Veering structures of the canonical decompositions of hyperbolic fibered two-bridge links

Naoki Sakata
Hiroshima University

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Main result

We completely determine, for each hyperbolic fibered two-bridge link, whether the canonical decomposition of its complement is veering.

Theorem (Epstein-Penner, 1988)

Each cusped hyperbolic manifold of finite volume admits a *canonical* decomposition into ideal polyhedra.

Theorem (Agol, 2011)

For each punctured surface bundle over S^1 with a pA monodromy, there exists a *unique* veering and "layered" ideal triangulation of the bundle.

Question

Are the veering ideal triangulations geometric?

Theorem (Hodgson-Rubinstein-Segerman-Tillmann, 2011)

Each veering triangulation admits a strict angle structure.

Theorem (Hodgson-Issa-Segerman, 2016)

$\exists$ a non-geometric veering ideal triangulation.

Question

Which canonical decompositions are veering?

Fact

The canonical decomposition of each once-punctured torus bundle over S^1 is veering and layered.

Theorem (S., 2015)

The canonical decomposition of each hyperbolic fibered two-bridge link complement is layered.

Theorem (S.)

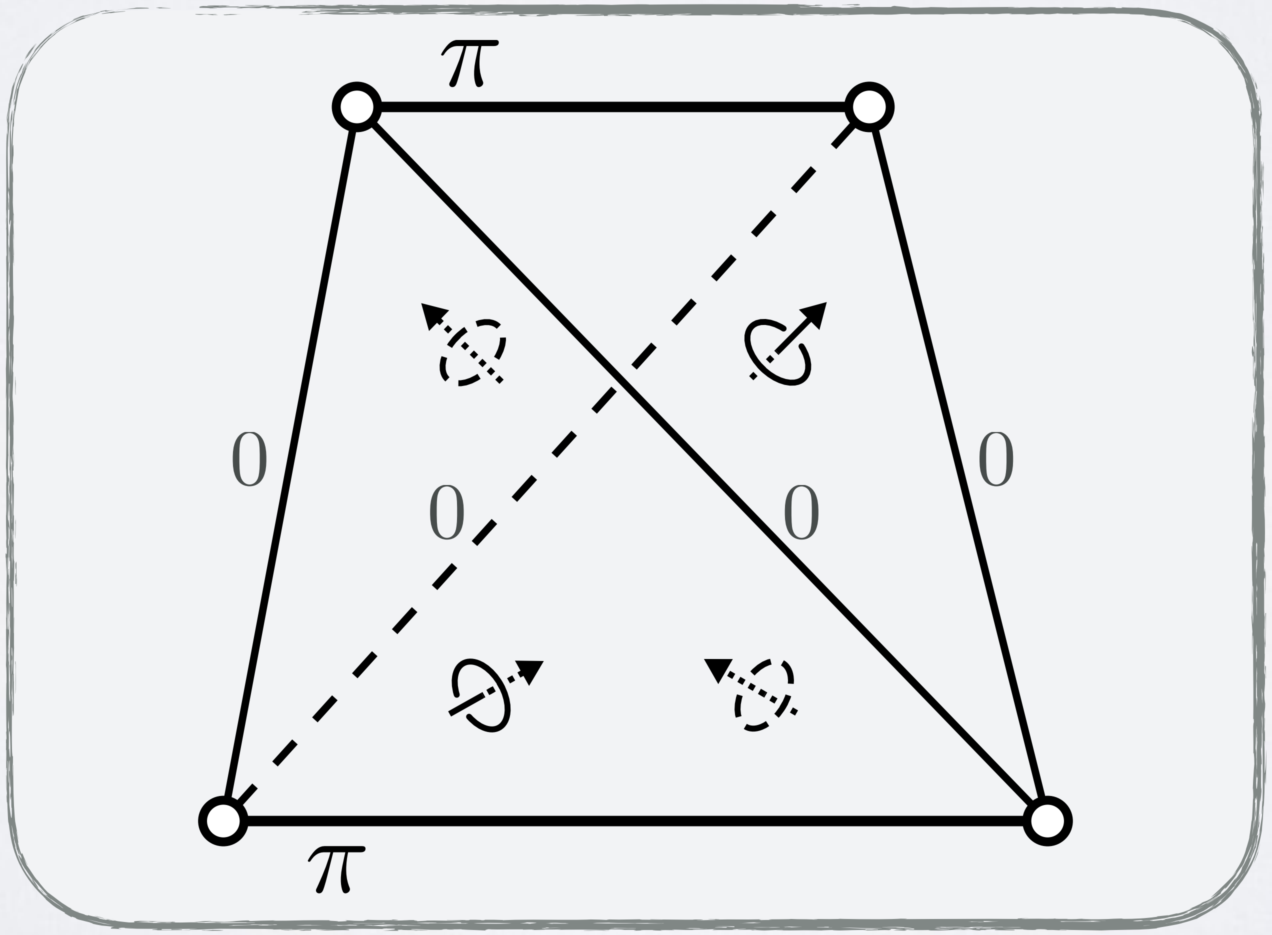
The canonical decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0 < |r| < 1/2)$ is veering $\iff$ the slope r has the continued fraction expansion $\pm[2, 2, \ldots, 2]$.

Taut angle structure (1)

an ideal tetrahedron is *taut*

$\overset{\text{def}}{\Longleftrightarrow}$

1. Each face is assigned a co-orientation so that two co-orientations point inwards and the others point outwards.
2. Each edge of the tetrahedron is assigned an angle of either π or 0 according to whether the co-orientations on the adjacent faces are same or different.



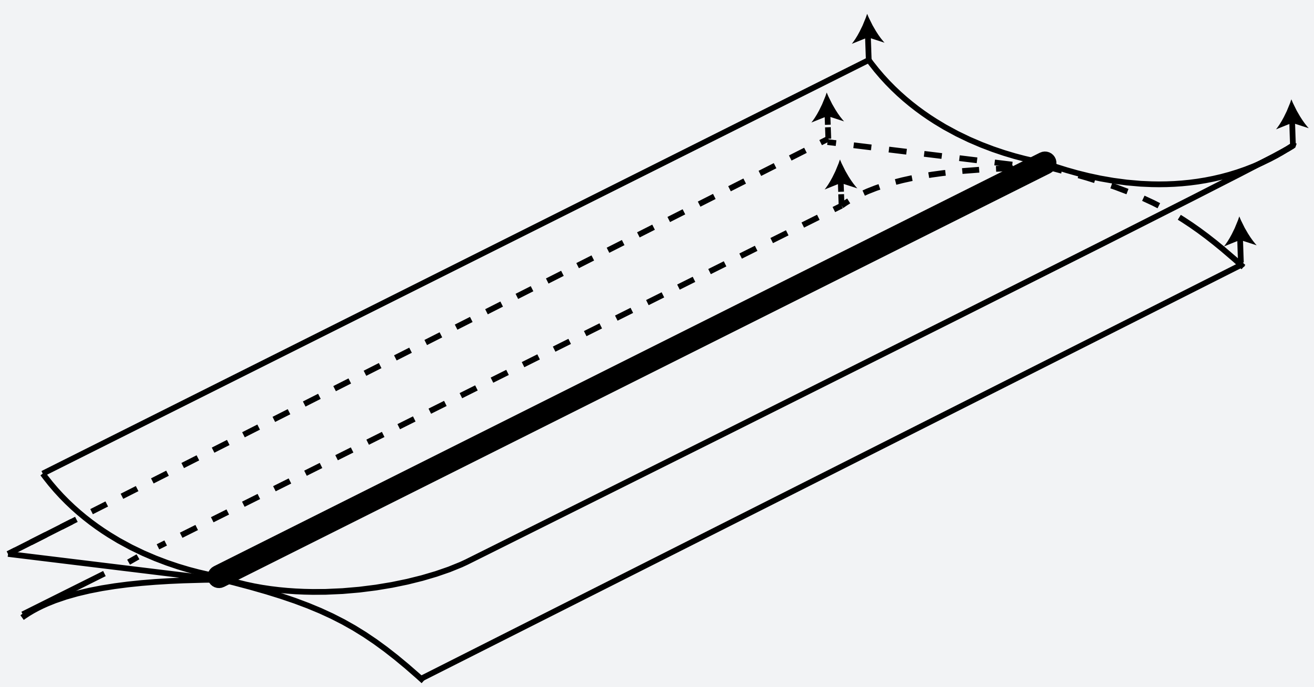
Taut angle structure (2)

M : a compact oriented 3-mfd with toral boundary

An ideal triangulation of $\mathring{M}$ is *taut*

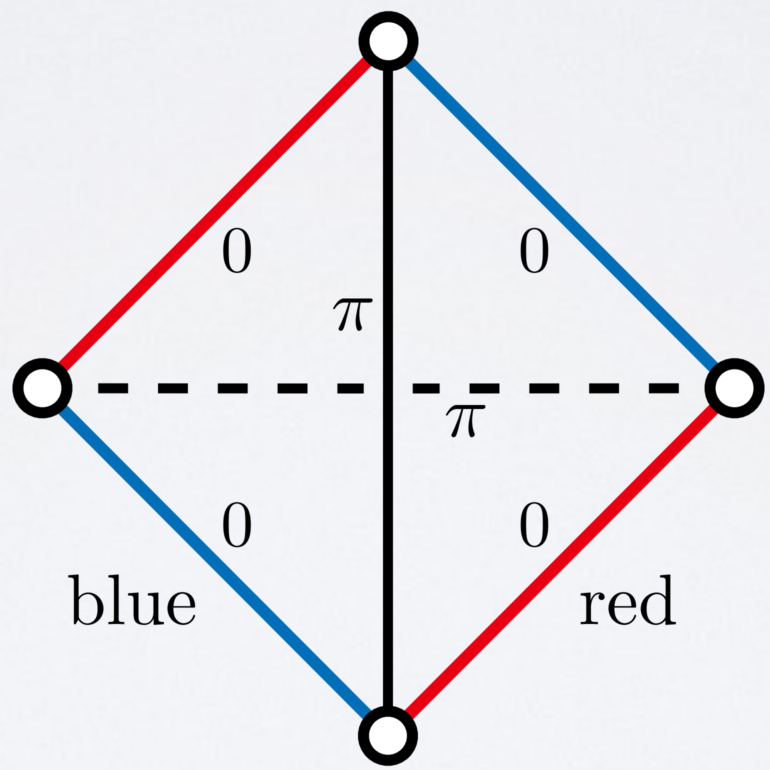
$\overset{\text{def}}{\Longleftrightarrow}$ 1. $\exists$ a co-orientation assigned to each faces s.t. each ideal tetrahedron is taut.

2. The sum of the angles around each edge is 2π.



Veering structure

A taut triangulation of $\mathring{M}$ is *veering*
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ an assignment of two colors, red and blue, to all ideal edges so that every ideal tetrahedron can be sent by an orientation-preserving homeomorphism to



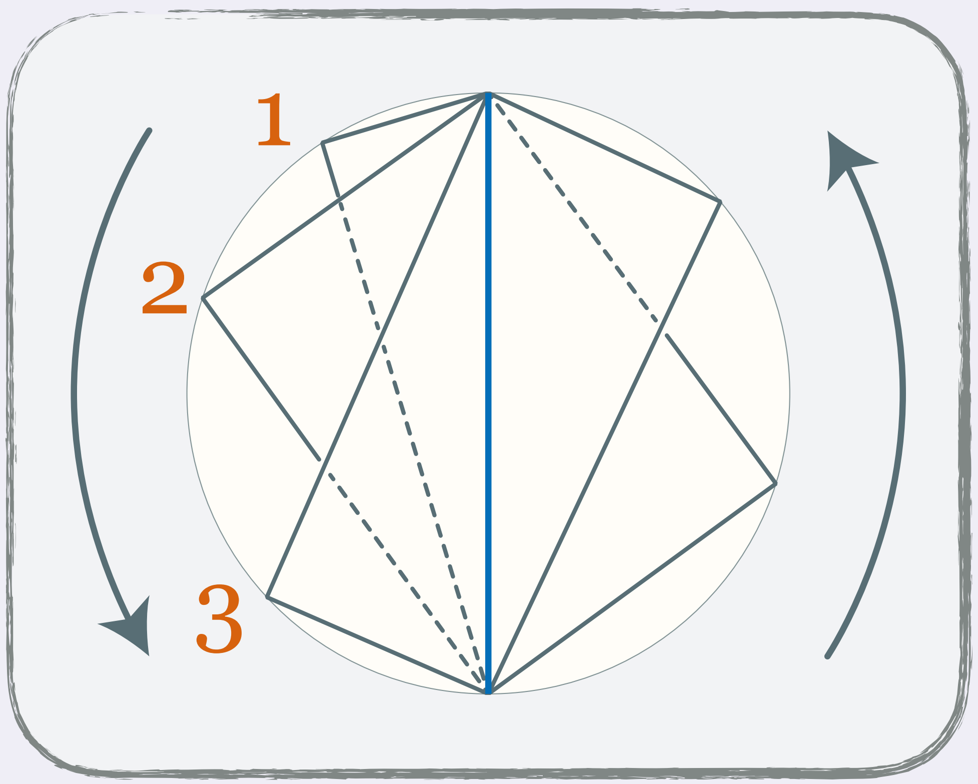
This is called a *veering structure* of the taut triangulation.

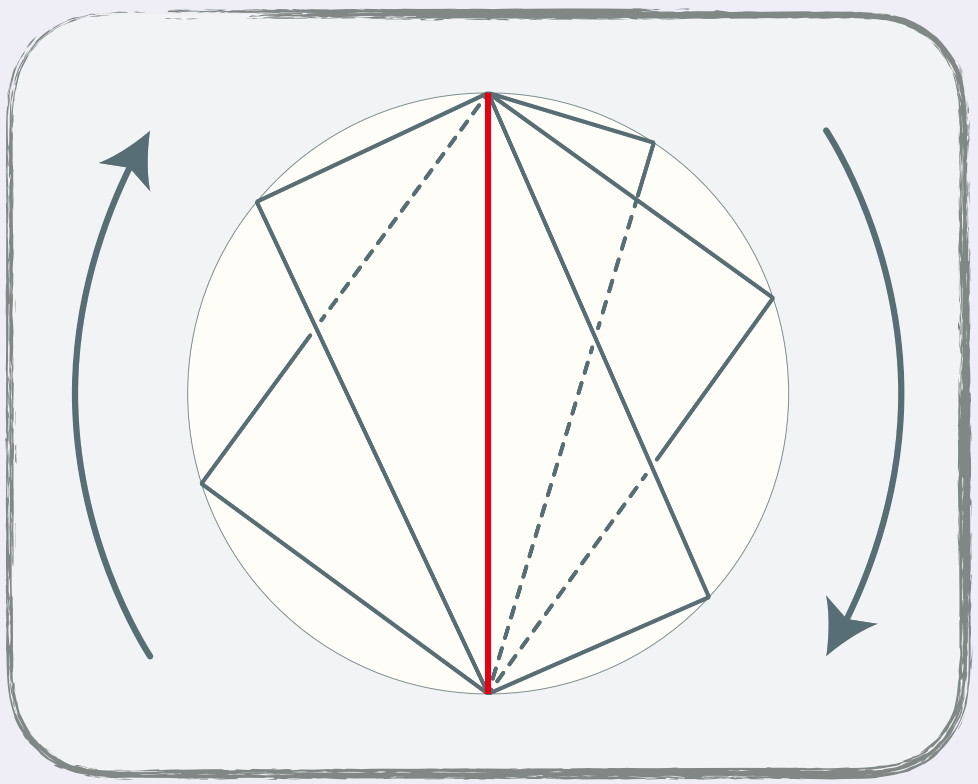
What is the meaning of veering

Theorem (Hodgson-Rubinstein-Segerman-Tillmann, 2011)

A taut triangulation of $\mathring{M}$ is veering

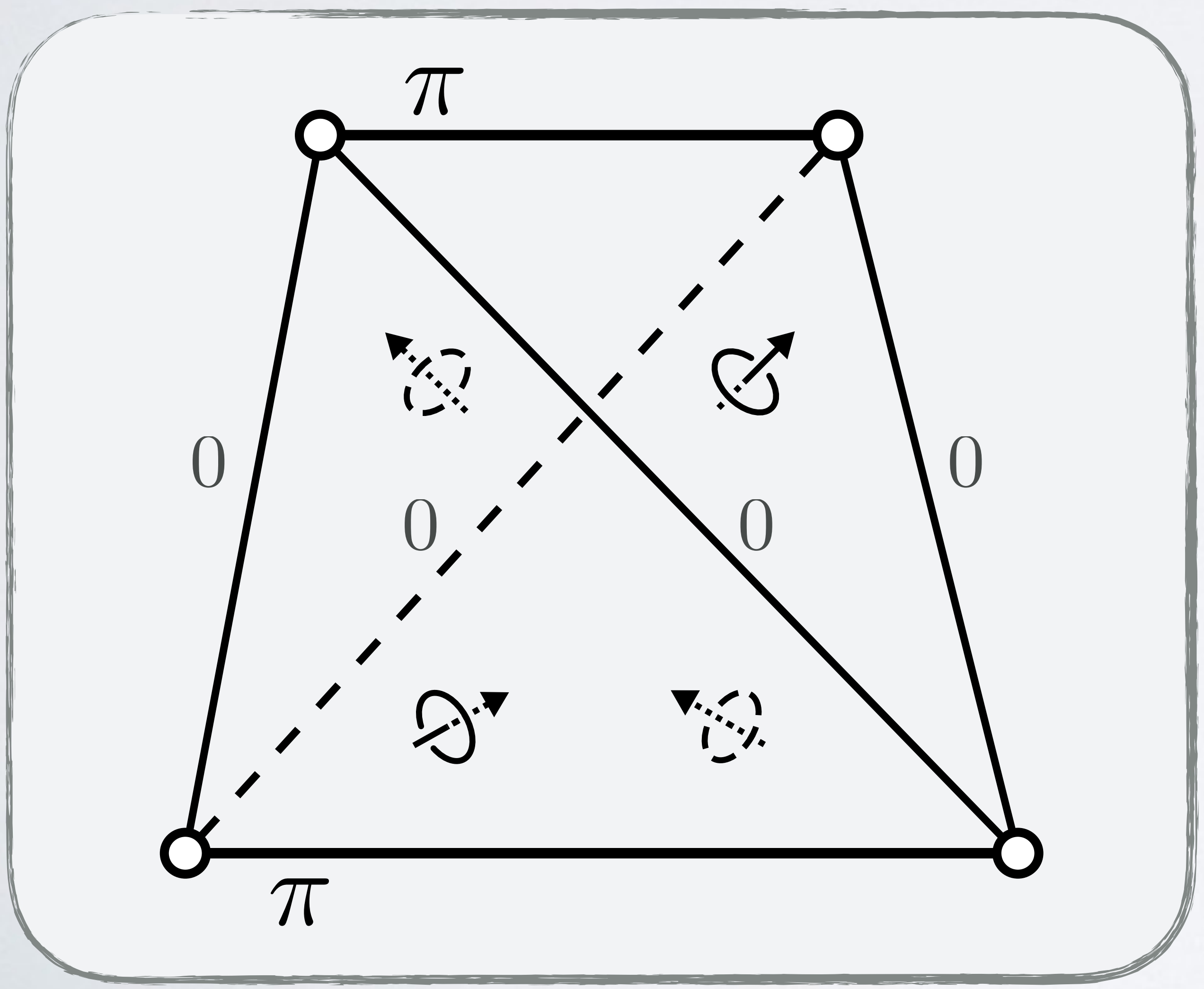
$\Longleftrightarrow$ Each edge of the taut triangulation is one of the following two types:





Veering tetrahedron and co-orientation

$\mathcal{D}$: taut triangulation of $\mathring{M}$



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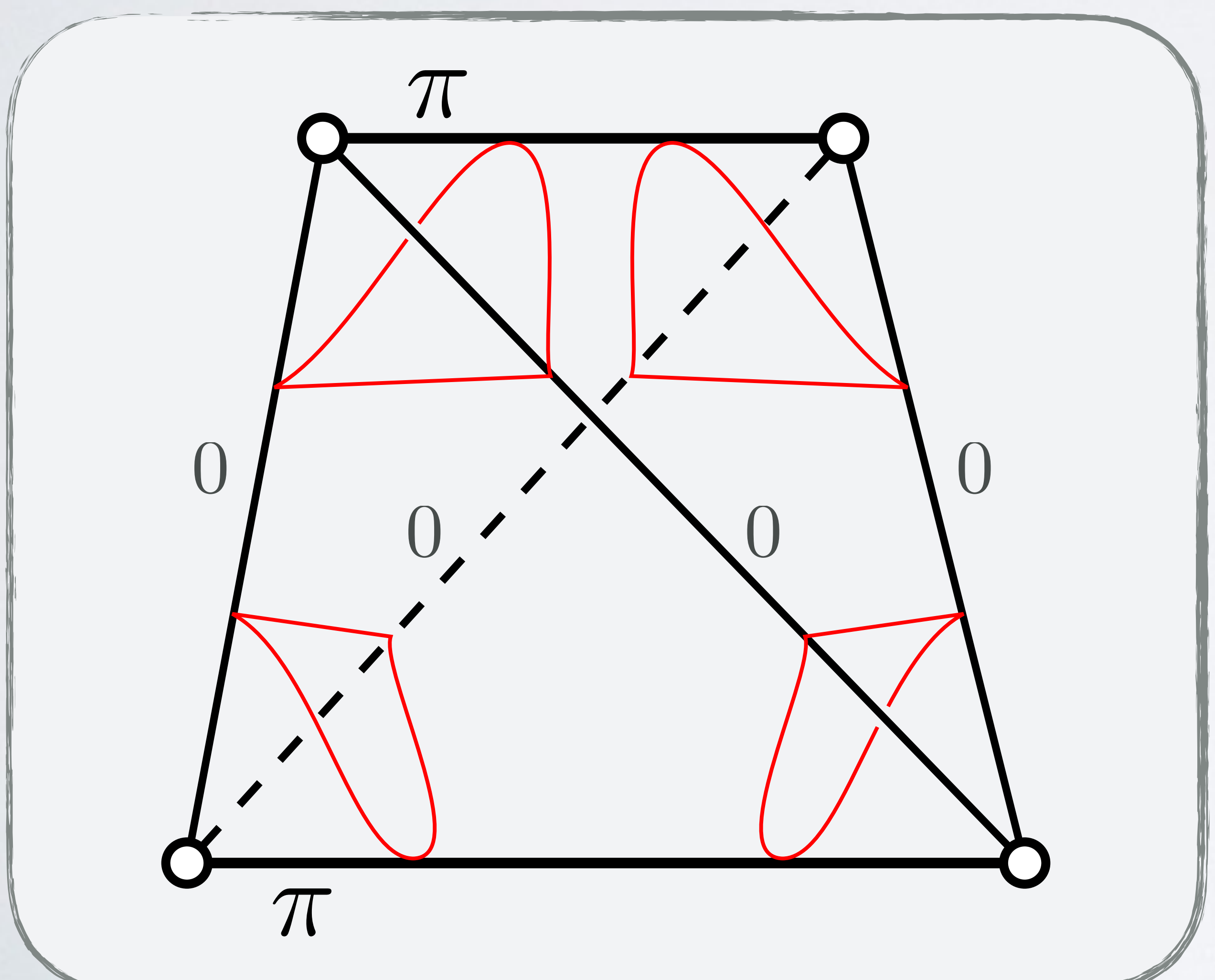
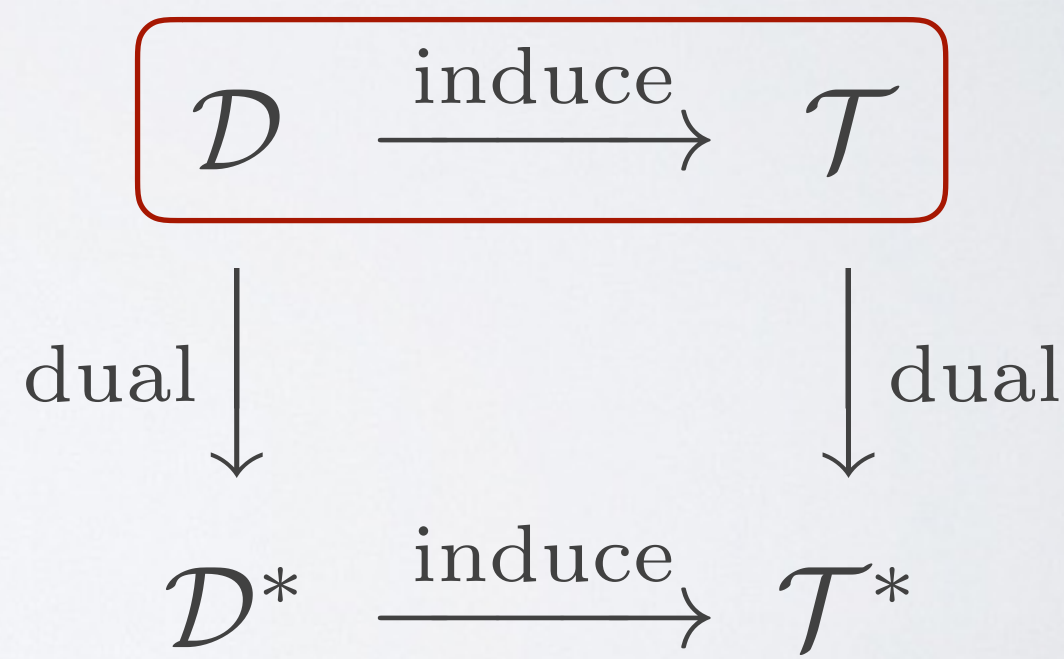
$\mathcal{D}$: taut triangulation of $\mathring{M}$

$\mathcal{T}$: triangulation of ∂M induced by $\mathcal{D}$





Veering tetrahedron and co-orientation

$\mathcal{D}$: taut triangulation of $\mathring{M}$

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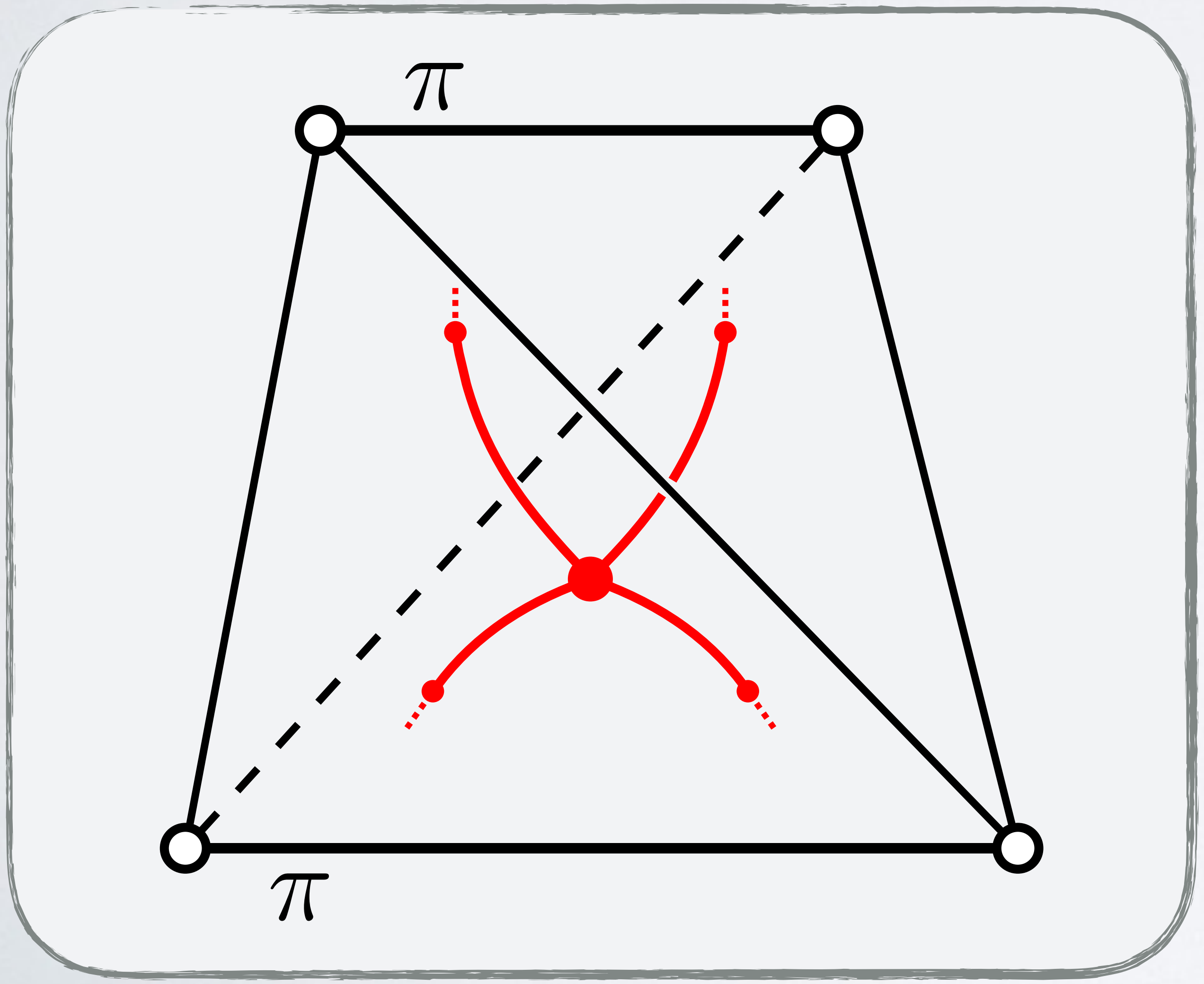
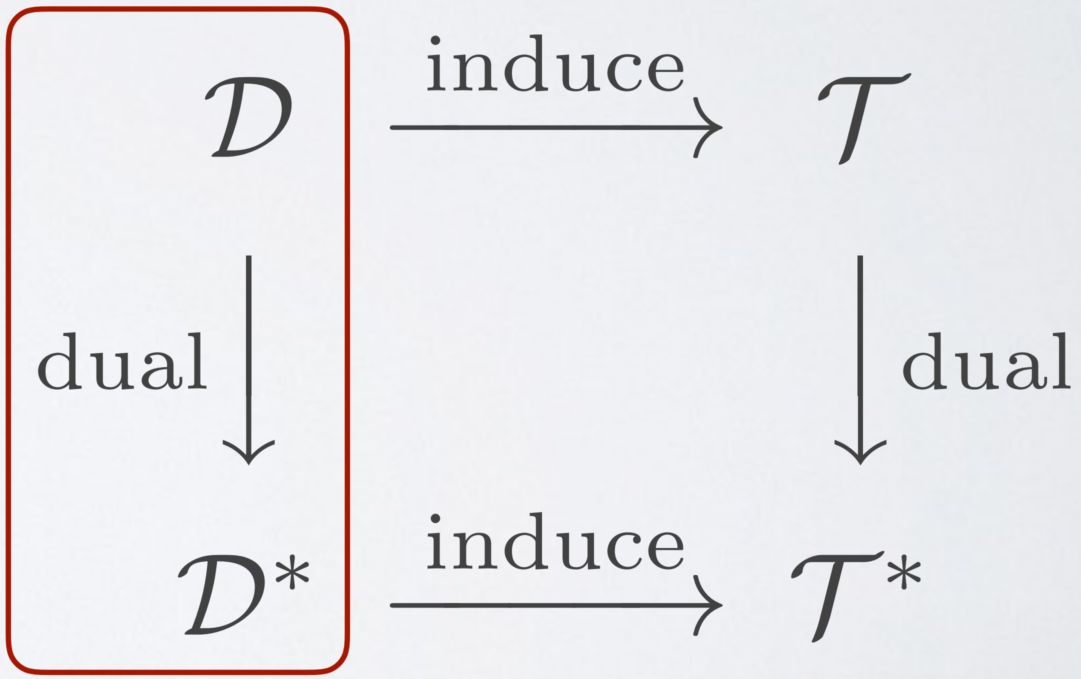
$\mathcal{D}^*$: 2-dim cell complex dual to $\mathcal{D}$





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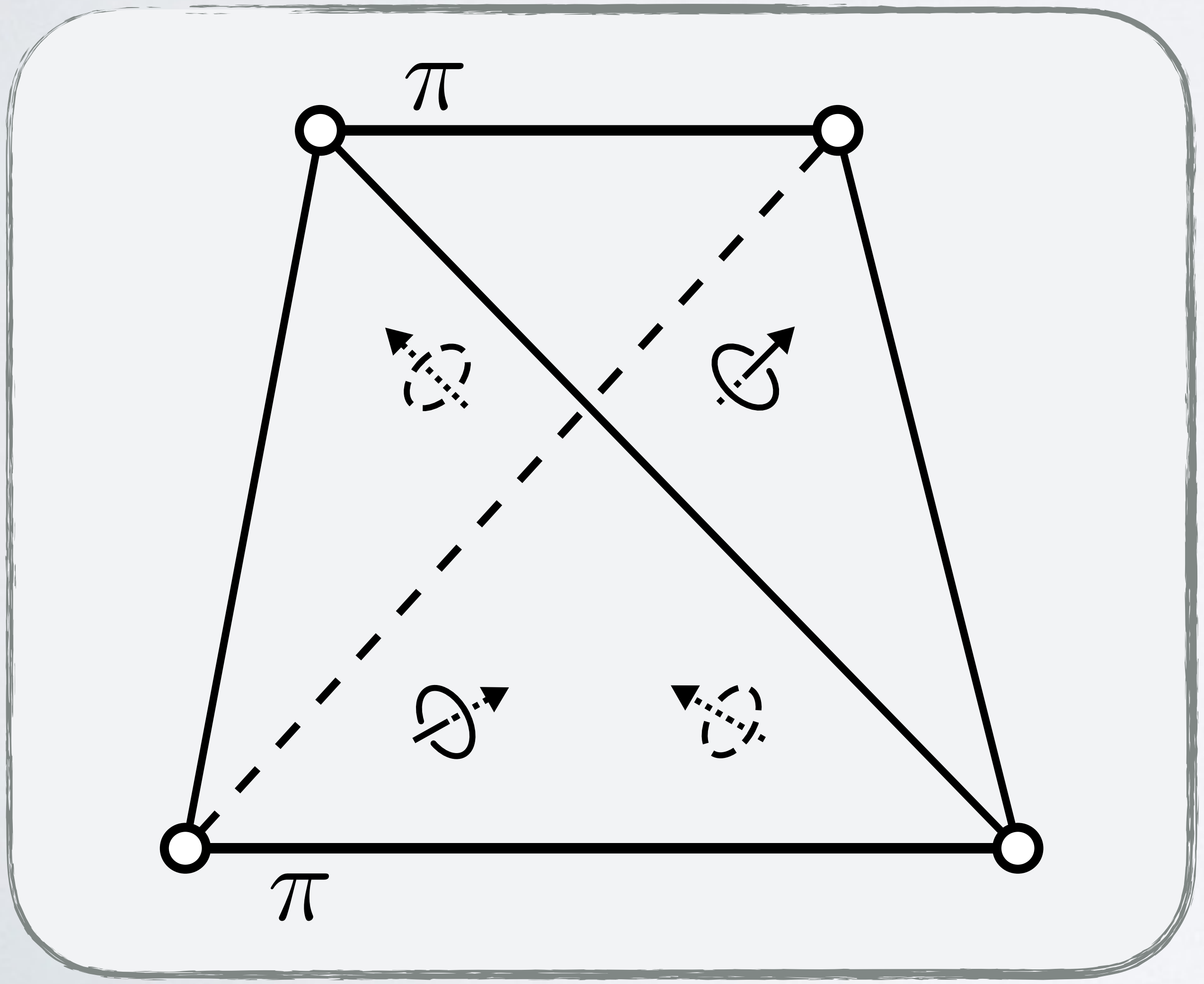

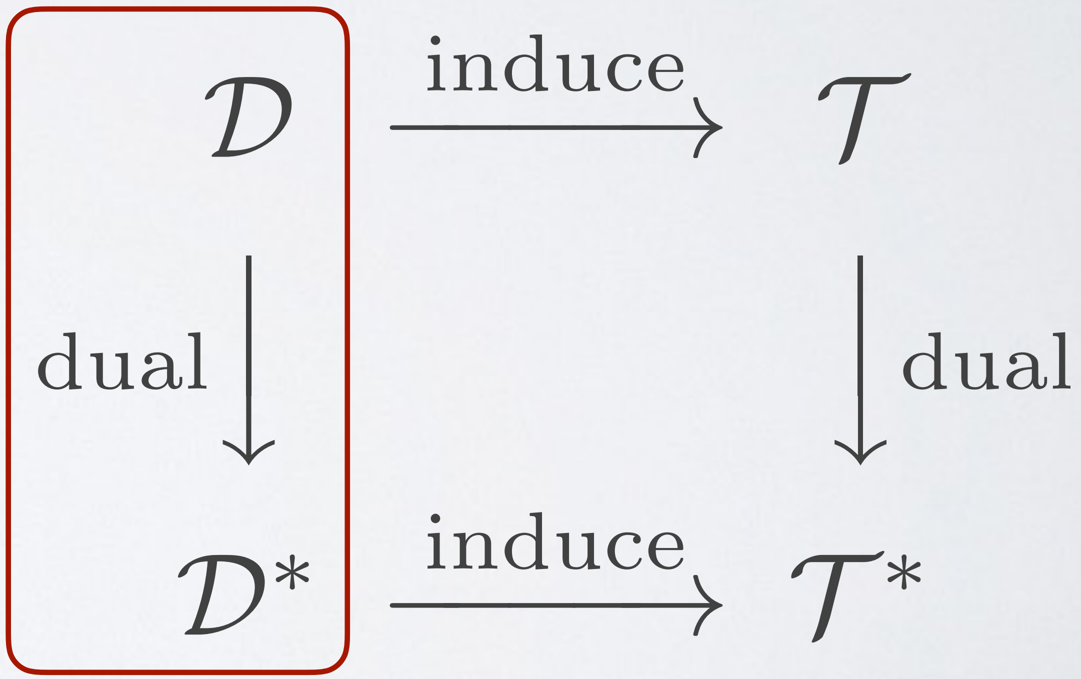

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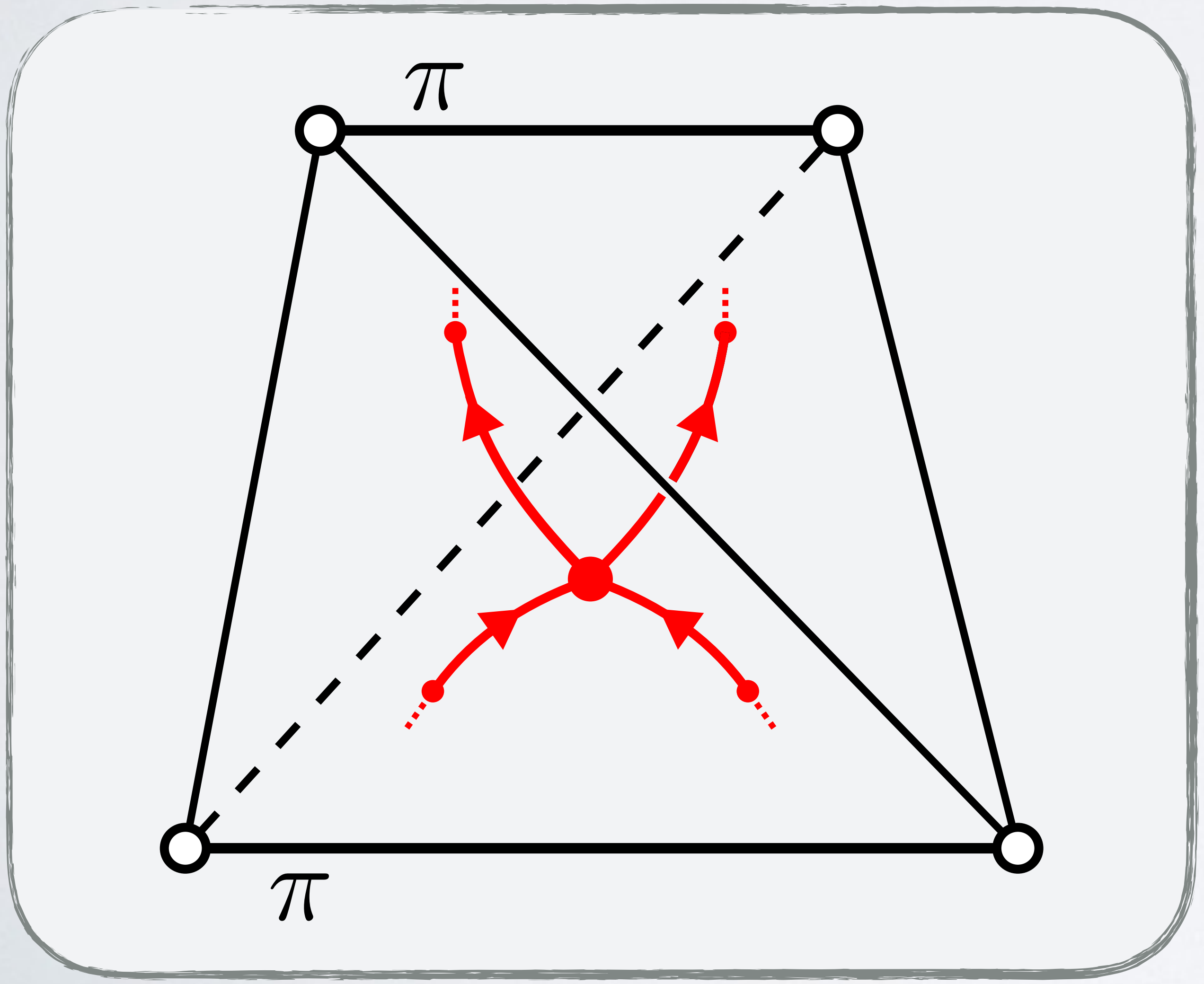

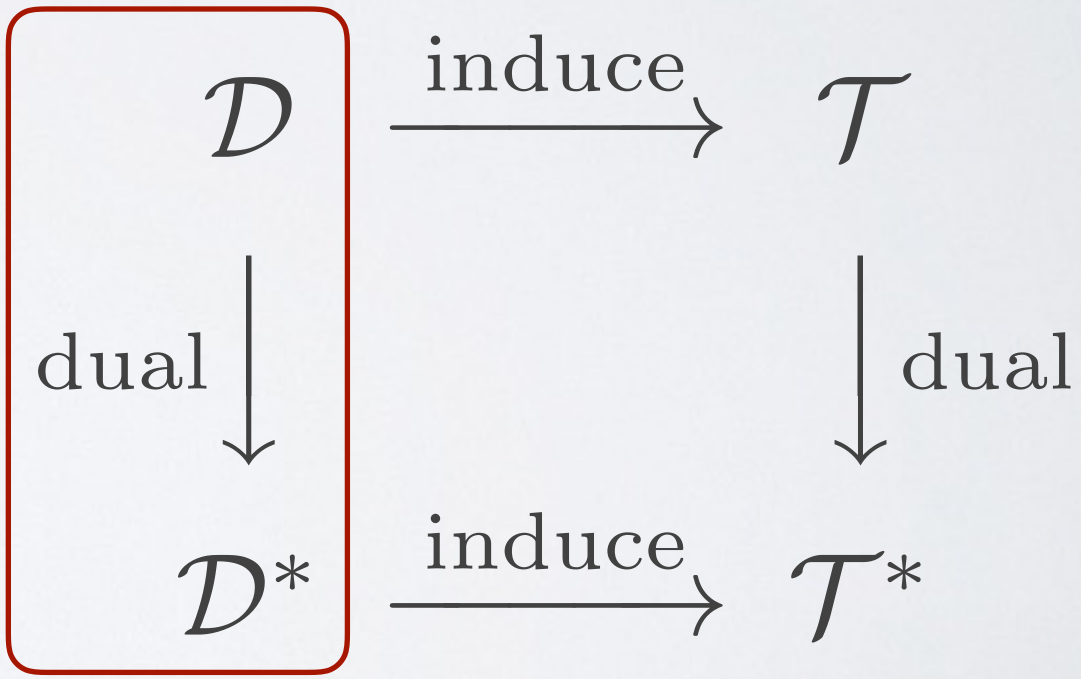

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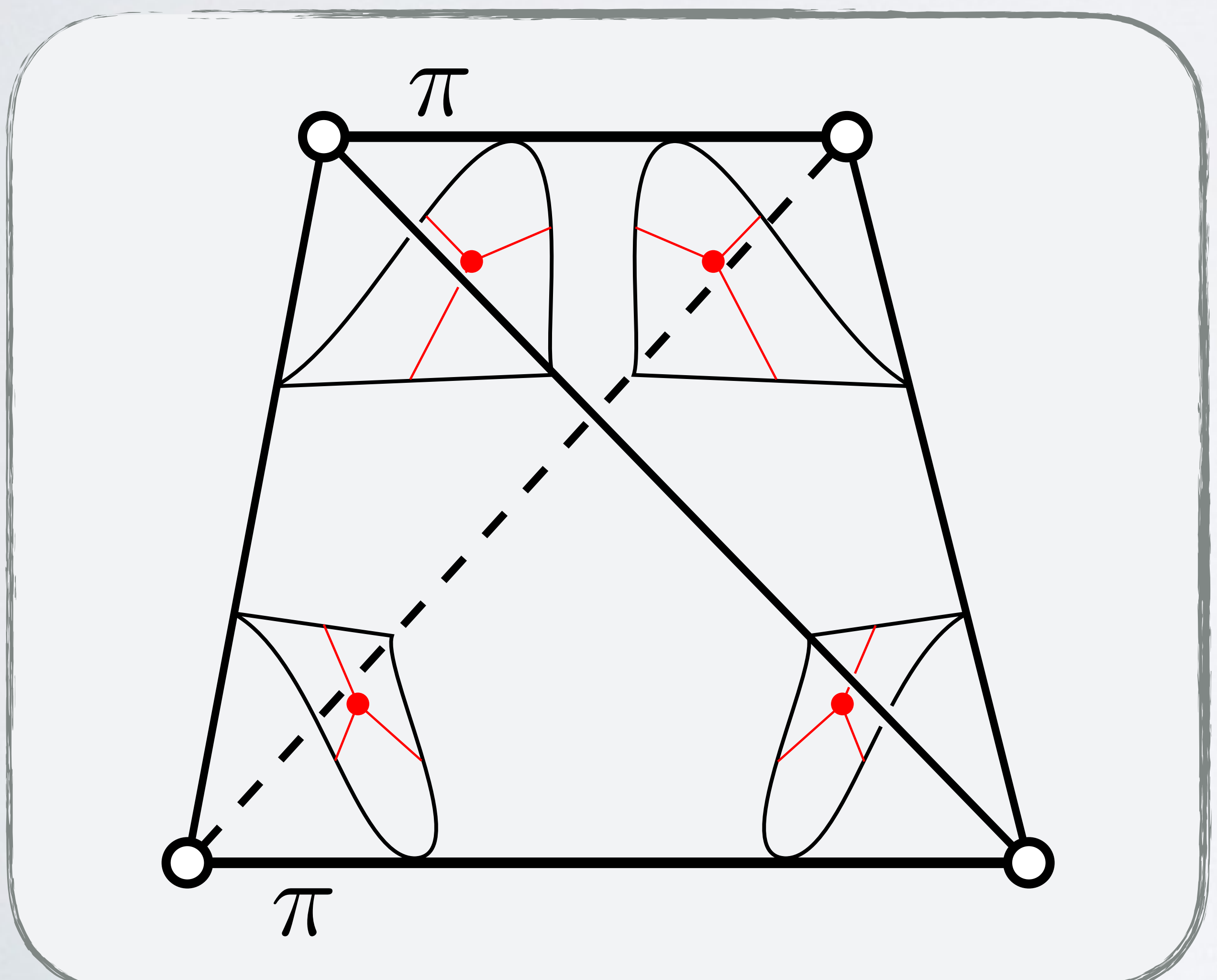

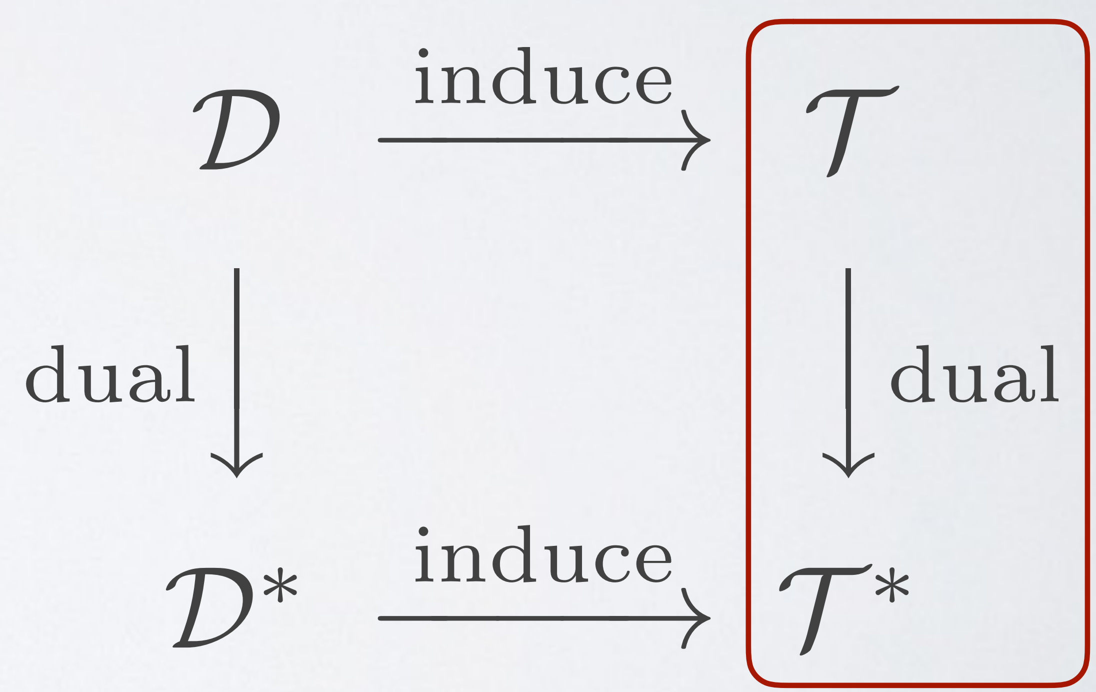

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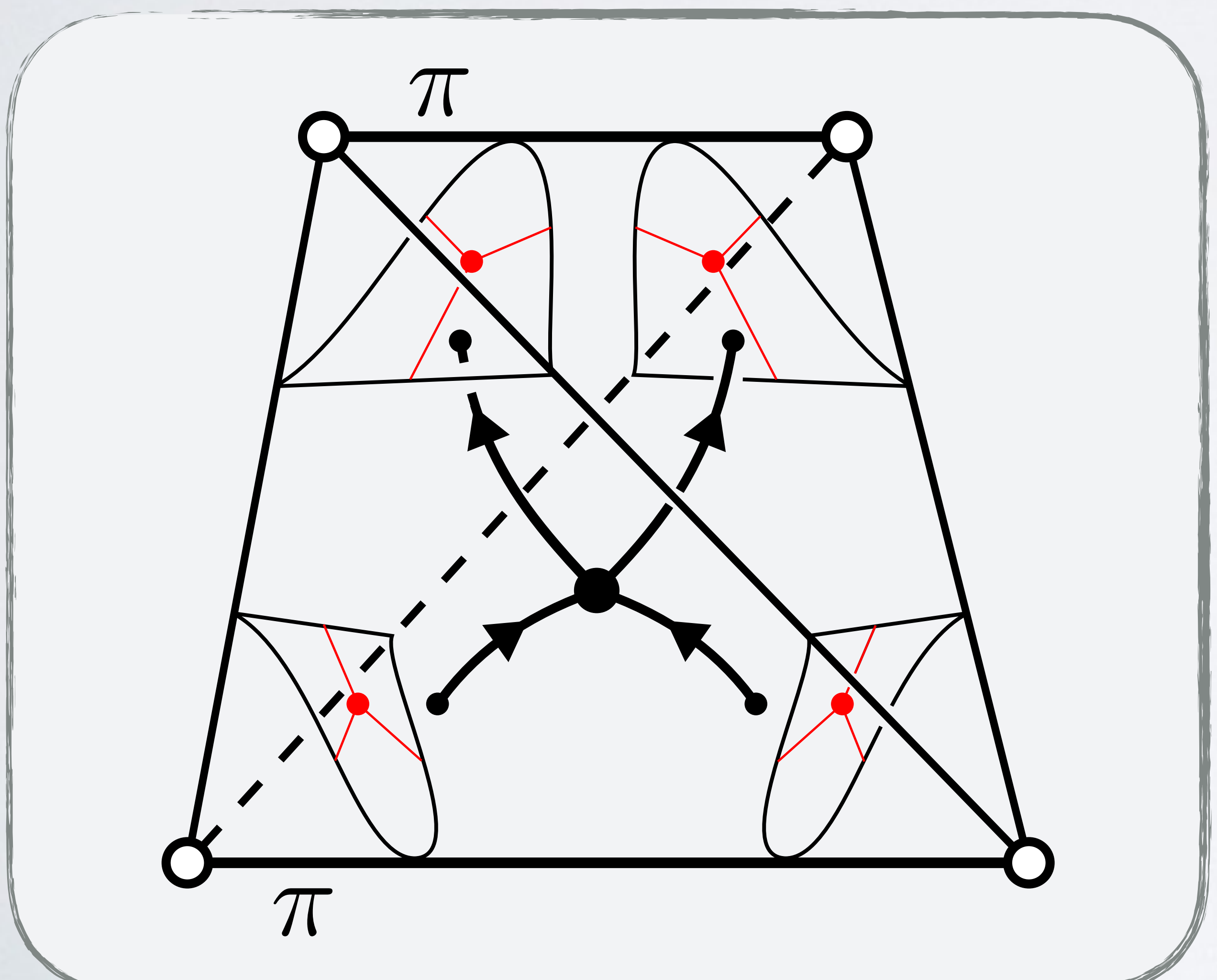

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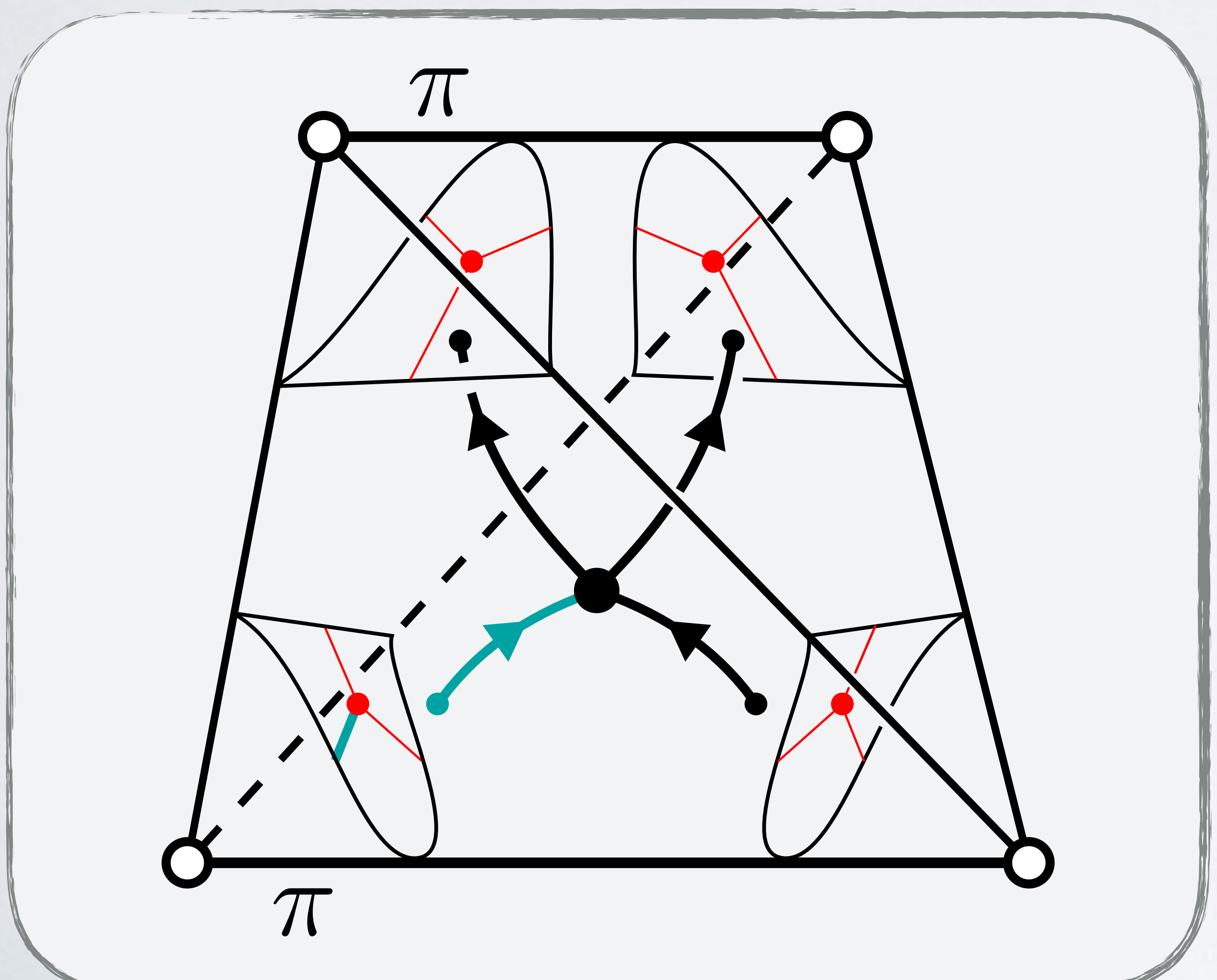

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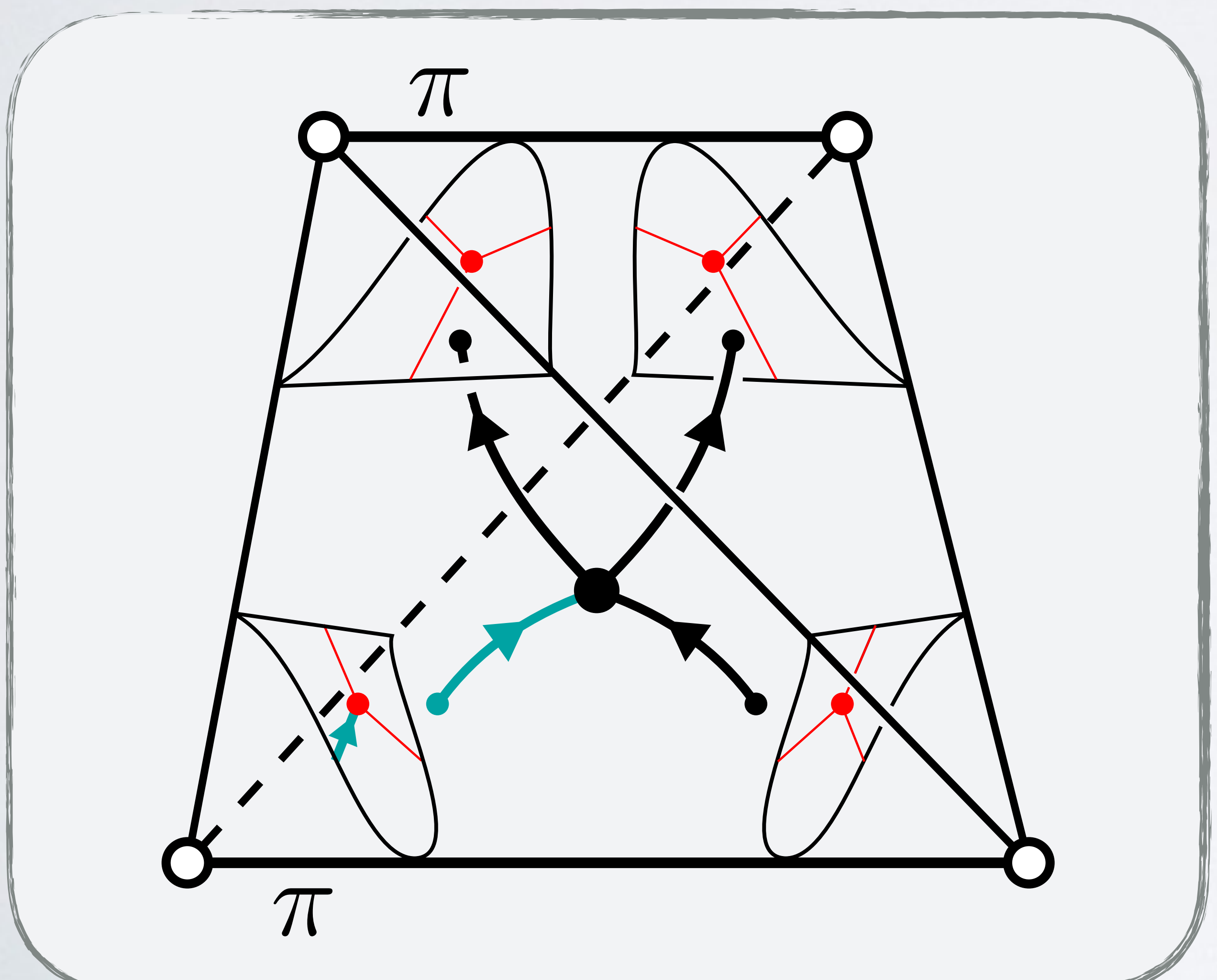

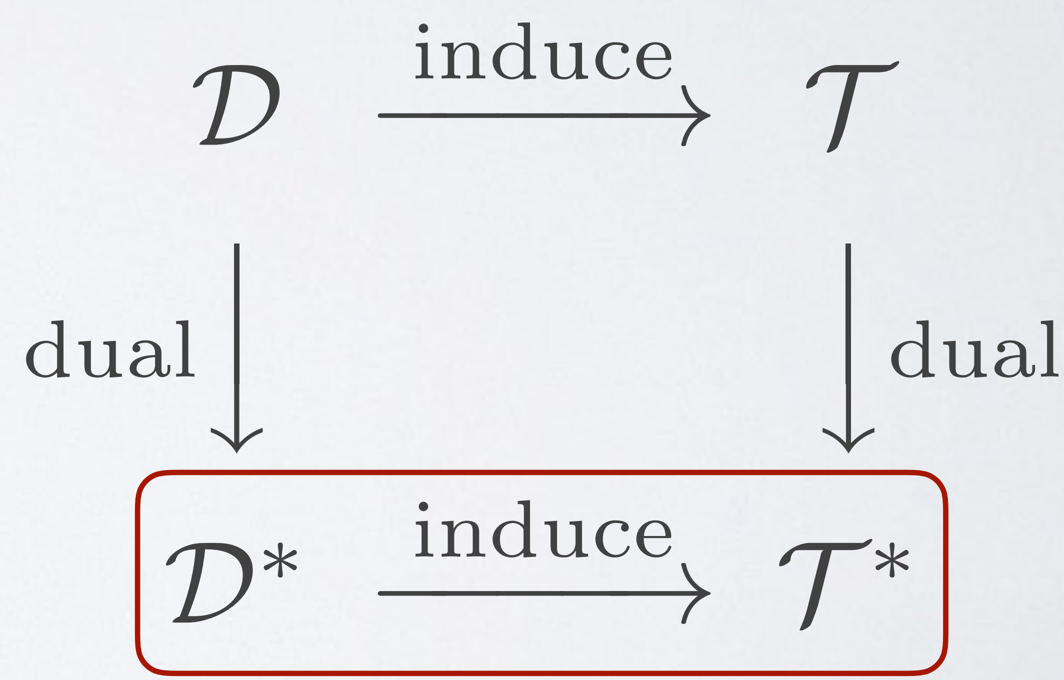

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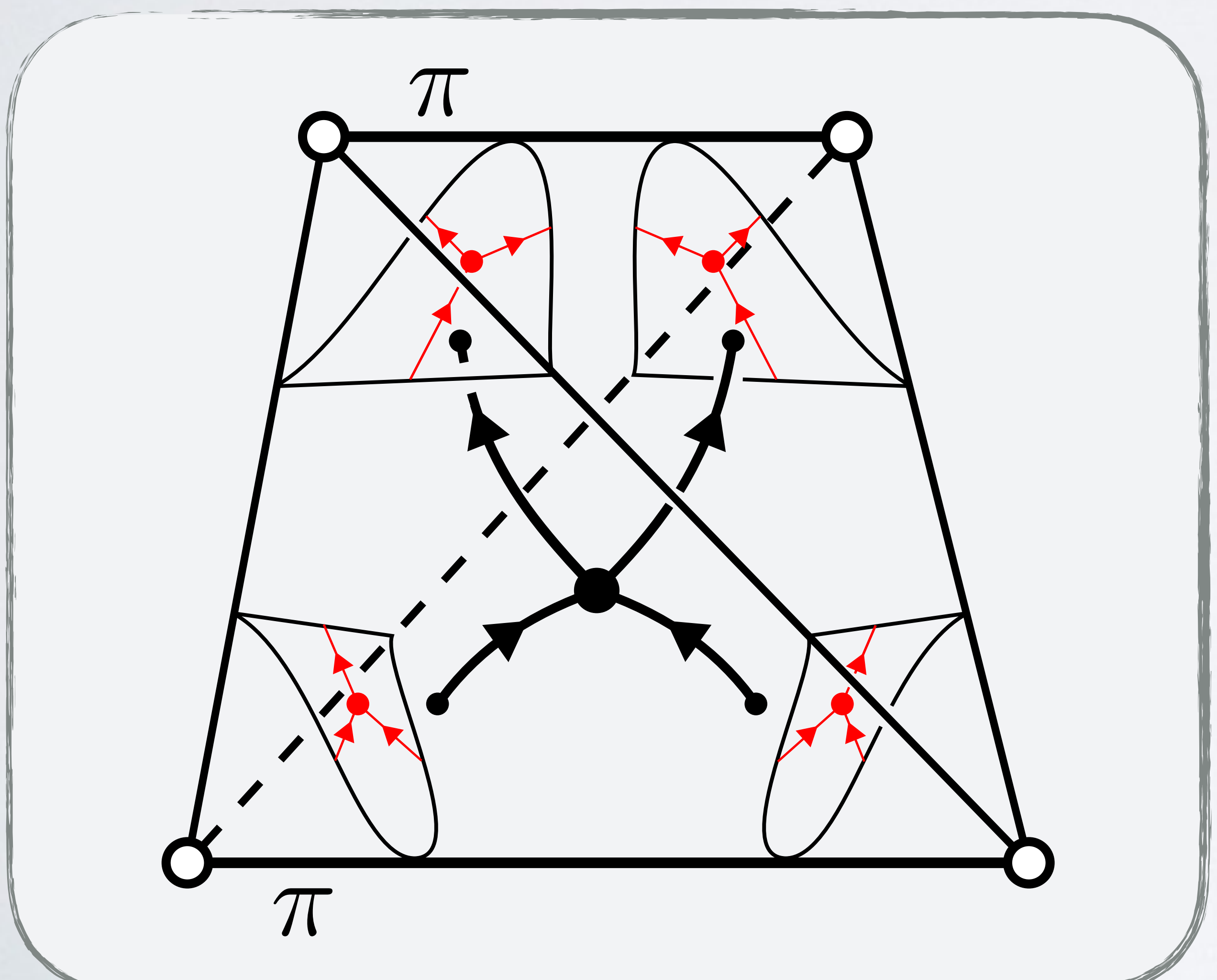

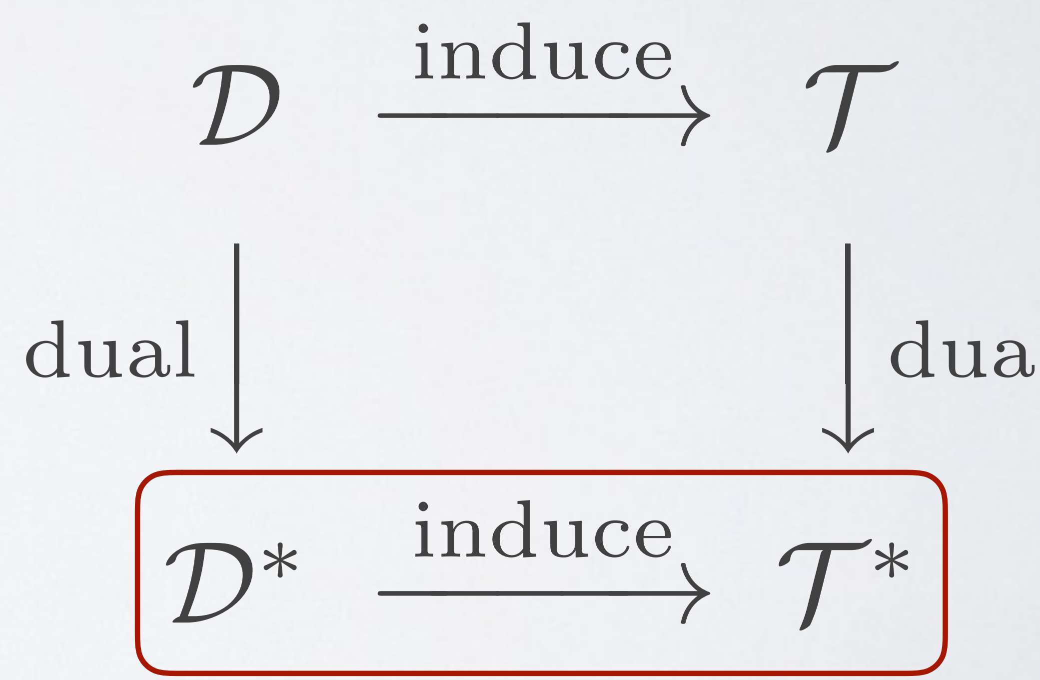

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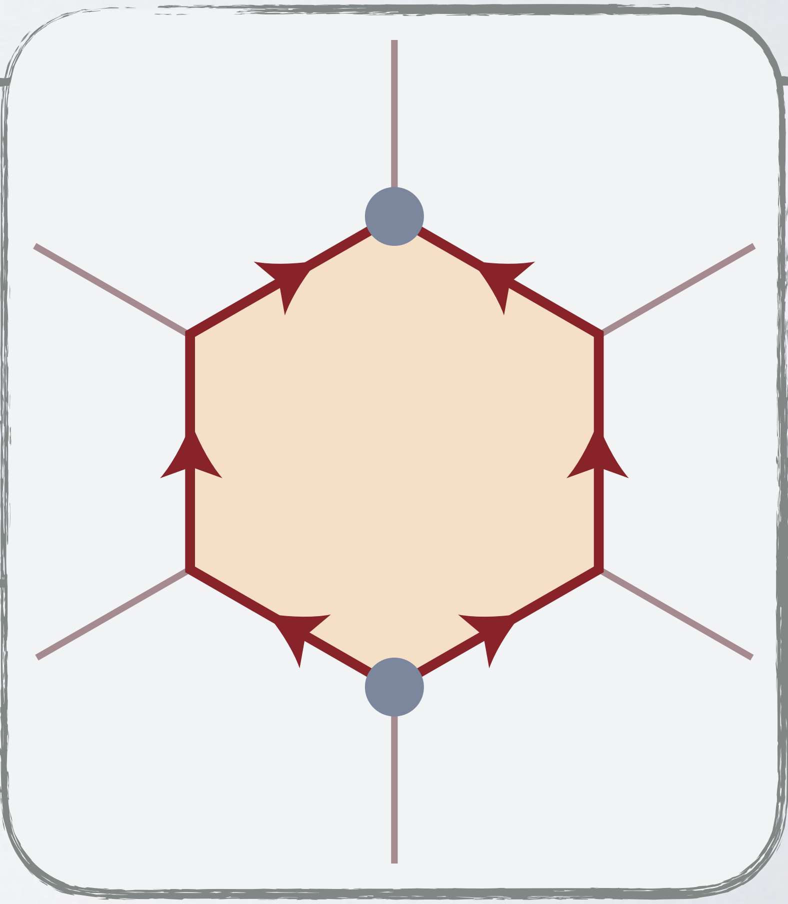
Fact

Each face of $\mathcal{T}^*$ has precisely one minimal vertex and precisely one maximal vertex.

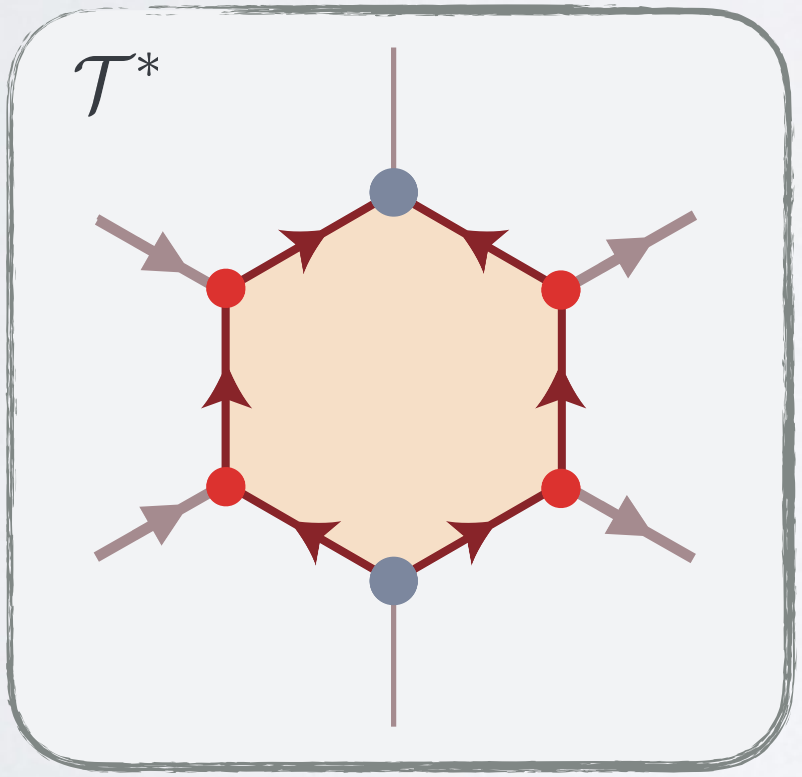


Veering tetrahedron and co-orientation

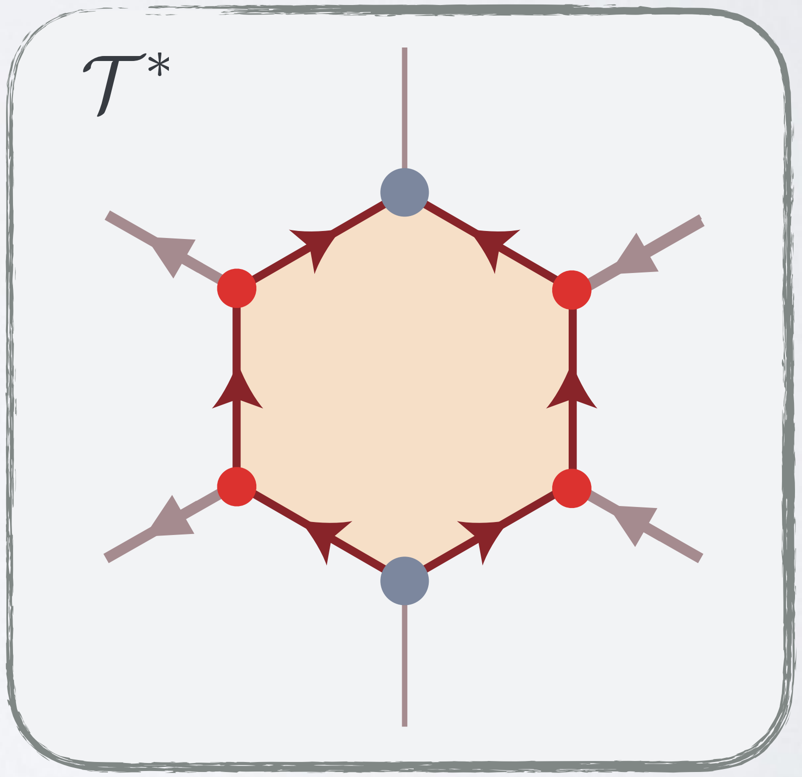
A face of $\mathcal{T}^*$ is *left-to-right* (resp. *right-to-left*)

$\overset{\text{def}}{\Longleftrightarrow}$

- The left-side of the face is "attractive" (resp. "repulsive").
- The right-side of the face is "repulsive" (resp. "attractive").



left-to-right face



right-to-left face

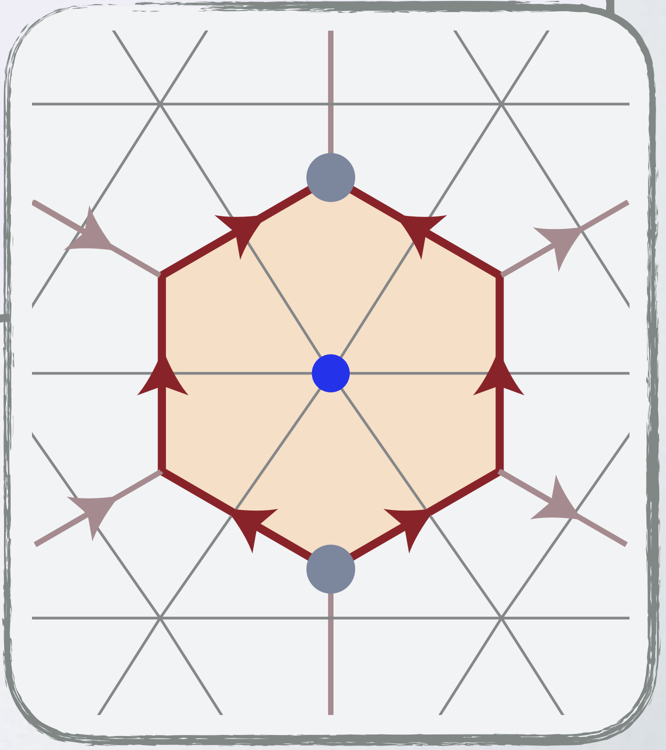
Method for checking if a triangulation is veering

Proposition

$\mathcal{D}$: taut triangulation of $\mathring{M}$

$\mathcal{D}$ is veering
$\iff$ Each face of $\mathcal{T}^*$ is either left-to-right or right-to-left.

Moreover, an ideal edge of $\mathcal{D}$ intersecting left-to-right (resp. right-to-left) face is blue-colored (resp. red-colored).



Idea of the proof of the main theorem

Recall (Theorem)

The canonical decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0 < |r| < 1/2)$ is veering $\Longleftrightarrow$ the slope r has the continued fraction expansion $\pm[2, 2, \ldots, 2]$.

Remark

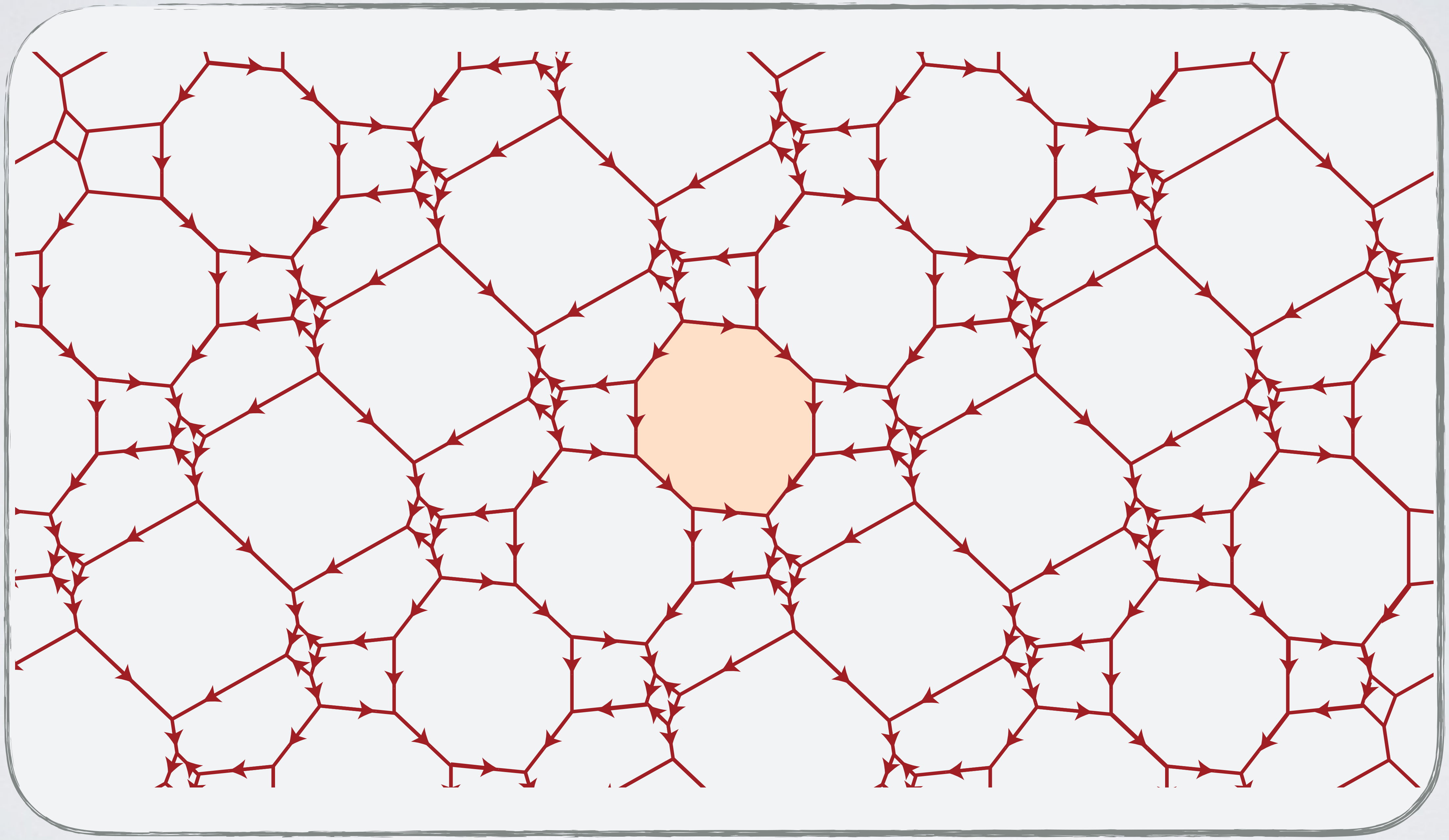
Guéritaud and Futer have proved that the canonical decompositions of the hyperbolic two-bridge link complements are equal to the ideal triangulations given by [Sakuma-Weeks].

Idea of the proof of the "only if" part

The canonical decomposition of $K(r)$ is NOT veering.

$(0 < |r| < 1/2,\ r \neq \pm[2, 2, \ldots, 2])$

$r = [2, 2, 2, -2]$



(given by SnapPy)

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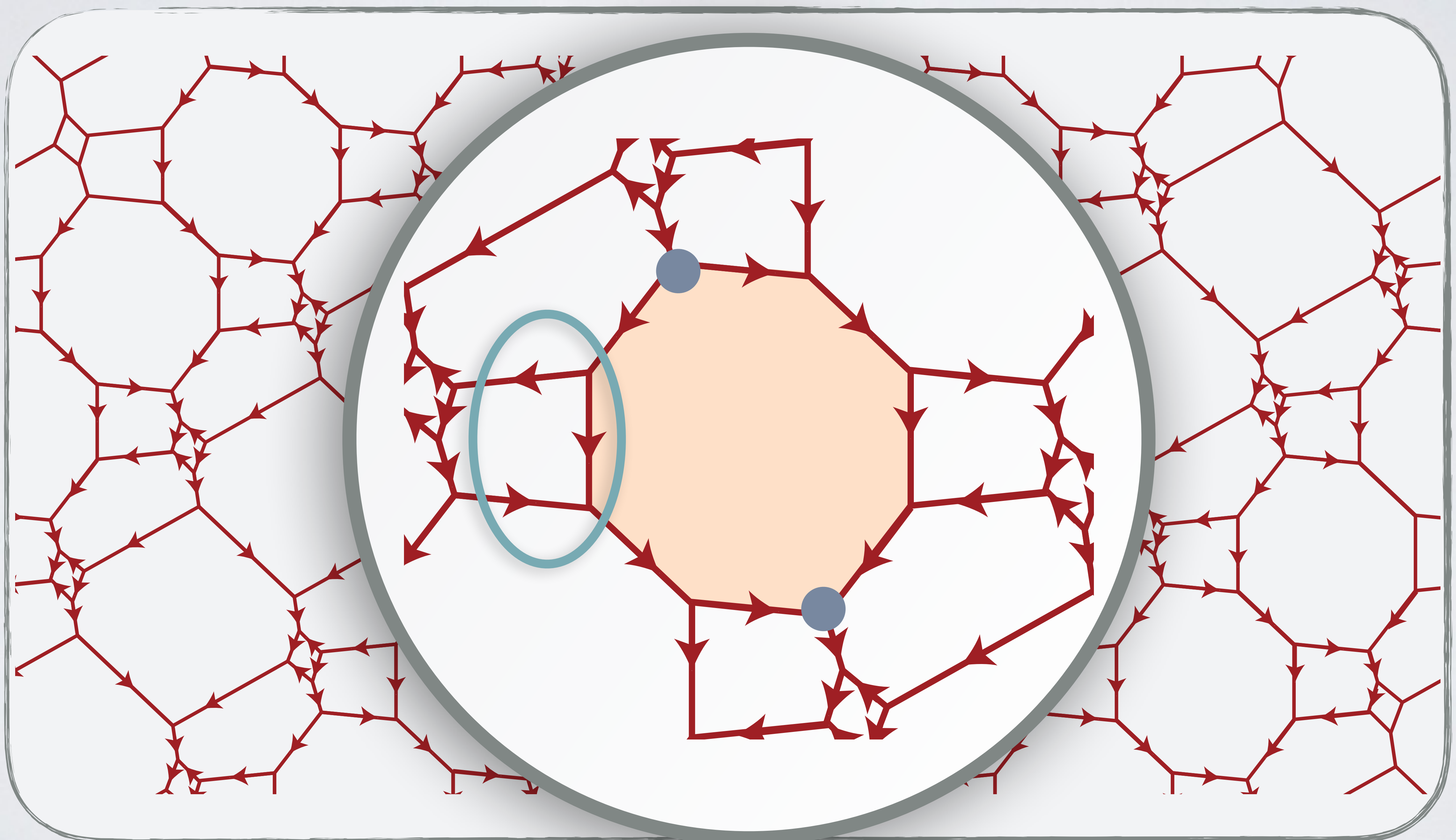

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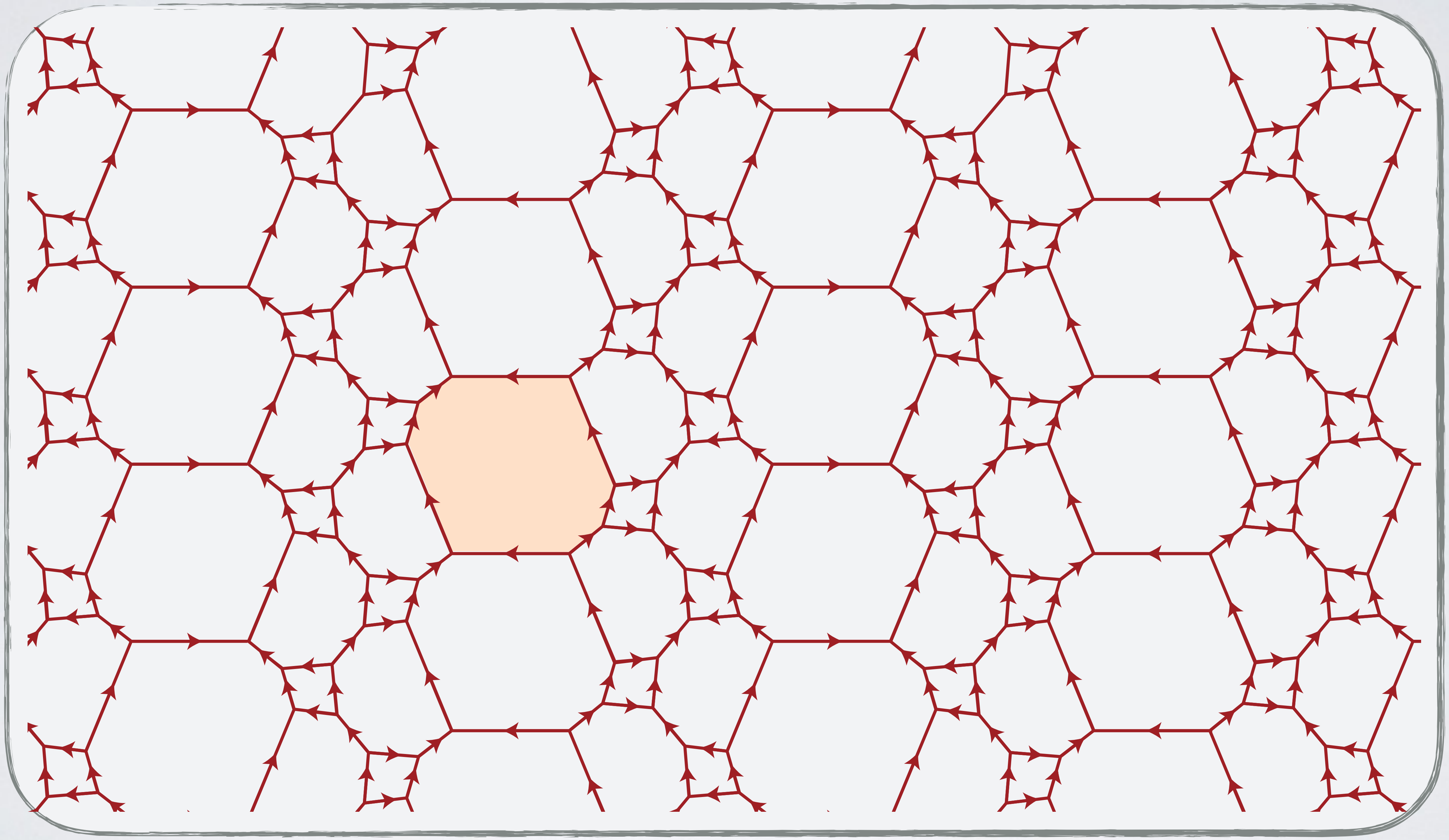

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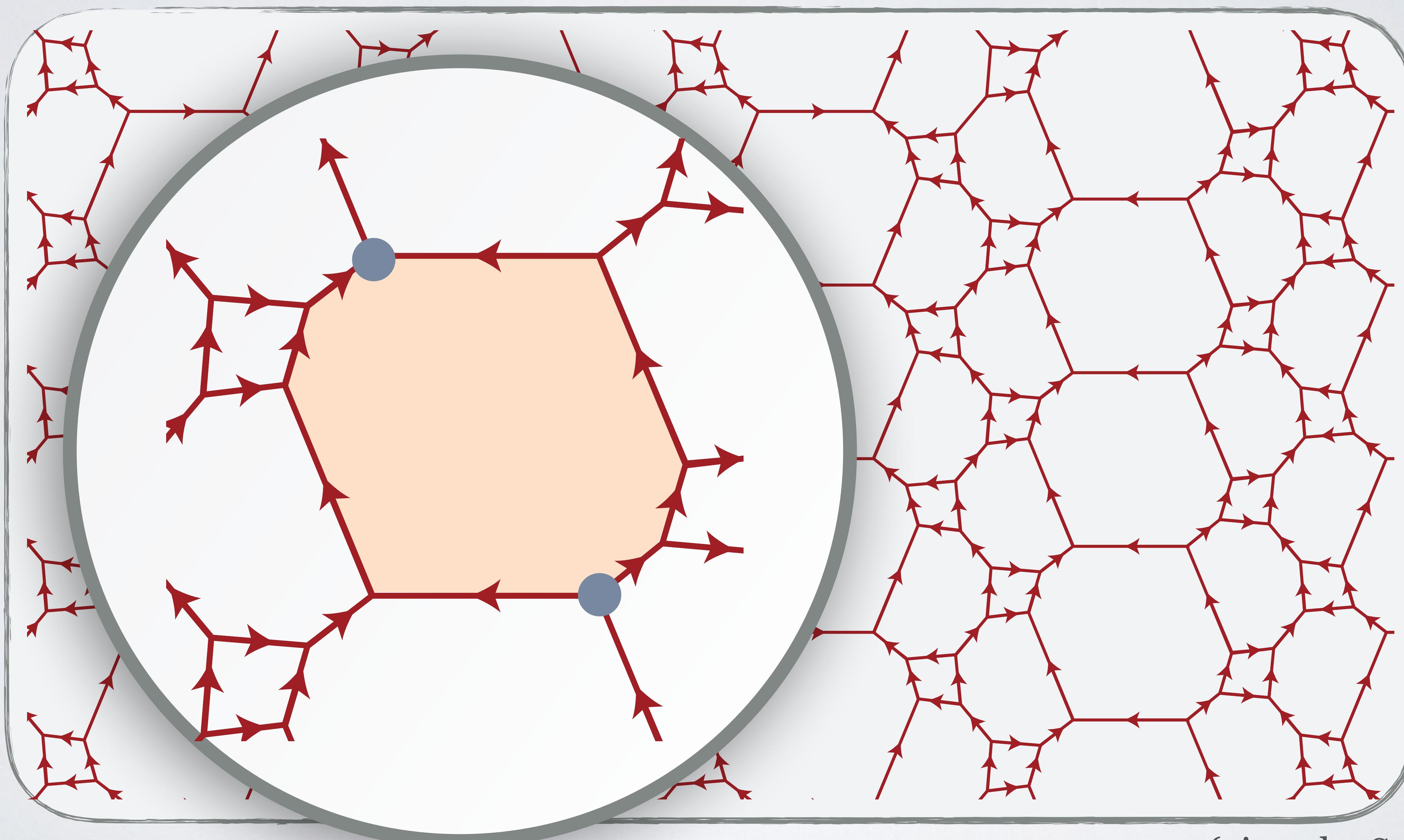

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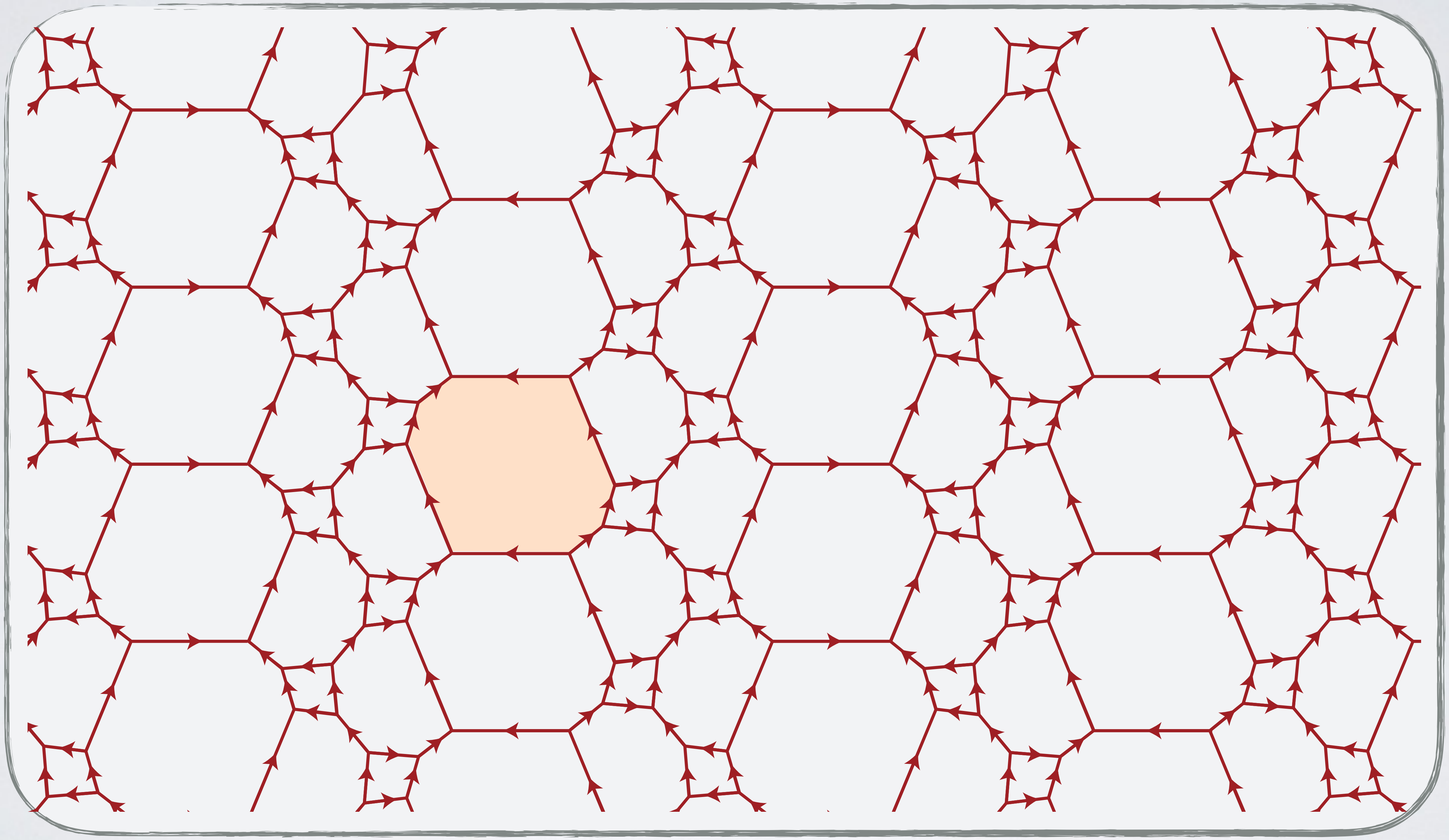
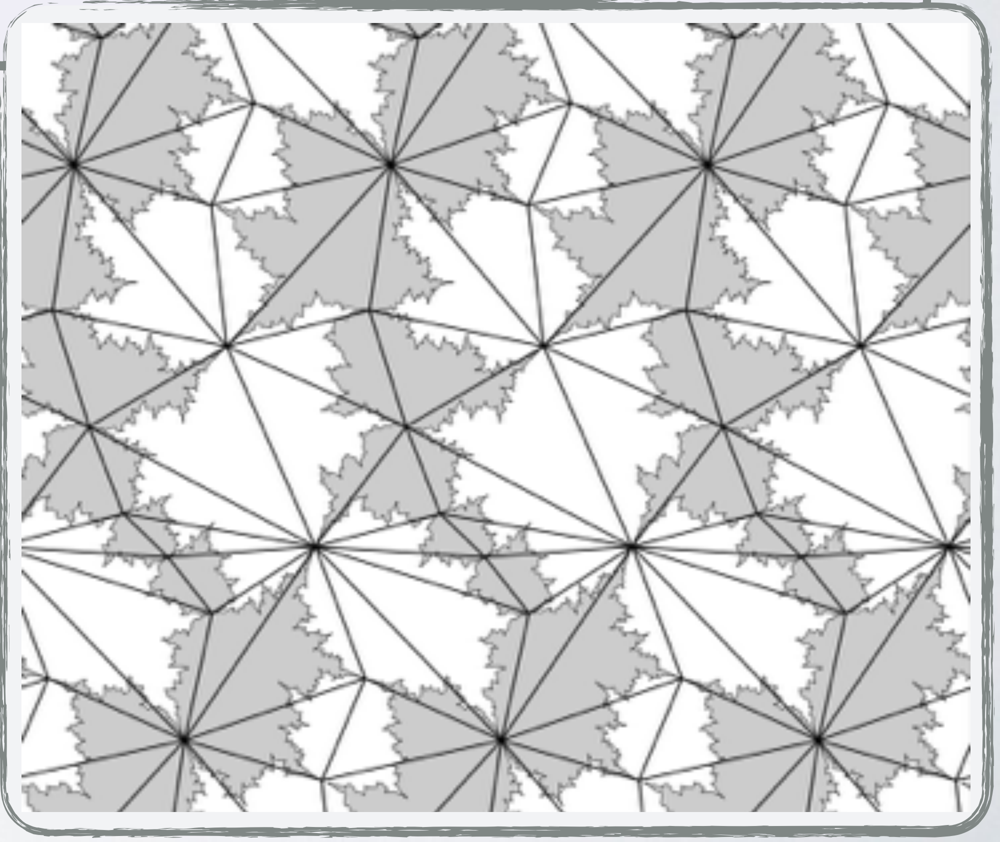


(given by SnapPy)

Future work (1)

Theorem (Dicks-Sakuma, 2010)

For a once-punctured torus bundle, the cusp triangulation induced by the canonical decomposition with the "layered structure" combinatorially determines the fractal tessellation with the "colored structure", and vice versa. In particular, two tessellations share the same vertex set.



Future work (2)

Guéritaud has established a beautiful relation between veering and layered triangulation of hyperbolic punctured surface bundles and the associated CT-maps.

↓ by using the main theorem

The fractal tessellation and the canonical decomposition of the complement of the two-bridge link $K(r)$ with $r = \pm[2, 2, \ldots, 2]$ are intimately related.

Future work (3)

Question

For r with $0 < |r| < 1/2$ and $r \neq \pm[2, 2, \ldots, 2]$, does there exist a relation between the fractal tessellation and the canonical decomposition of the complement of a hyperbolic fibered two-bridge link $K(r)$?

Thank you for your attention!