Corks, exotic 4-manifolds and knot concordance

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I. Background and Main results

Exotic 4-manifolds represented by framed knots

Application to knot concordance

II. Brief review of corks

III. Proof of the main results

1.A. Exotic framed knots

Problem

Does every smooth 4-manifold admit an exotic (i.e. homeo but non-diffeo) smooth structure?

We consider a special class of 4-manifolds:

A framed knot (i.e. knot $+$ integer) in S^3 gives a 4-mfd by attaching 2-handle $D^2 \times D^2$ to D^4 along the framed knot.

A pair of framed knots in S^3 is said to be **exotic** if they represent homeo but non-diffeo 4-mfds.

Problem

Find exotic pairs of framed knots!

Remark. $\exists$ framed knot admitting NO exotic framed knot

1.A. Exotic framed knots

Problem

Find exotic pairs of framed knots!

Theorem (Akbulut '91)

$\exists$ an exotic pair of -1-framed knots.

Theorem (Kalmár-Stipsicz '13)

$\exists$ an infinite family of exotic pairs of -1-framed knots.

Remark. Framings of these examples are all -1.
For each pair, one 4-mfd is Stein, but the other is non-Stein.

1.A. Exotic framed knots

Theorem (Y)

$\forall n \in \mathbb{Z}$, $\exists$ infinitely many exotic pairs of n-framed knots.
Furthermore, both knots in each pair gives Stein 4-mfds.

Moreover, we give machines which produce vast examples.

Recall:

A knot P in $S^1 \times D^2$ induces a **satellite map**

$P : \{\text{knot in } S^3\} \to \{\text{knot in } S^3\}$

by identifying reg. nbd of a knot with $S^1 \times D^2$ via 0-framing.

1.A. Exotic framed knots

Machines producing vast examples:

Main Theorem (Y)

$\forall n \in \mathbb{Z}$, $\exists$ satellite maps P_n, Q_n s.t.
for any knot K in S^3 with

$$2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \widehat{tb}(K),$$

n-framed $P_n(K)$ and $Q_n(K)$ are an exotic pair.

Remark.

For each n, there are many K satisfying the assumption.
If K satisfies the assumption, then $P_n(K)$ and $Q_n(K)$ satisfy.

1.A. Different viewpoint: exotic satellite maps

For a satellite map $P : \{\text{knot}\} \to \{\text{knot}\}$ and $n \in \mathbb{Z}$,
we define a **4-dimensional n-framed satellite map**

$$P^{(n)} : \{\text{knot in } S^3\} \to \{\text{smooth 4-mfd}\}$$

by $P^{(n)}(K) =$ 4-manifold represented by n-framed $P(K)$.

$P^{(n)}$ and $Q^{(n)}$ are called smoothly the same,
if $P^{(n)}(K)$ and $Q^{(n)}(K)$ are diffeo for any knot K

New difference between smooth and topological categories:

Theorem (Y)

$\forall n \in \mathbb{Z}$, $\exists$ 4-dim n-framed satellite maps
which are topologically the same but smoothly distinct.

1.A. Different viewpoint: exotic satellite maps

For a satellite map $P : \{\text{knot}\} \to \{\text{knot}\}$ and $n \in \mathbb{Z}$,
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1.B. Application to knot concordance

n-surgery on a knot K in S^3 := boundary of the 4-mfd represented by n-framed K.

Two oriented knots K_0, K_1 are **concordant**

if $\exists\ S^1 \times I \hookrightarrow S^3 \times I$ s.t. $S^1 \times i = K_i \times i \quad (i = 0, 1)$.

Conjecture (Akbulut-Kirby 1978)

If 0-surgeries on two knots in S^3 give the same 3-mfd, then the knots (with relevant ori) are concordant.

Remark. Quotation from Kirby's problem list ('97):
all known concordance invariants of the two knots
are the same.

1.B. Application to knot concordance

Conjecture (Akbulut-Kirby 1978)

If 0-surgeries on two knots in S^3 give the same 3-mfd, then the knots (with relevant ori) are concordant.

Theorem (Cochran-Franklin-Hedden-Horn 2013)
$\exists$ infinitely many pairs of non-concordant knots with homology cobordant 0-surgeries.

Theorem (Abe-Tagami)
If the slice-ribbon conjecture is true, then the Akbulut-Kirby conjecture is false.

1.B. Application to knot concordance

Conjecture (Akbulut-Kirby 1978)

If 0-surgeries on two knots in S^3 give the same 3-mfd, then the knots (with relevant ori) are concordant.

Theorem (Y)

$\exists$ infinitely many counterexamples to AK conjecture.

In fact, our exotic 0-framed knots are counterexamples.

Corollary (Y)

Knot concordance invariants $g_4, \tau,\ s$ are NOT invariants of 3-manifolds given by 0-surgeries on knots.

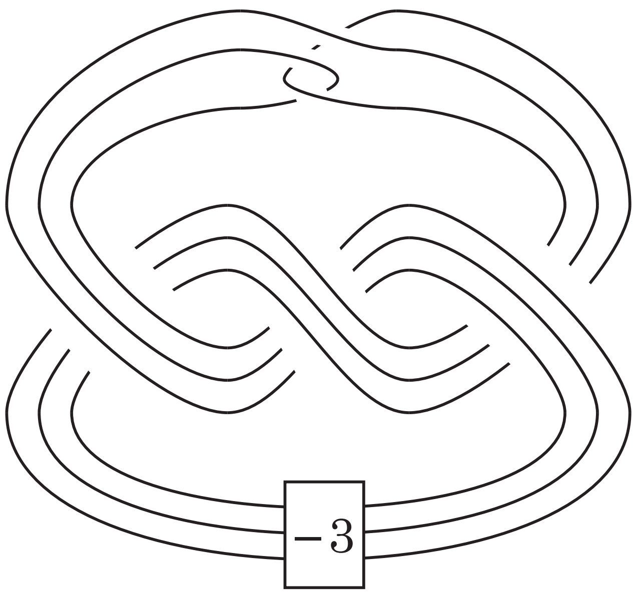
1.B. Application to knot concordance

Conjecture (Akbulut-Kirby 1978)

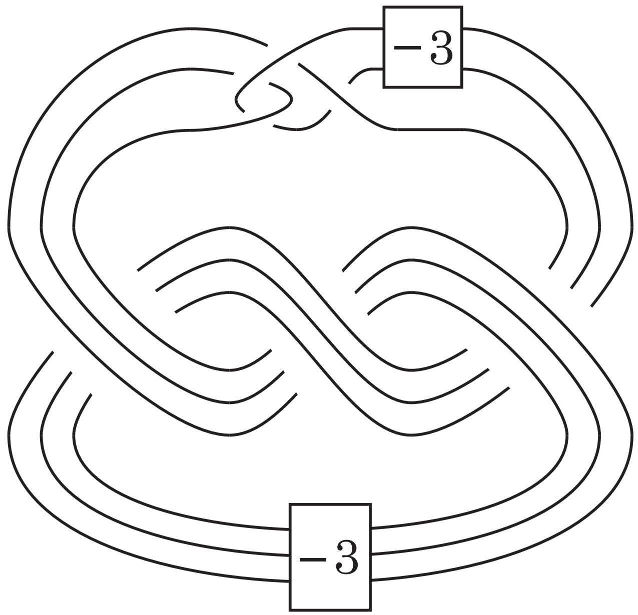
If 0-surgeries on two knots in S^3 give the same 3-mfd, then the knots (with relevant ori) are concordant.

Simple counterexample

$P_0(T_{2,3})$



$Q_0(T_{2,3})$



1.B. Application to knot concordance

Conjecture (Akbulut-Kirby 1978)

If 0-surgeries on two knots in S^3 give the same 3-mfd, then the knots (with relevant ori) are concordant.

Question.

If two 0-framed knots in S^3 give the same smooth 4-mfd, are the knots (with relevant ori) concordant?

Remark

Abe-Tagami's proof shows the answer is no, if the slice-ribbon conjecture is true.

2. Brief review of corks

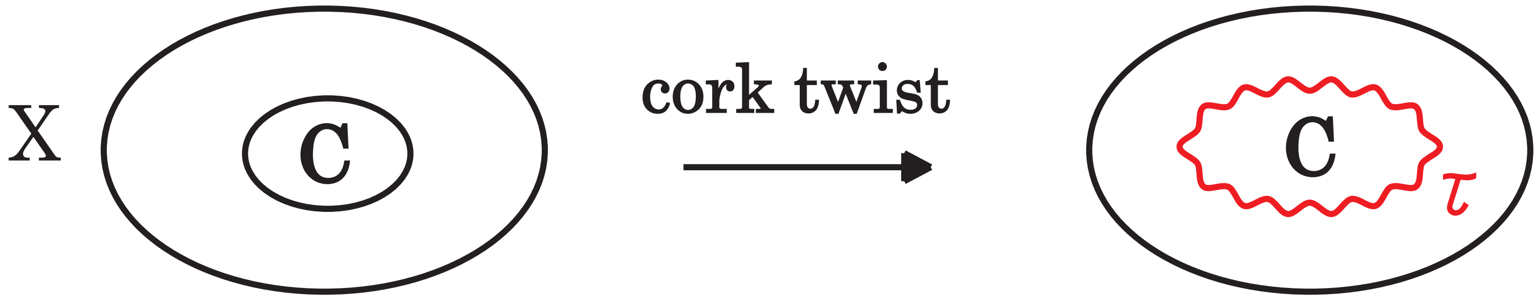
C : cpt contractible 4-mfd, $\tau : \partial C \to \partial C$: involution,

Definition

(C, τ) is a **cork** $\Leftrightarrow$ τ extends to a self-homeo of C,
but cannot extend to any self-diffeo of C.

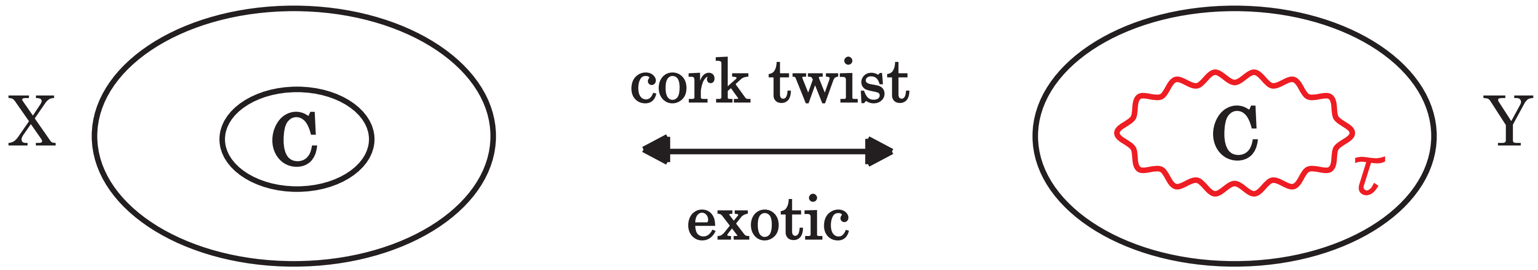
Suppose $C \subset X^4$.
The following operation is called a **cork twist** of X:

$$X \rightsquigarrow (X - C) \cup_\tau C.$$



2. Brief review of corks

Theorem(Curtis-Freedman-Hsiang-Stong '96, Matveyev '96)
X, Y : simp. conn. closed ori. smooth 4-mfds
If Y is an exotic copy of X,
then Y is obtained from X by a cork twist.



Smooth structures are determined by corks !!

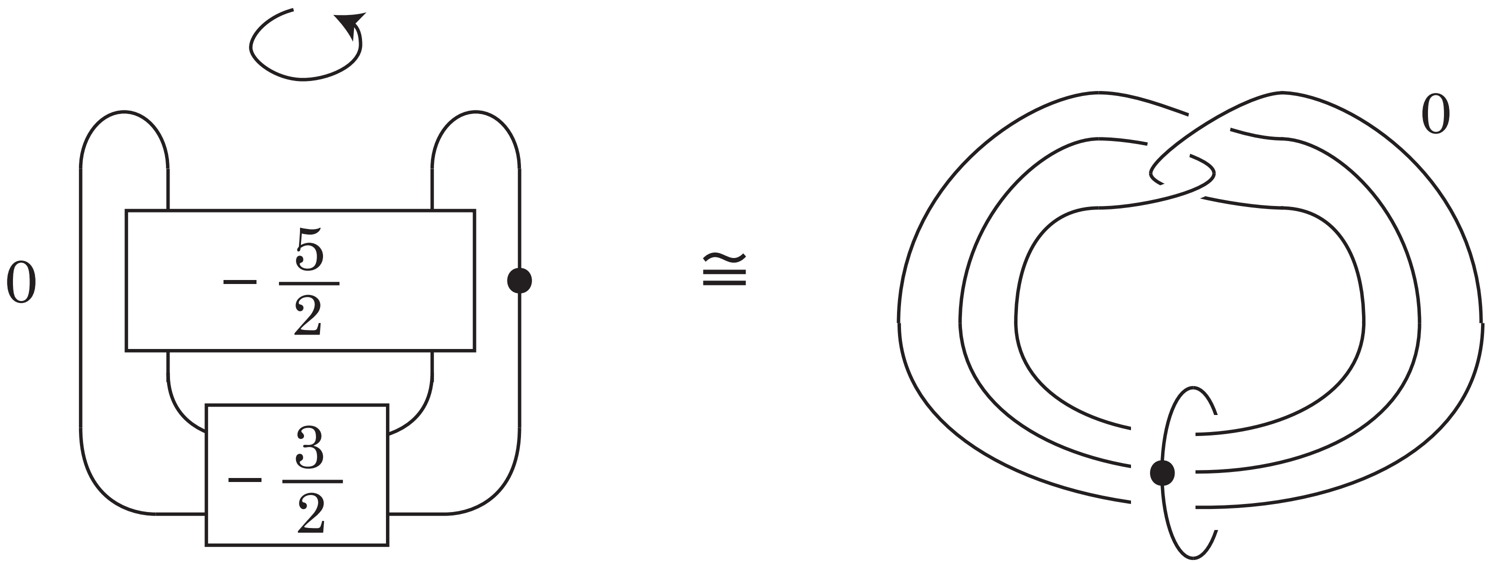
Remark
Cork twists do NOT always produce exotic smooth structures.

2. Brief review of corks: examples

Definition $L = K_0 \sqcup K_1$ is a symmetric Mazur link if

- K_0 and K_1 are unknot, $lk(K_0, K_1) = 1$.
- $\exists$ involution of S^3 which exchanges K_0 and K_1.

A symmetric Mazur link L gives
a contractible 4-mfd C_L and an involution $\tau_L : \partial C_L \to \partial C_L$.



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- K_0 and K_1 are unknot, $lk(K_0, K_1) = 1$.
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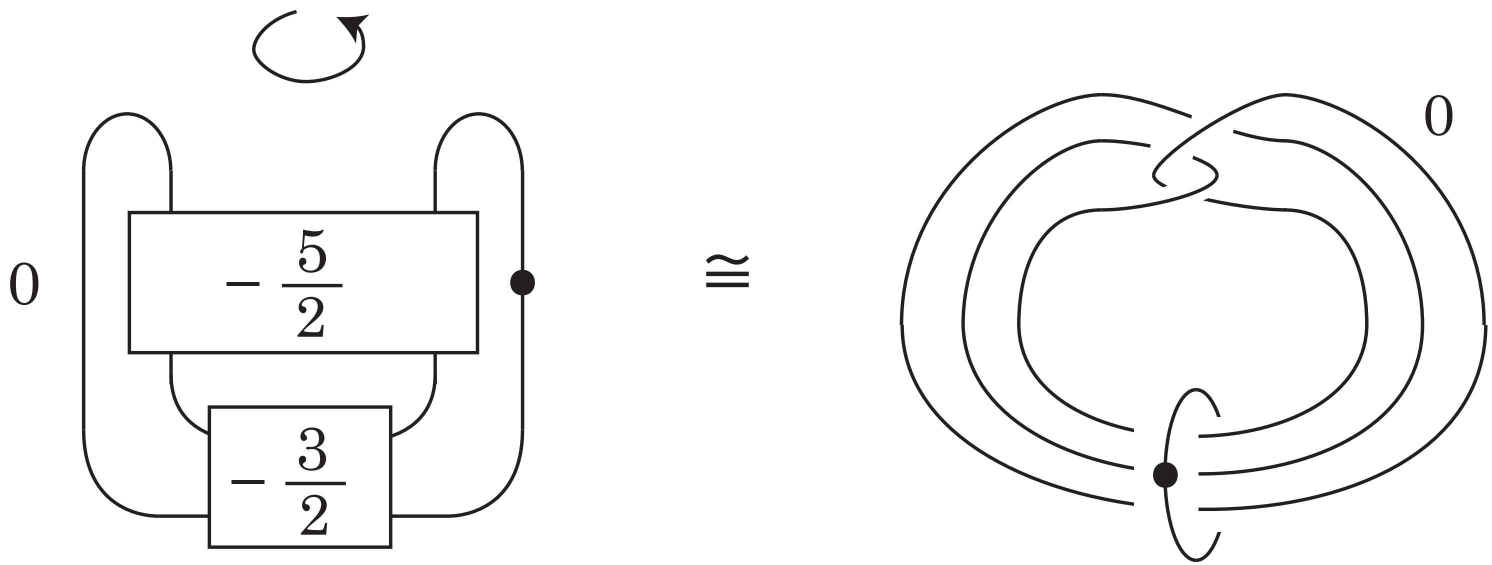
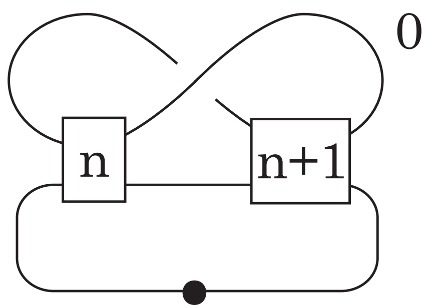
Theorem (Akbulut '91) There exists a cork.



2. Brief review of corks: examples

Theorem (Akbulut-Matveyev '97, cf. Akbulut-Karakurt '12)
For a symmetric Mazur link L, (C_L, τ_L) is a cork if C_L becomes a Stein handlebody in a 'natural way'.

Theorem (Akbulut '91, Akbulut-Y '08).
(W_n, f_n) is a cork for $n \geq 1$.

Wn :=


Theorem(Y)
For a symmetric Mazur link L, (C_L, τ_L) is NOT a cork if L becomes a trivial link by one crossing change.

2. Brief review of corks: examples

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For a symmetric Mazur link L, (C_L, τ_L) is a cork
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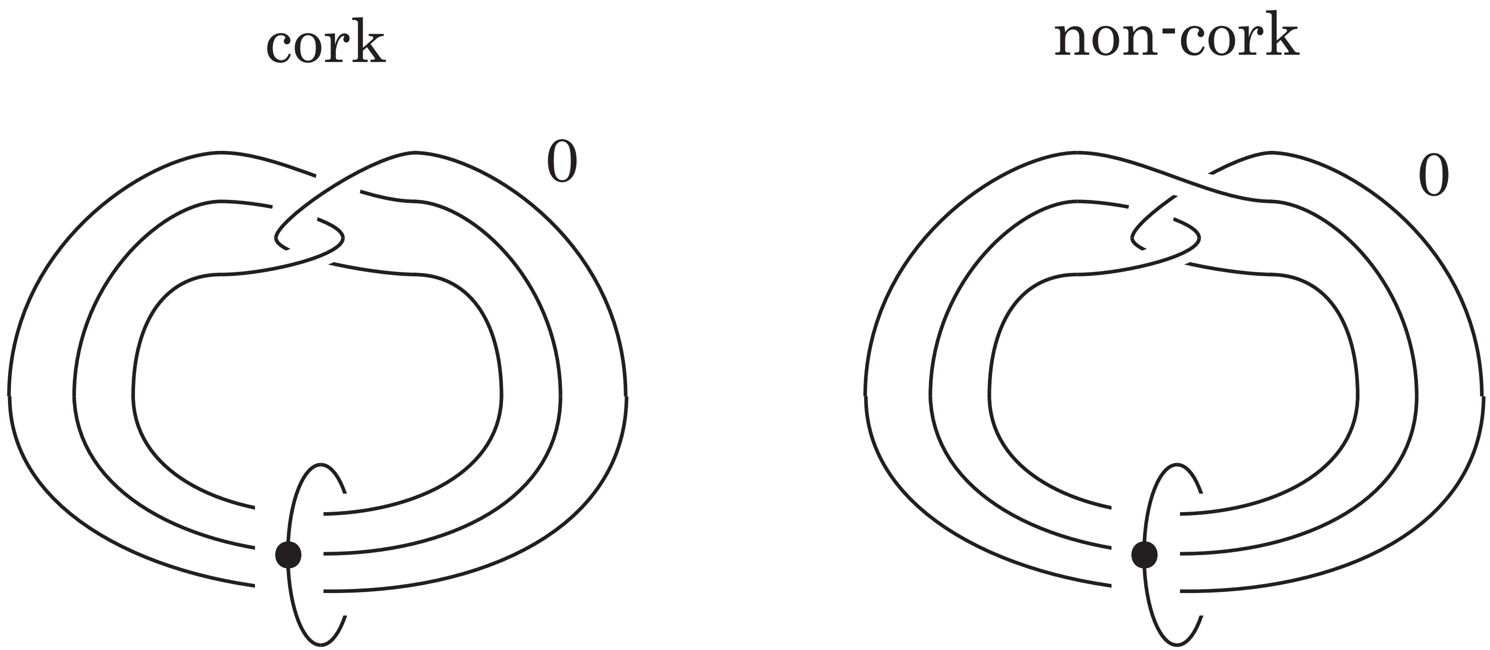
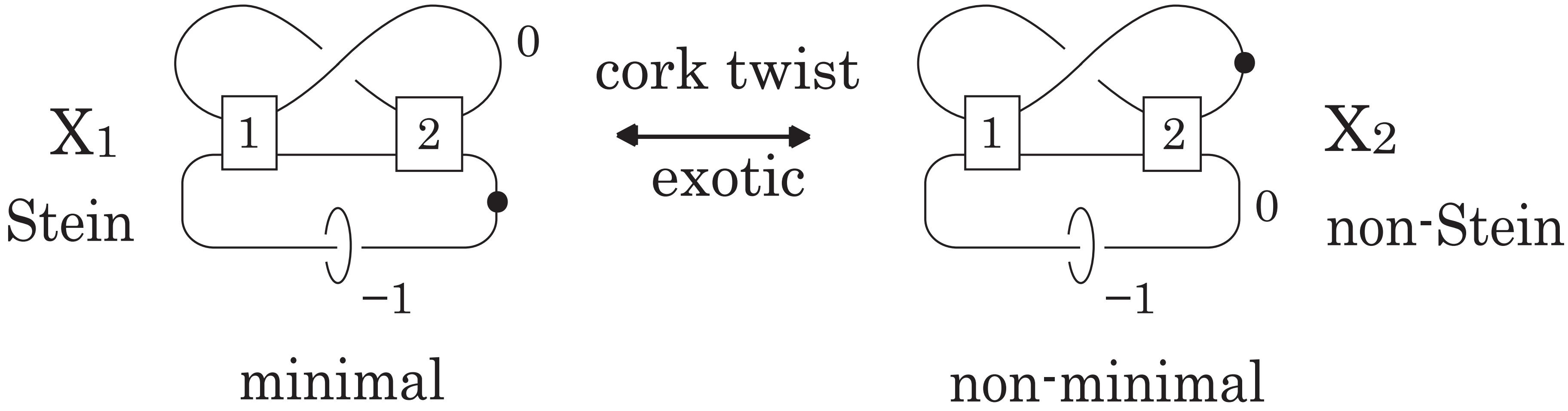


2. Brief review of corks: applications

Theorem (Akbulut '91, Akbulut-Matveyev 97')

$\exists$ exotic pair of simp. conn. 4-manifold with $b_2 = 1$.



2. Brief review of corks: applications

2-handlebody := handlebody consisting of 0-, 1-, 2-handles.

Thm (Akbulut-Y '13)

$\forall X$: 4-dim cpt ori 2-handlebody with $b_2(X) \neq 0$, $\forall n \in \mathbb{N}$,

$\exists X_1, X_2, \ldots, X_n$: 4-mfds admitting Stein str. s.t.

- $X_1, X_2, \ldots, X_n$ are pairwise exotic.
- $H_*(X_i) \cong H_*(X)$, $\pi_1(X_i) \cong \pi_1(X)$, $Q_{X_i} \cong Q_X$, $H_*(\partial X_i) \cong H_*(\partial X)$.
- Each X_i can be embedded into X.

Cor (Akbulut-Y '13)

For a large class of 4-manifolds with ∂,
their topological invariants are realized as
those of arbitrarily many pairwise exotic 4-mfds

2. Brief review of corks: applications

Thm (Akbulut-Y '13)

Z, Y: cpt conn. ori. 4-mfds, $Y \subset Z$.

$Z - \text{int}\, Y$ is a 2-handlebody with $b_2 \neq 0$.

Then $\forall n \in \mathbb{N}$, $\exists Y_1, Y_2, \dots, Y_n \subset Z$: cpt 4-mfds s.t.

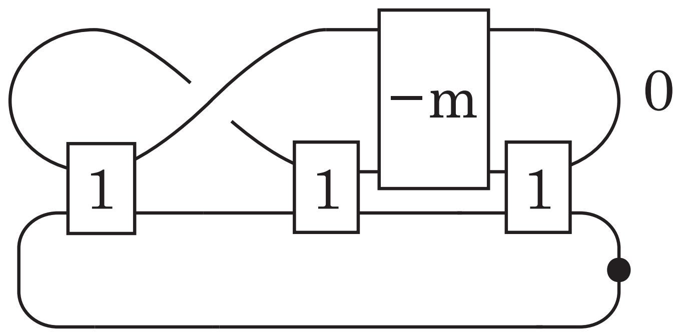
- Y_i is diffeo to Y_j $(\forall i \neq j)$.
- (Z, Y_i) is homeo but non-diffeo to (Z, Y_j) $(i \neq j)$.
- $H_*(Y_i) \cong H_*(Y)$, $\pi_1(Y_i) \cong \pi_1(Y)$, $Q_{Y_i} \cong Q_Y$, $H_*(\partial Y_i) \cong H_*(\partial Y_i)$.

Cor (Akbulut-Y '13) Every cpt. ori. 4-manifold Z admits arbitrarily many pairwise exotic embedding of a 4-mfd into Z.

3. Proof: new presentations of cork twists

Lemma (Y). (V_m, g_m) is a cork for $m \geq 0$.

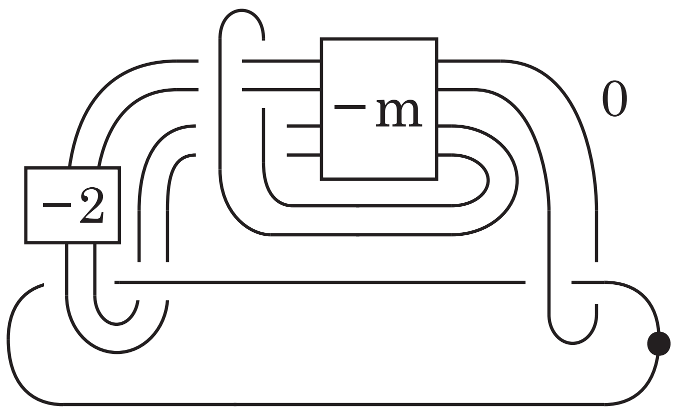
$\mathrm{V_m}$:



Remark. (V_{-1}, g_{-1}) is NOT a cork.

Definition

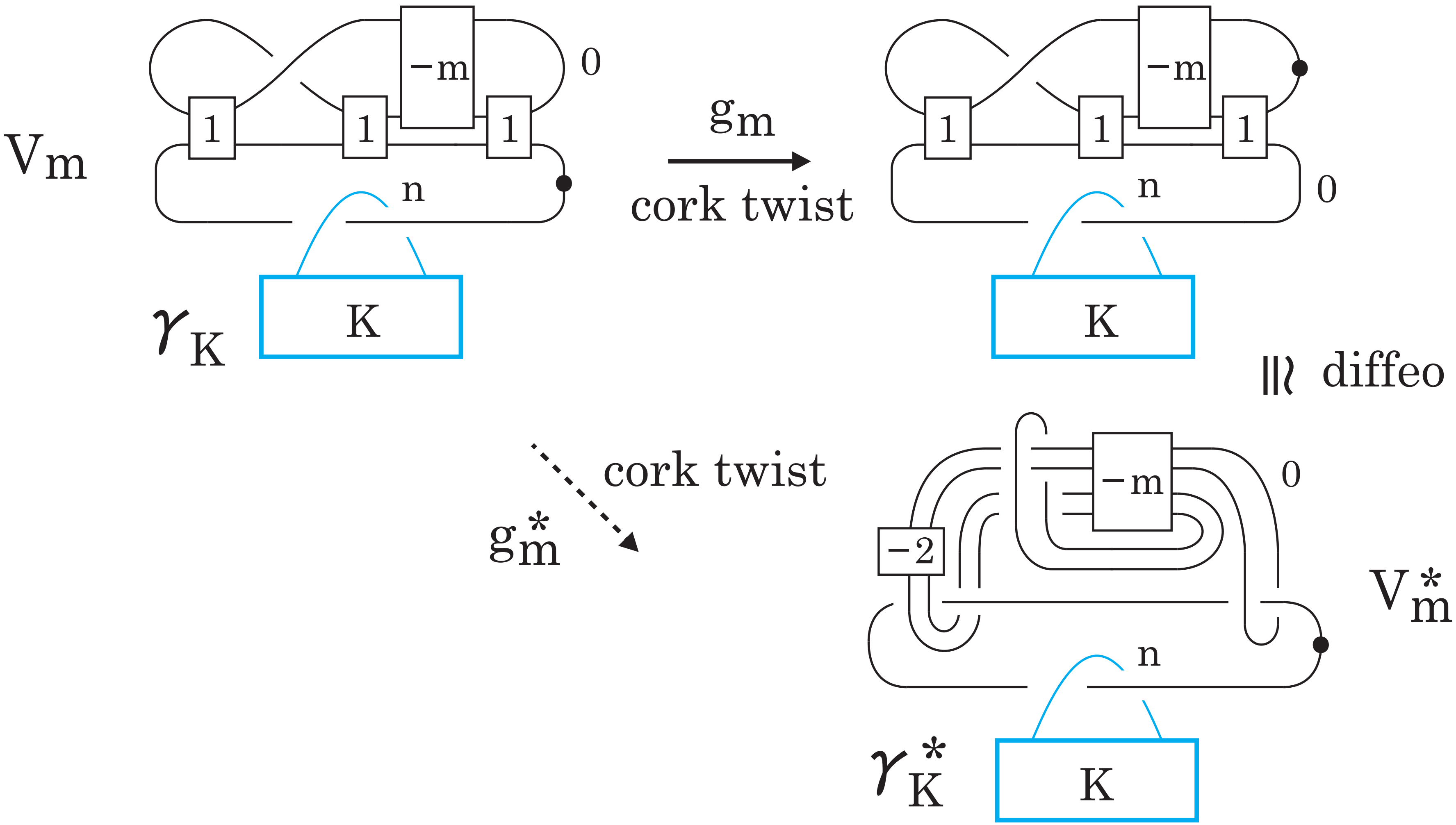
$\mathrm{V_m^*}$:



Theorem (Y) [hook surgery]

There exists a diffeomorphism $g^*_m : \partial V_m \to \partial V^*_m$ s.t.

- g^*_m sends the knot γ_K to γ^*_K for any knot K in S^3.
- $g^*_m \circ g_m^{-1} : \partial V_m \to \partial V^*_m$ extends to a diffeo $V_m \to V^*_m$.



Theorem (Y) [hook surgery]

There exists a diffeomorphism $g_m^* : \partial V_m \to \partial V_m^*$ s.t.

- g_m^* sends the knot γ_K to γ_K^* for any knot K in S^3.
- $g_m^* \circ g_m^{-1} : \partial V_m \to \partial V_m^*$ extends to a diffeo $V_m \to V_m^*$.

Corollary X : 4-mfd, $V_m \subset X$.

The cork twist $(X - V_m) \cup_{g_m} V_m$ is diffeomorphic to the hook surgery $(X - V_m) \cup_{g_m^*} V_m^*$.

3. Proof: satellite maps

Machines producing vast examples:

Main Theorem (Y)

$\forall n \in \mathbb{Z}$, $\exists$ satellite maps P_n, Q_n s.t.
for any knot K in S^3 with

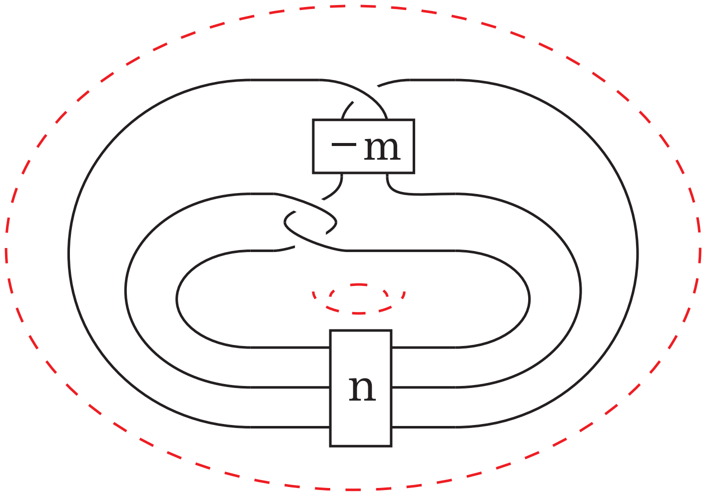
$$2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \widehat{tb}(K),$$

n-framed $P_n(K)$ and $Q_n(K)$ are an exotic pair.

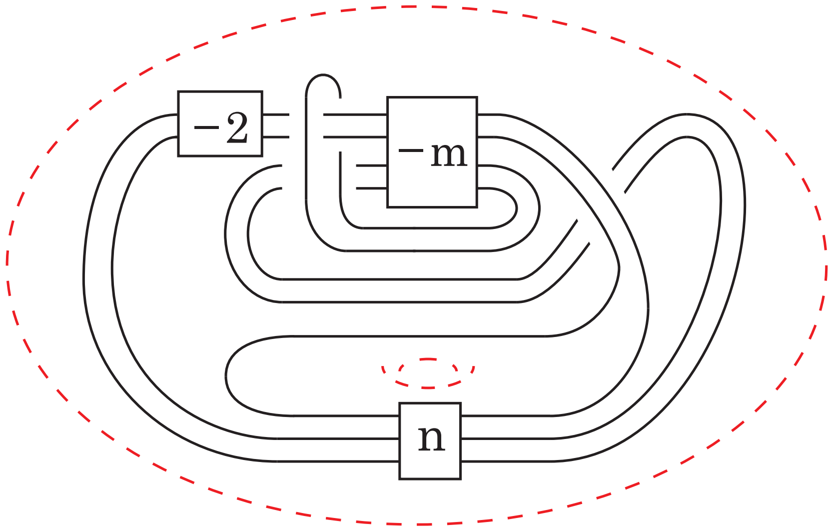
3. Proof: satellite maps

$P_{m,n},\ Q_{m,n}$: (pattern) knots in $S^1 \times D^2$

$\mathrm{P_{m,n}}$:

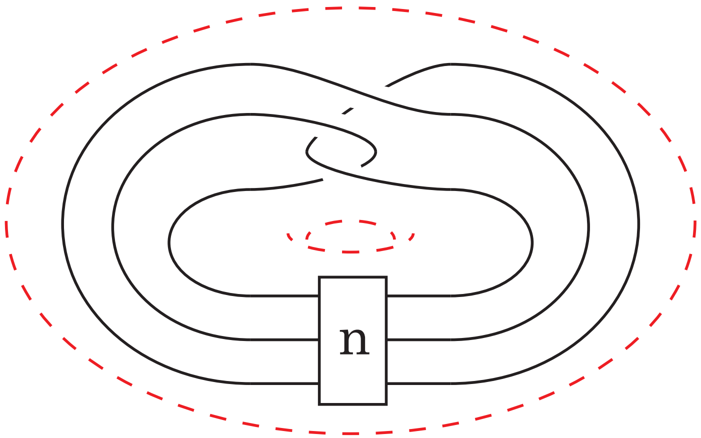


$\mathrm{Q_{m,n}}$:

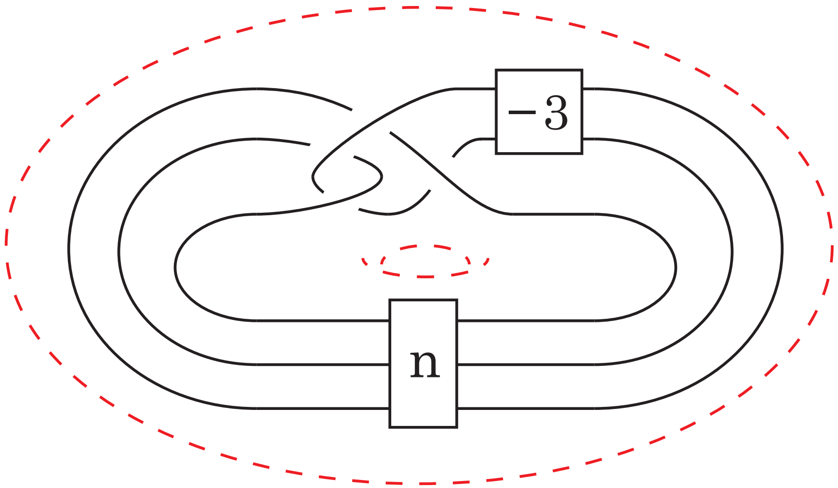


The case $m = 0$:

$\mathrm{P_{0,n}}$:



$\mathrm{Q_{0,n}}$:



3. Proof: satellite maps

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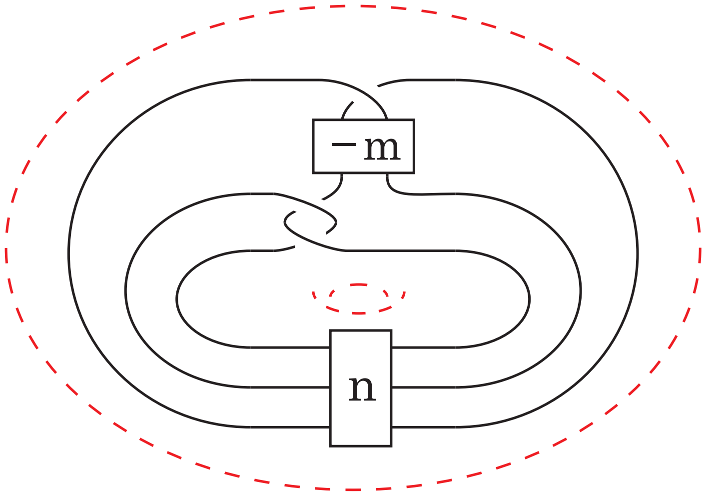
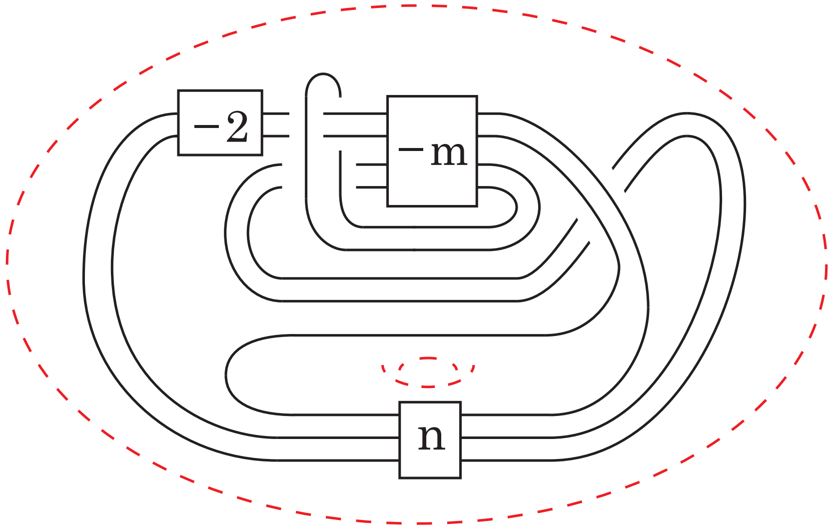
$\mathrm{P_{m,n}}$:



$\mathrm{Q_{m,n}}$:



Remark.

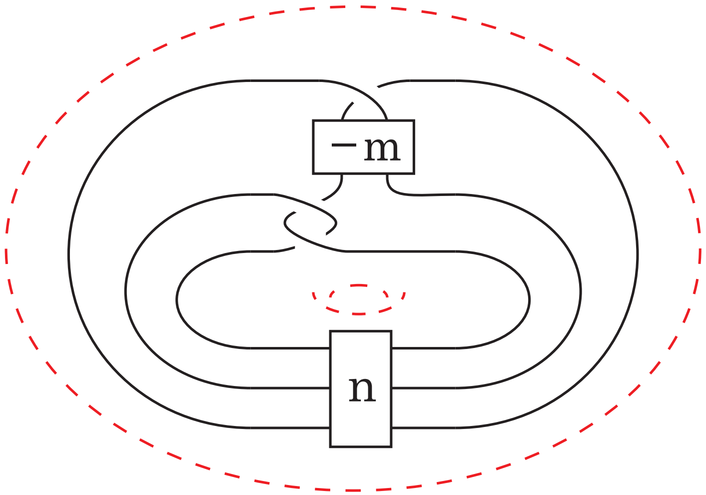
- $Q_{m,n}(K)$ is concordant to K.
- $g_4(Q_{m,n}(K)) = g_4(K)$, $\quad g_4(P_{m,n}(K)) \leq g_4(K) + 1$.

Definition.

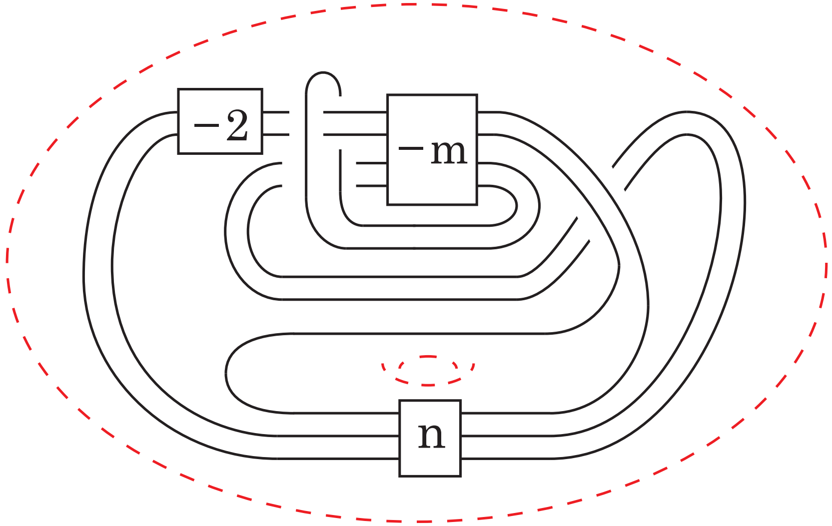
$P^{(n)}_{m,n}(K) :=$ 4-manifold represented by n-framed $P_{m,n}(K)$.

$Q^{(n)}_{m,n}(K) :=$ 4-manifold represented by n-framed $Q_{m,n}(K)$.

$P_{m,n}$:

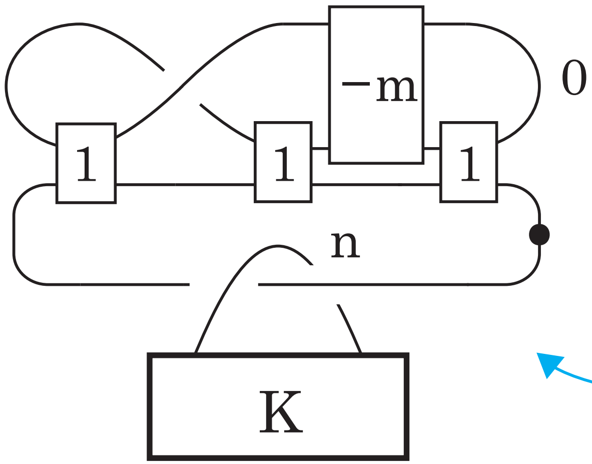


$Q_{m,n}$:

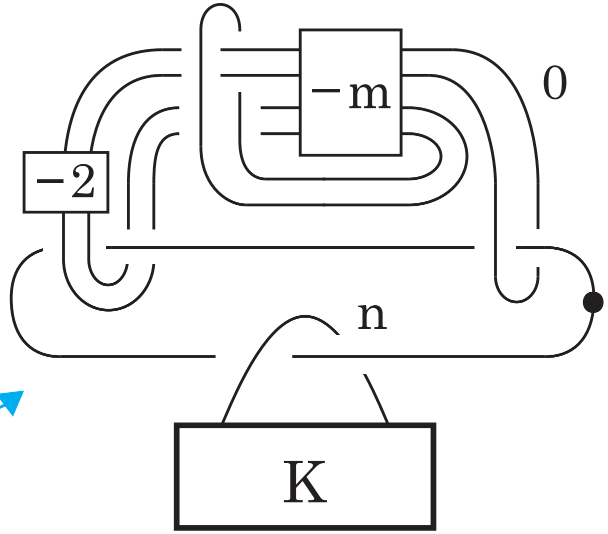


Lemma.

$P^{(n)}_{m,n}(K) \cong$



$Q^{(n)}_{m,n}(K) \cong$



Therefore $P^{(n)}_{m,n}(K)$ is homeo to $Q^{(n)}_{m,n}(K)$

$\mathcal{L}(K) :=$ {Legendrian knot isotopic to K}

$\overline{ad}(K) := \max\{ad(\mathcal{K}) := tb(\mathcal{K}) - 1 + |r(\mathcal{K})| \mid \mathcal{K} \in \mathcal{L}(K)\}$

$\widehat{tb}(K) := \max\{tb(\mathcal{K}) \mid \mathcal{K} \in \mathcal{L}(K),\ ad(\mathcal{K}) = \overline{ad}(K)\}$

$g_s^{(n)}(K) := \min\{g(\Sigma) \mid [\Sigma]$ is a generator of $H_2(K^{(n)})\}$

Fact (adjunction inequality).

For $n < \widehat{tb}(K)$, $\quad \overline{ad}(K) \leq 2g_s^{(n)}(K) - 2$.

Main Theorem (Y)

Fix $m \geq 0$. Assume a knot K and $n \in \mathbb{Z}$ satisfies

$$2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \widehat{tb}(K).$$

Then $P_{m,n}^{(n)}(K)$ and $Q_{m,n}^{(n)}(K)$ are homeo but not diffeo.

Main Theorem (Y)

Fix $m \geq 0$. Assume a knot K and $n \in \mathbb{Z}$ satisfies

$$2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \widehat{tb}(K).$$

Then $P_{m,n}^{(n)}(K)$ and $Q_{m,n}^{(n)}(K)$ are homeo but not diffeo.

By finding Legendrian realization of $P_{m,n}(K)$, we see

$\overline{ad}(P_{m,n}(K)) \geq \overline{ad}(K) + 2, \quad \widehat{tb}(P_{m,n}(K)) \geq n + 2.$

$$\Longrightarrow g_s^{(n)}(P_{m,n}(K)) = g_4(K) + 1$$

Since $g_s^{(n)}(Q_{m,n}(K)) \leq g_4(K), \quad P_{m,n}^{(n)}(K) \not\cong Q_{m,n}^{(n)}(K).$