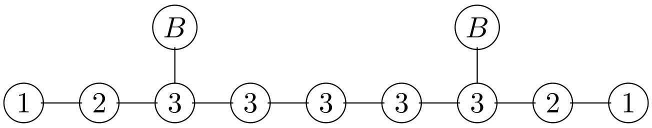
$5 - \mathrm{III}_3$: $G_{\mathrm{KFS}}^{\frac{3}{2}+2}$



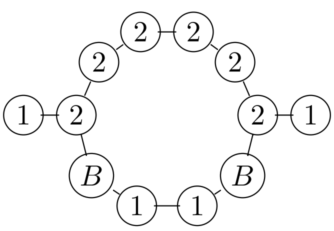
II_{4-4}: $H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$



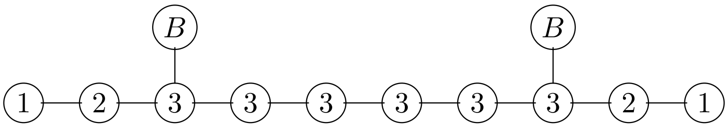
$5 - \mathrm{III}_4$: $G_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$



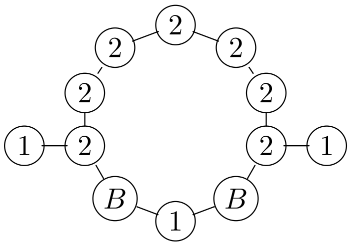
II_{4-5}: $H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$



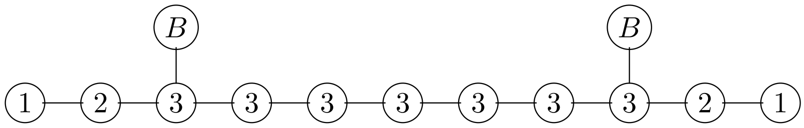
$5 - \mathrm{III}_5$: $G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$



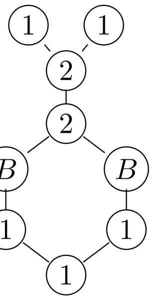
$\mathrm{II}_{3-6}\colon H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$



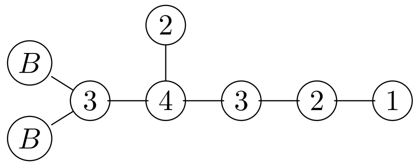
$5-\mathrm{III}_6\colon G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$



$\mathrm{II}_{5-1}\colon H_{\mathrm{NY}}^{A_5}$

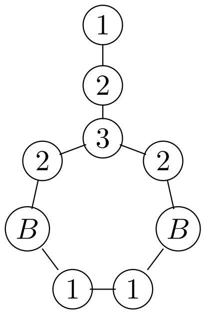


$\mathrm{IV}-\mathrm{III}^*-(-1)\colon G_{\mathrm{NY}}^{A_5}$



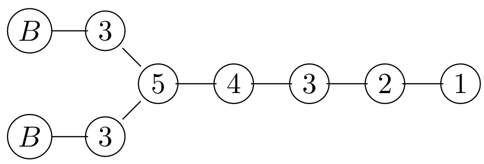
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{IV}^*-\mathrm{II}_4\colon H_{\mathrm{NY}}^{A_4}$



$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{IX}-4\colon G_{\mathrm{NY}}^{A_4}$



$$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Spectral curve fibration of Sasano equations with respect to H

Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$H_{\mathrm{Ss}}^{D_6}$	31,22,22,1111	$\mathrm{I}_3 - \mathrm{I}_0^* - 0$	$A_2 - D_4 - 0$	$(3) \times (2)^2$	2
$H_{\mathrm{Ss}}^{D_5}$	(111)(1),22,22	$\mathrm{I}_3 - \mathrm{I}_1^* - 0$	$A_2 - D_5 - 0$	$(3) \times (4)$	2
$H_{\mathrm{Ss}}^{D_4}$	(2)(2),(111)(1)	$\mathrm{I}_3 - \mathrm{I}_2^* - 0$	$A_2 - D_6 - 0$	$(3) \times (2)^2$	2
$H_{\mathrm{KSs}}^{\frac{3}{2}+2}$	$(1)_2 11$, (2)(2)]	$\mathrm{I}_3 - \mathrm{I}_3^* - 0$	$A_2 - D_7 - 0$	$(3) \times (4)$	2
$H_{\mathrm{KSs}}^{\frac{4}{3}+2}$	$(1)_3 1$, (2)(2)	$\mathrm{I}_3 - \mathrm{I}_4^* - 0$	$A_2 - D_8 - 0$	$(3) \times (2)^2$	2
$H_{\mathrm{KSs}}^{\frac{5}{4}+2}$	$(1)_4$, (2)(2)	$\mathrm{I}_3 - \mathrm{I}_5^* - 0$	$A_2 - D_9 - 0$	$(3) \times (4)$	2
$H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$	$(2)_2(1)_4$	$\mathrm{I}_2 - \mathrm{I}_6^* - 0$	$A_1 - D_{10} - 0$	$(2) \times (2)^2$	2

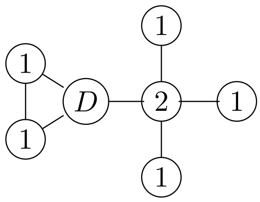
The singular fibers at $H = \infty$ of spectral curve fibrations
of autonomous 4-dimensional (degenerate) Sasano equations

Spectral curve fibration of Sasano equations with respect to G

Ham.	spectral type	N-U type	Φ	Ogg
$G_{\mathrm{Ss}}^{D_6}$	31,22,22,1111	I	$(2)^2$	4
$G_{\mathrm{Ss}}^{D_5}$	(111)(1),22,22	III_1 (par[4])	(4)	5
$G_{\mathrm{Ss}}^{D_4}$	(2)(2),(111)(1)	III_2 (par[4])	$(2)^2$	5
$G_{\mathrm{KSs}}^{\frac{3}{2}+2}$	$(1)_2 11$, (2)(2)	III_3 (par[4])	(4)	5
$G_{\mathrm{KSs}}^{\frac{4}{3}+2}$	$(1)_3 1$, (2)(2)	III_4 (par[4])	$(2)^2$	5
$G_{\mathrm{KSs}}^{\frac{5}{4}+2}$	$(1)_4$, (2)(2)	III_5 (par[4])	(4)	5
$G_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$	$(2)_2(1)_4$	III_6 (par[4])	$(2)^2$	5

The singular fibers at $G = \infty$ of spectral curve fibrations of autonomous 4-dimensional (degenerate) Sasano equations

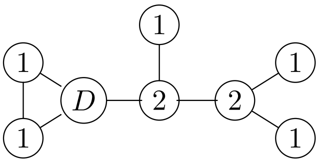
$\mathrm{I}_3-\mathrm{I}_0^*-0$: $H_{\mathrm{Ss}}^{D_6}$



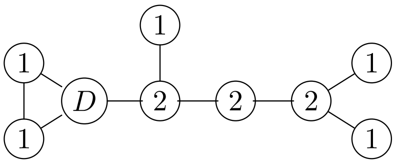
$$\begin{pmatrix} -1 & 0 & -p & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $\mathrm{I_n}-\mathrm{I_p^*}-\mathrm{m}$

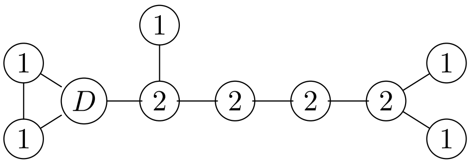
$\mathrm{I}_3-\mathrm{I}_1^*-0$: $H_{\mathrm{Ss}}^{D_5}$



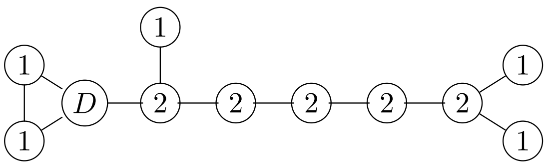
$\mathrm{I}_3-\mathrm{I}_2^*-0$: $H_{\mathrm{Ss}}^{D_4}$



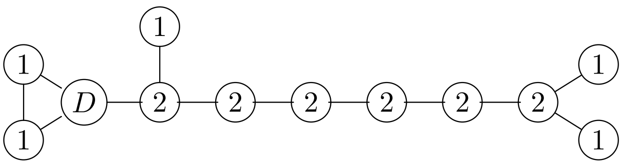
$\mathrm{I}_3-\mathrm{I}_3^*-0$: $H_{\mathrm{KSs}}^{\frac{3}{2}+2}$



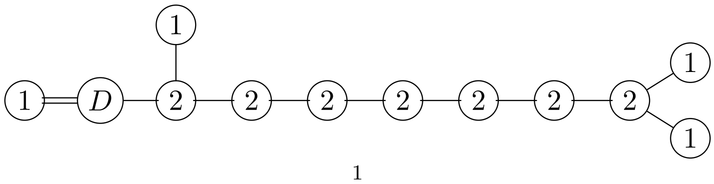
$\mathrm{I}_3-\mathrm{I}_4^*-0$: $H_{\mathrm{KSs}}^{\frac{4}{3}+2}$



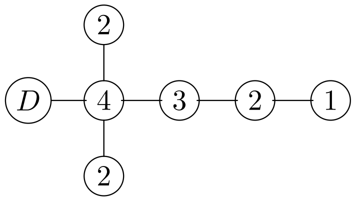
$\mathrm{I}_3-\mathrm{I}_5^*-0$: $H_{\mathrm{KSs}}^{\frac{5}{4}+2}$



$\mathrm{I}_2-\mathrm{I}_6^*-0$: $H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$

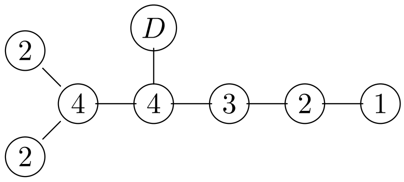


VI: $G_{\mathrm{Ss}}^{D_6}$



$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

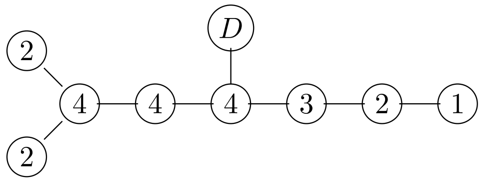
$4-\mathrm{III}_1$: $G_{\mathrm{Ss}}^{D_5}$



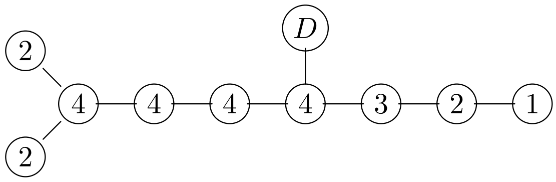
$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & n & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $\mathrm{III_n}$

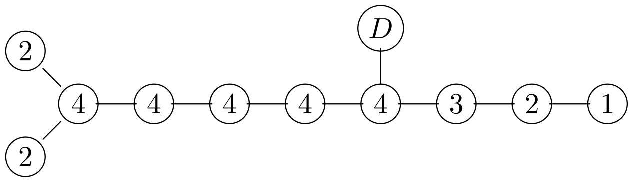
$4-\mathrm{III}_2$: $G_{\mathrm{Ss}}^{D_4}$



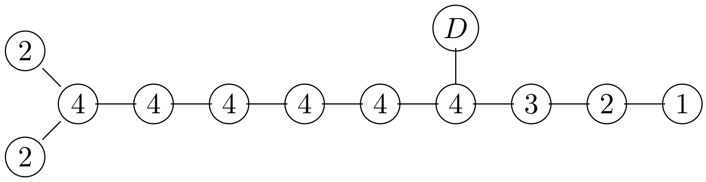
$4-\mathrm{III}_3$: $G_{\mathrm{KSs}}^{\frac{3}{2}+2}$



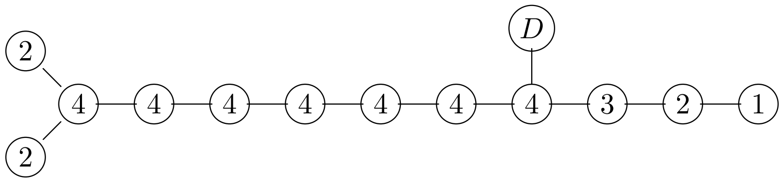
$4-\mathrm{III}_4$: $G_{\mathrm{KSs}}^{\frac{4}{3}+2}$



$4-\mathrm{III}_5$: $G_{\mathrm{KSs}}^{\frac{5}{4}+2}$



$4-\mathrm{III}_6\colon G^{\frac{3}{2}+\frac{5}{4}}_{\mathrm{KSs}}$



Spectral curve fibration of Matrix Painlevé equations with respect to H

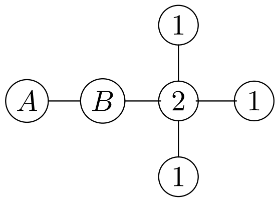
Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$H_{\mathrm{VI}}^{\mathrm{Mat}}$	22,22,22,211	$\mathrm{I}_0 - \mathrm{I}_0^* - 1$	$\mathrm{I}_0 - D_4 - 1$	$(2)^2$	14
$H_{\mathrm{V}}^{\mathrm{Mat}}$	(2)(11),22,22	$\mathrm{I}_0 - \mathrm{I}_1^* - 1$	$\mathrm{I}_0 - D_5 - 1$	(4)	14
$H_{\mathrm{III(D_6)}}^{\mathrm{Mat}}$	(2)(2),(2)(11)	$\mathrm{I}_0 - \mathrm{I}_2^* - 1$	$\mathrm{I}_0 - D_6 - 1$	$(2)^2$	14
$H_{\mathrm{III(D_7)}}^{\mathrm{Mat}}$	$(2)(2), (11)_2$	$\mathrm{I}_0 - \mathrm{I}_3^* - 1$	$\mathrm{I}_0 - D_7 - 1$	(4)	14
$H_{\mathrm{III(D_8)}}^{\mathrm{Mat}}$	$(2)_2, (11)_2$	$\mathrm{I}_0 - \mathrm{I}_4^* - 1$	$\mathrm{I}_0 - D_8 - 1$	$(2)^2$	14
$H_{\mathrm{IV}}^{\mathrm{Mat}}$	((2))((11)),22,	$\mathrm{I}_0 - \mathrm{IV}^* - 1$	$\mathrm{I}_0 - E_6 - 1$	(3)	14
$H_{\mathrm{II}}^{\mathrm{Mat}}$	(((2)))(((11)))	$\mathrm{I}_0 - \mathrm{III}^* - 1$	$\mathrm{I}_0 - E_7 - 1$	(2)	14
$H_{\mathrm{I}}^{\mathrm{Mat}}$	$(((((11)))))_2$	$\mathrm{I}_0 - \mathrm{II}^* - 1$	$\mathrm{I}_0 - E_8 - 1$	0	14

The singular fibers at $H = \infty$ of spectral curve fibrations
of autonomous 4-dimensional Matrix Painlevé equations

Ham.	spectral type	N-U type	Dynkin	Φ	Ogg
$G_{\mathrm{VI}}^{\mathrm{Mat}}$	22,22,22,211	$2\mathrm{I}_0^* - 0$	$2D_4 - 0$	$(2)^2$	24a
$G_{\mathrm{V}}^{\mathrm{Mat}}$	(2)(11),22,22	$2\mathrm{I}_1^* - 0$	$2D_5 - 0$	(4)	24
$G_{\mathrm{III(D_6)}}^{\mathrm{Mat}}$	(2)(2),(2)(11)	$2\mathrm{I}_2^* - 0$	$2D_6 - 0$	$(2)^2$	24
$G_{\mathrm{III(D_7)}}^{\mathrm{Mat}}$	$(2)(2), (11)_2$	$2\mathrm{I}_3^* - 0$	$2D_7 - 0$	(4)	24
$G_{\mathrm{III(D_8)}}^{\mathrm{Mat}}$	$(2)_2, (11)_2$	$2\mathrm{I}_4^* - 0$	$2D_8 - 0$	$(2)^2$	24
$G_{\mathrm{IV}}^{\mathrm{Mat}}$	((2))((11)),22,	$2\mathrm{IV}^* - 0$	$2E_6 - 0$	(3)	26
$G_{\mathrm{II}}^{\mathrm{Mat}}$	(((2)))(((11)))	$2\mathrm{III}^* - 0$	$2E_7 - 0$	(2)	27
$G_{\mathrm{I}}^{\mathrm{Mat}}$	$(((((11)))))_2$	$2\mathrm{II}^* - 0$	$2E_8 - 0$	0	28

The singular fibers at $G = \infty$ of spectral curve fibrations
of autonomous 4-dimensional Matrix Painlevé equations

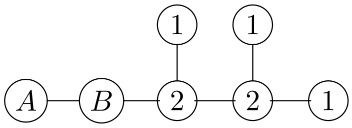
$\mathrm{I}_0-\mathrm{I}_0^*-1$: $H^{\mathrm{Mat}}_{\mathrm{VI}}$



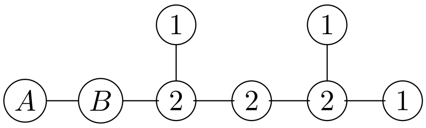
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{I}_0-\mathrm{I}_\mathrm{n}^*-\mathrm{m}$

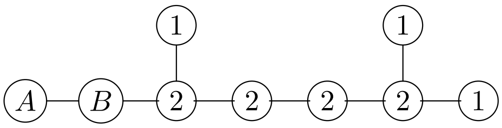
$\mathrm{I}_0-\mathrm{I}_1^*-1$: $H^{\mathrm{Mat}}_{\mathrm{V}}$



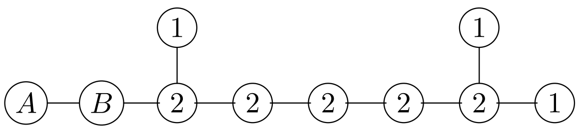
$\mathrm{I}_0-\mathrm{I}_2^*-1$: $H^{\mathrm{Mat}}_{\mathrm{III}(D_6)}$



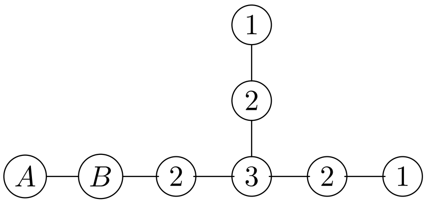
$\mathrm{I}_0-\mathrm{I}_3^*-1$: $H^{\mathrm{Mat}}_{\mathrm{III}(D_7)}$



$\mathrm{I}_0-\mathrm{I}_4^*-1$: $H^{\mathrm{Mat}}_{\mathrm{III}(D_8)}$

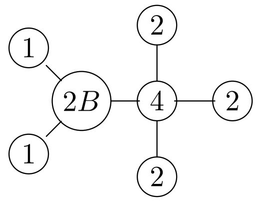


$\mathrm{I}_0-\mathrm{IV}^*-1$: $H^{\mathrm{Mat}}_{\mathrm{IV}}$



$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

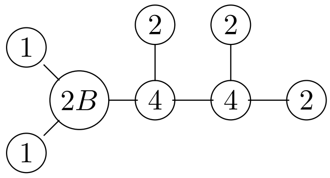
$2\mathrm{I}_0^*-0$: $G^{\mathrm{Mat}}_{\mathrm{VI}}$



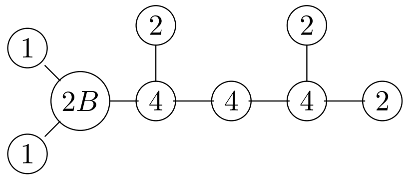
$$\begin{pmatrix} 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $2\mathrm{I}_\mathrm{n}^*-\mathrm{m}$

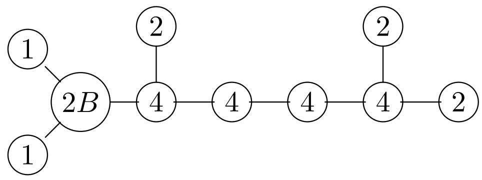
$2\mathrm{I}_1^*-0$: $G^{\mathrm{Mat}}_{\mathrm{V}}$



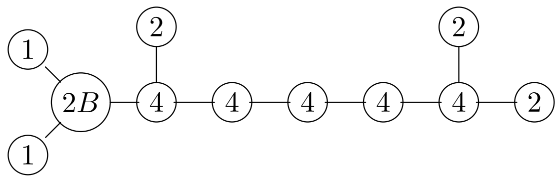
$2\mathrm{I}_2^*-0$: $G^{\mathrm{Mat}}_{\mathrm{III}(D_6)}$



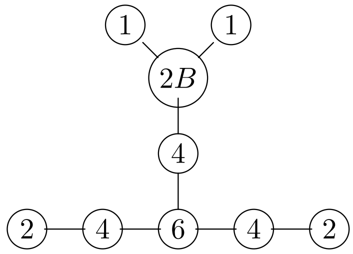
$2\mathrm{I}_3^*-0$: $G^{\mathrm{Mat}}_{\mathrm{III}(D_7)}$



$2\mathrm{I}_4^*-0$: $G^{\mathrm{Mat}}_{\mathrm{III}(D_8)}$

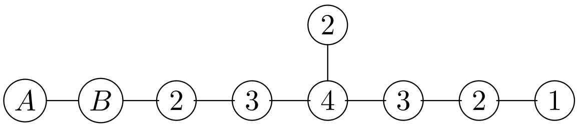


$2\mathrm{IV}^*-0$: $G^{\mathrm{Mat}}_{\mathrm{IV}}$



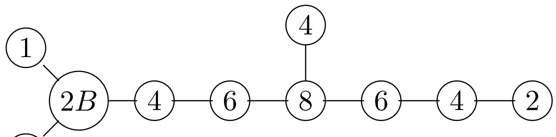
$$\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{III}^* - 1$: $H_{\mathrm{II}}^{\mathrm{Mat}}$



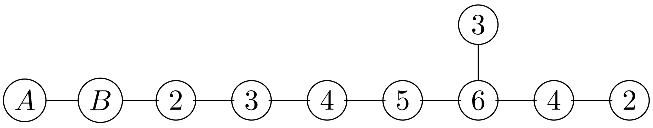
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2\mathrm{III}^* - 0$: $G_{\mathrm{II}}^{\mathrm{Mat}}$



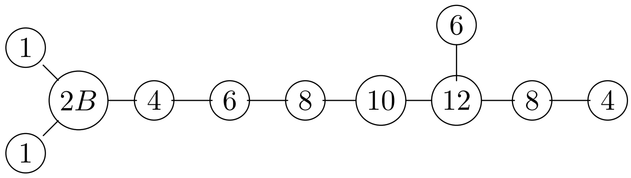
$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{II}^* - 1$: $H_{\mathrm{I}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2\mathrm{II}^* - 0$: $G_{\mathrm{I}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark:compactification of the affine Liouville tori and degeneration of curves of genus two

- The generic fiber of moment map is an **affine part of an Abelian surface**.
- The affine part of an Abelian variety corresponds to the Taylor series with 4 free parameters.
- Such affine surfaces can be **compactified by adjoining divisors** (corresponding to the Laurent solutions with 3 free parameters) and **points** (corresponding to the Laurent solutions with 2 free parameters).
- Each irreducible components of the divisors to be adjoined is a **curve of genus 2** (except the case of the Matrix Painlevé equations).
- Such genus 2 component have **the same Namikawa-Ueno types degeneration** at $H_1 = \infty$ and $H_2 = \infty$, as in the case of the spectral curve fibrations.

Isomonodromic deformation equations:

$$\frac{\partial A(x,\tilde{t})}{\partial t} - \delta \frac{\partial B(x,\tilde{t})}{\partial x} + \left[A(x,\tilde{t}), B(x,\tilde{t})\right] = 0$$

$$\Big\downarrow \ \boxed{\text{Isospectral limit } (\delta \to 0)}$$

Isospectral deformation equations: $\frac{\partial A(x)}{\partial t} + [A(x), B(x)] = 0$

$$\Big\downarrow$$

Spectral curve: $\det\left(yI - A(x)\right) = 0$

$$\Big\downarrow$$

Spectral curve fibration: $f(x, y, h) = 0$, where h is a non-Casimir conserved quantity ($h \in \mathbb{P}^1$) + compactification + minimal desingularization

$$\Big\downarrow \ \boxed{\text{Tate or Liu's algorithm}}$$

Singular fibers of the spectral curve fibration (Kodaira or Namikawa-Ueno type)

Future work

- The spaces we actually want to know are the 4-dimensional phase spaces.
- We can think of these phase spaces as the relative compactified Jacobian of the spectral curve fibrations.
- How can we distinguish these 4-dimensional phase spaces?
- What are their Mordell-Weil lattices?
- Can we classify a certain class of 4-dimensional integrable systems (autonomous limit of the 4-dimensional Painlevé-type equations) from geomtry?
- Can we classify the 4-dimensional Painlevé-type equations from geometry?

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