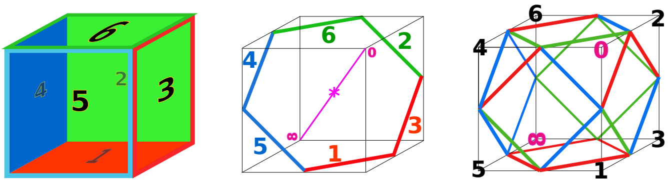
Construction of highly symmetric Riemann surfaces, related manifolds and some exceptional objects, II

Iain Aitchison

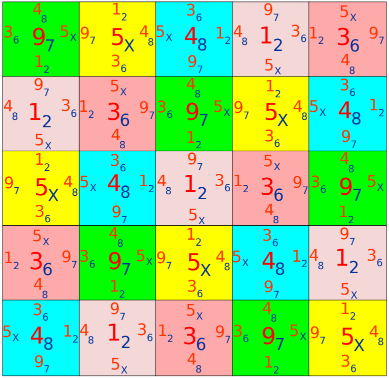


Hiroshima University March 2018

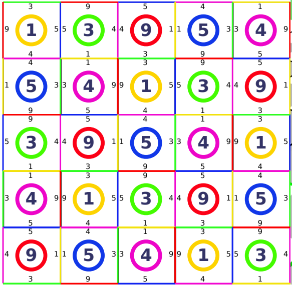
Strategy for application of transcription complexes

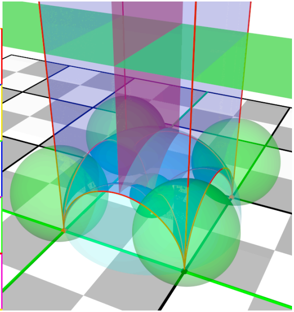
- Obtain highly symmetric objects with labels permuted by a natural symmetry group of the object
- The label structure, combined with the combinatorial symmetry, may encode interesting mathematical structures
- This encoding may lead to new representations of mathematical structure, and relationships between structures
- Today:
  - Construction of some special objects ($p = 7, 11$)
  - Klein's quartic, Weeks-Thurston-Christie 8-component link
  - New picture for Mathieu group M_{24}, Steiner $S(5, 8, 24)$ (Golay code, octads)
  - New Arnold Trinity involving Thurston's 8-component link

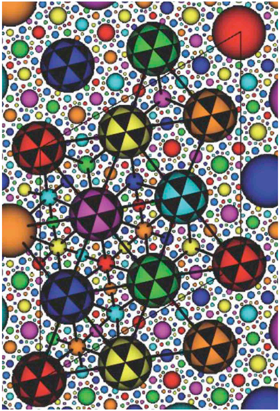
Warm-up – 5-coloured torus



Expanding horoballs create six 5-coloured cusp tori in a quotient 3-manifold







Thurston link is an analogue, planar tessellation by hexagons

Choosing any square, reverse order of 4 neighbours, change square colour

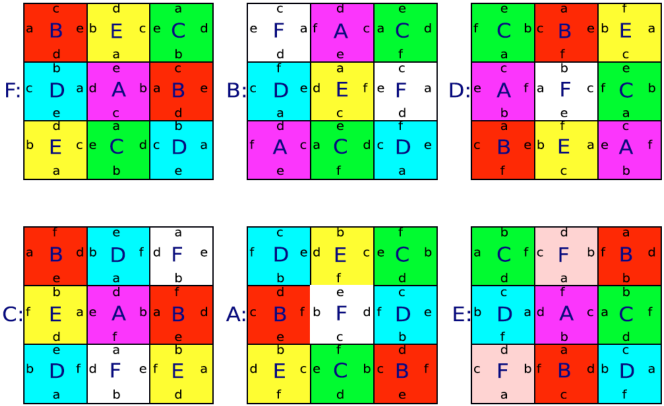
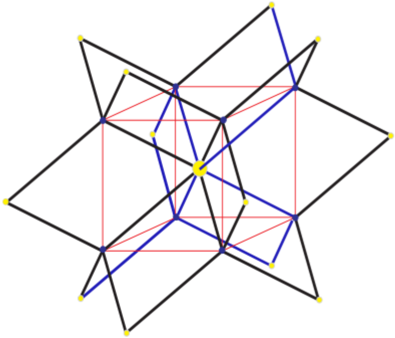
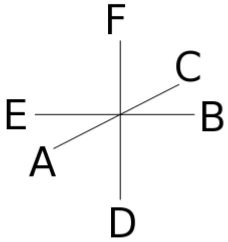


Figure: Six coloured cusps, each giving two repeating 5-cycles

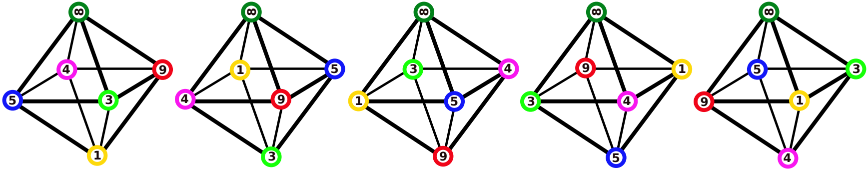
From one cusp, four cusps visible at a vertex, the other on the other side. Dual picture is an octahedron, one at each vertex in hyperbolic 3-space





5-coloured vertices give 5 regular ideal octahedra defining quotient manifold, 4 at each edge

40 triangles identified in pairs, creating two immersed totally-geodesic non-orientable hemi-icosahedra



Two other 6-component links known with order 120 symmetry – one described by Weeks, another by Matsuda (Sepak Takraw). Volumes differ.

Arnold Trinity = this manifold, Klein Quartic ($p = 5$), and a genus 26 surface with 60 11-gons of degree 3, 660 automorphisms ($p = 5$). (Already studied by Klein?) Transcription rule

Theorem (Aitchison)

Generalizing this construction gives a 3-manifold for $p = 5, 7, 11$, but not for larger primes p

Symmetric Riemann surfaces: context for Klein's Quartic

Finite groups can be the full group of isometries of some Riemann surface.

- Genus 1 – elliptic curve E. (8 triangles)
  2-fold branched cover of S^2 over vertices of the tetrahedron
- Genus 2 – the Bolza surface has order $48 = |S_4|$. (16 triangles)
  2-fold branched cover of S^2 over vertices of the octahedron
- Genus 3 the order is maximized by the Klein quartic (first Hurwitz surface), with order $168 = |PSL_2(7)|$; $= PSL_2(7)$ the second-smallest non-abelian simple group. (24 heptagons)
- Genus 4, Bring's surface, order $|S_5| = 120$. (24 pentagons)
  3-fold irregular branched cover of S^2 over vertices of the icosahedron
- Genus 7 the Macbeath surface (second Hurwitz surface) has order 504 $= |PSL(2,8)|$; $PSL_2(8)$ the fourth-smallest non-abelian simple group.

Klein's Quartic (1878)

- $x^3y + y^3z + z^3x = 0$: genus 3 algebraic curve
- Hyperbolic geometry:
  24 regular seven-gons $2\pi/3$-angled
  84 edges
  56 vertices degree 3
- (Dually: 56 triangles, 84 edges, 24 degree-7 vertices)
- Seven-gon gives 14 $(2,3,7)$ triangles: $14 \times 24 = 168 \times 2 = 336$
  allowing orientation-reversing reflection group symmetries
- Maximal possible symmetry group order: 168
  (Hurwitz: Maximum order for genus $g \geq 3 = 84(g-1)$)

Face transcription: Hexagon example : $p = 7$

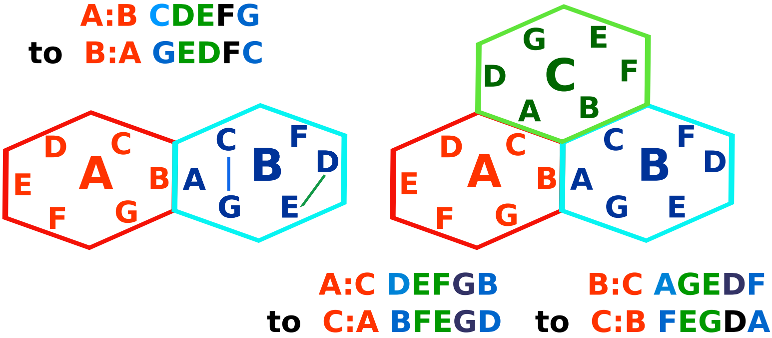
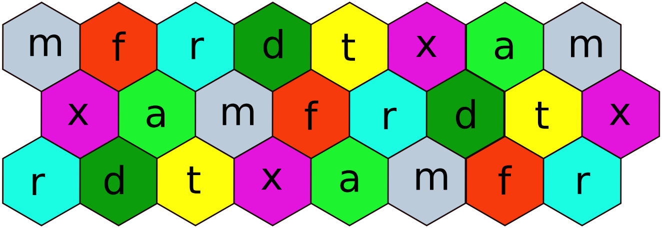
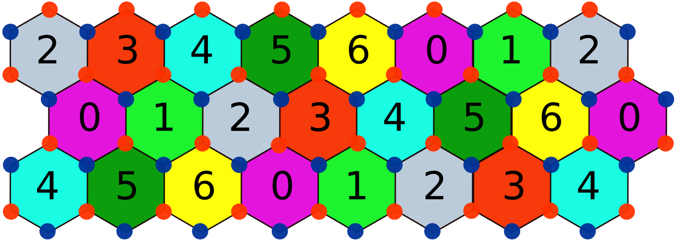


Figure: Working around a vertex, the pattern repeats: degree three vertices.

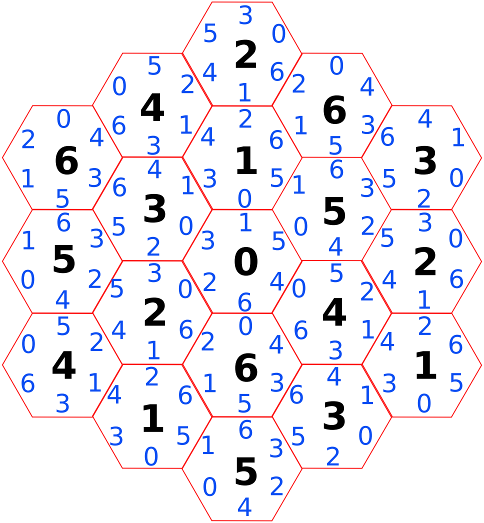
7-coloured tessellation of the Euclidean plane by continuation :

Abstract labels can be replaced by integers mod 7, as occured before. Graph bipartite, vertices define Fano, A-Fano triples





Integers modulo 7 defined by labels (addition, multiplication)



Hexagons – Face Transcription Rule $A{:}B\ CDEFG \ \rightarrow\ B{:}A\ GEDFC$

$A{:}B\ CDEFG \ \rightarrow\ B{:}A\ GEDFC$ generates (i) 7-coloured torus (ii) WTC link as 28 ideal regular tetrahedra, and (iii) the Klein quartic

Theorem

A: The transcription rules generating the WTC link encode the transcription rules generating the Klein quartic. Conversely B. The transcription rules generating the Klein quartic encode the transcription rules generating the WTC link.

KQ: $P{:}Q\ RSTUVW \ \rightarrow\ Q{:}P\ WTSVUR$ generates Klein quartic and WTC link complement.

All tetrahedra and face identifications are encoded in the face-colourings

Constructing the Weeks-Thurston-Christie (WTC) link

As before: 8 cusps labelled by $\mathbb{Z}/7\mathbb{Z} \cup \infty$ (projective line)
Take a 7-coloured torus, a union of 7 hexagons.
Assign colour '∞' to $C_\infty = T^2 \times [0, \infty)$.
Interpret the 7 hexagon colours of original $T^2 := T^2_\infty$ as 7 distinct coloured cusps $C_i := T^2_i \times [0, \infty)$.

Theorem

(a) Given the 7-coloured torus T^2_0, there is a canonical 7-colouring of each torus T^2_i such that exactly the colours $0, \ldots, \hat{i}, \ldots$ occur: the colour i is replaced by the colour ∞. (b) The resulting 3-manifold M^3_F is the union of 28 regular hyperbolic tetrahedra (vertices at ∞) with six tetrahedra meeting along each edge. (c) M^3_F is canonically a complete hyperbolic 3-manifold of volume $14 Vol_8$, where Vol_8 is the volume of the figure-8 knot complement in S^3. (d) Faces of the tetrahedra canonically define an immersed totally-geodesic 24-punctured Klein Quartic.

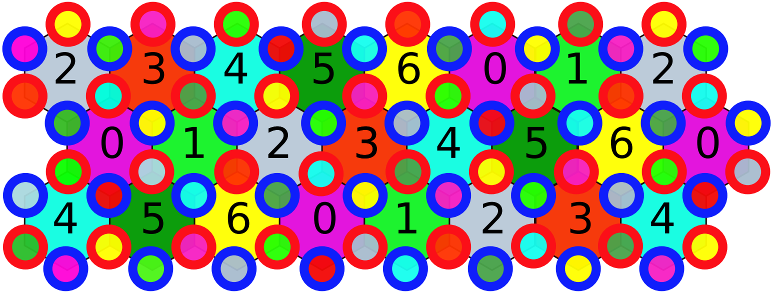
Paths to left define a vertex labelling. Paths bisecting hexagons can be labelled accordingly

Vertices bipartite, define abstract non-associative binary multiplication. Encoded is the Fano plane

rc	1	2	3	4	5	6	0	$\emptyset$
1	$*$	**4**	**0**	$*2$	**6**	$*5$	$*3$	$3\emptyset$
2	$*4$	$*$	**5**	**1**	$*3$	**0**	$*6$	$4\emptyset$
3	$*0$	$*5$	$*$	**6**	**2**	$*4$	**1**	$5\emptyset$
4	**2**	$*1$	$*6$	$*$	**0**	**3**	$*5$	$6\emptyset$
5	$*6$	**3**	$*2$	$*0$	$*$	**1**	**4**	$0\emptyset$
6	**5**	$*0$	**4**	$*3$	$*1$	$*$	**2**	$1\emptyset$
0	**3**	**6**	$*1$	**5**	$*4$	$*2$	$*$	$2\emptyset$

This matrix defines OCTONION multiplication, and also encodes the Galois field $GF(8)$, and the linear Hamming 8-code with check bit.

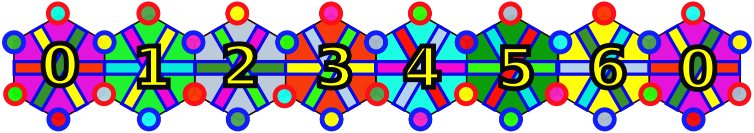
Binary composition – all hexagons adjoin all others. Any ordered pair defines a unique product by taking the third at a Fano (blue) vertex, signed with anticlockwise positive

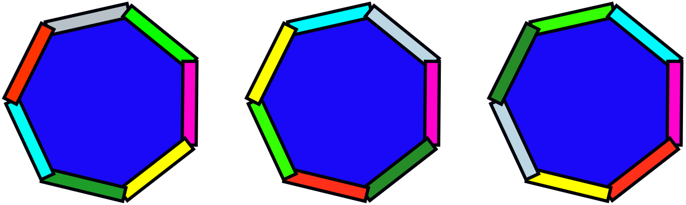


Vertices naturally 7-coloured, eg 561, 023 vertices labelled 4, which is furthest away - neither immediate nor next neighbour.

Hexagons have three bisectors, new edges joining opposite-edge midpoints. These are naturally coloured by the single Fano vetex color on a unique side ('Anti'-Fano product of the other two)

Paths bisecting hexagons naturally inherit labels. These paths continue to create three geodesics on the torus, each with 7 edges naturally 7-coloured. Colourings differ as the squares of a 7-cycle, an operation of order three.





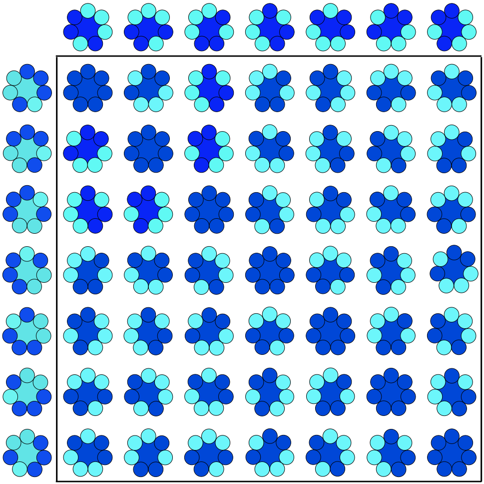
$$(0413265)^2 = (0123456), \quad (0123456)^2 = (0246153), \quad (0246153)^2 = (0413265),$$

Hamming 8-code words from $\mathbb{Z}/2\mathbb{Z}$ interpretation: row, column labels define check-bit, clockwise, anticlockwise defines $+, -$. Total of 16 linear code words over $\mathbb{Z}/2\mathbb{Z}$

1	1	1	1	1	1	1	1
1	**0**	**1**	**1**	0	**1**	0	0
1	0	**0**	**1**	**1**	0	**1**	0
1	0	0	**0**	**1**	**1**	0	**1**
1	**1**	0	0	**0**	**1**	**1**	0
1	0	**1**	0	0	**0**	**1**	**1**
1	**1**	0	**1**	0	0	**0**	**1**
1	**1**	**1**	0	**1**	0	0	**0**

This pattern also encodes the Galois field $GF(8)$

Hamming $\mathbb{Z}/2\mathbb{Z}$ 8-code can be considered as Fano lines $+ \infty$, and complements. Symmetric difference of set addition $= \mathbb{Z}/2\mathbb{Z}$ arithmetic



Array encoding multiplication and addition in the Galois field $GF(8)$ over $\mathbb{Z}/2\mathbb{Z}$, $x^3 = 1 + x$, $x^7 = 1$

0	1	1	1	1	1	1	1
$x = x^1$	**0**	**0**	**0**	1	**0**	1	1
x^2	1	**0**	**0**	**0**	1	**0**	1
x^3	1	1	**0**	**0**	**0**	1	**0**
x^4	**0**	1	1	**0**	**0**	**0**	1
x^5	0	**0**	1	1	**0**	**0**	**0**
x^6	**0**	1	**0**	1	1	**0**	**0**
$x^7 = 1 = x^0$	**0**	**0**	1	**0**	1	1	**0**

Example: $x^3 + x^6 + 0 = x^4$. Add three rows including 0 to find the result.

From planar hexagons to Galois field $GF(8)$ – details

$$\alpha^7 - 1 = 0 \quad \Rightarrow \quad 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$$

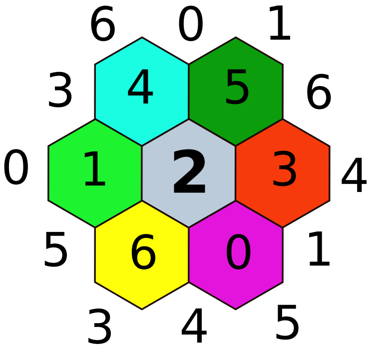
$$1 + \alpha + \alpha^3 = 0 \quad \Rightarrow \quad \alpha^2 + \alpha^4 + \alpha^5 + \alpha^6 = 0 \quad \Rightarrow \quad \alpha^6 = \alpha^2 + \alpha^4 + \alpha^5$$

$$\alpha^6 = \alpha^2 + \alpha^4 + \alpha^5 \quad \Rightarrow \quad \alpha = \alpha^4 + \alpha^6 + \alpha^0$$

n	1	2	3	4	5	6	0			
α^n	α^1	α^2	α^3	α^4	α^5	α^6	α^0	*sum*		$=$
α^1	0	**0**	**0**	1	**0**	1	1	$\alpha = \alpha^4 + \alpha^6 + \alpha^0$	**460**	$= \alpha^1$
α^2	1	0	**0**	**0**	1	**0**	1	$\alpha^2 = \alpha^5 + \alpha^0 + \alpha^1$	**501**	$= \alpha^2$
α^3	1	1	0	**0**	**0**	1	**0**	$\alpha^3 = \alpha^6 + \alpha^1 + \alpha^2$	**612**	$= \alpha + 1$
α^4	**0**	1	1	0	**0**	**0**	1	$\alpha^4 = \alpha^0 + \alpha^2 + \alpha^3$	**023**	$= \alpha^2 + \alpha$
α^5	1	**0**	1	1	0	**0**	**0**	$\alpha^5 = \alpha^1 + \alpha^3 + \alpha^4$	**134**	$= \alpha^2 + \alpha + 1$
α^6	**0**	1	**0**	1	1	0	**0**	$\alpha^6 = \alpha^2 + \alpha^4 + \alpha^5$	**245**	$= \alpha^2 + 1$
α^0	**0**	**0**	1	**0**	1	1	0	$\alpha^0 = \alpha^3 + \alpha^5 + \alpha^6$	**356**	$= 1$

Non-zero elements $GF(8) - 0$ cyclic order 7. Addition in $GF(8)$ by row comparison - positions where entries are equal. Eg α^3, α^6 agree in 230, sum α^4

8 cusps from local pattern reflection, then applying same transcription. Pattern is the same, labels/colours change



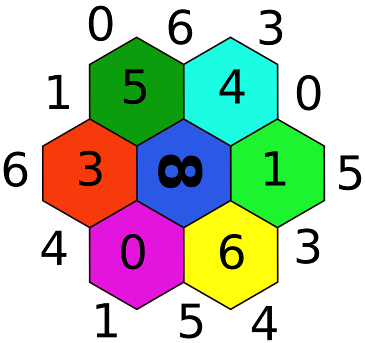


Figure: Choose a hexagon: reflect and relabel this colour: example 2, ∞

7 hexagons: 7 neighbouring face arrangements

- Take seven face-labeled hexagons

$$0 : 132645 \qquad 1 : 243056 \qquad 2 : 354160 \qquad 3 : 46520146$$

$$4 : 506312 \qquad 5 : 610423 \qquad 6 : 021534$$

- Relabel the central hexagon with ∞.
- Reverse the order of the cyclic arrangement:

$$\infty : 546231 \qquad \infty : 650342 \qquad \infty : 061453 \qquad \infty : 102564$$

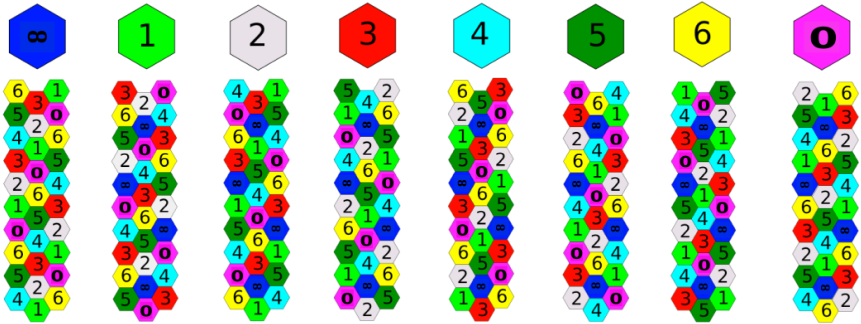
$$\infty : 213605 \qquad \infty : 324016 \qquad \infty : 435120$$

- Use the same face transcription rule

$$A : B\,CDEFG \qquad \to \qquad B : A\,GEDFC$$

  to tessellate the plane
- Get 8 distinct 7-coloured tori, 8 tessellations of the plane by labeled hexagons, characterised by omitting one of the 8 possible labels.

8 cusp pictures: relabel fundamental domain hexagons



The Eightfold Way: 8 tessellations by face transcription

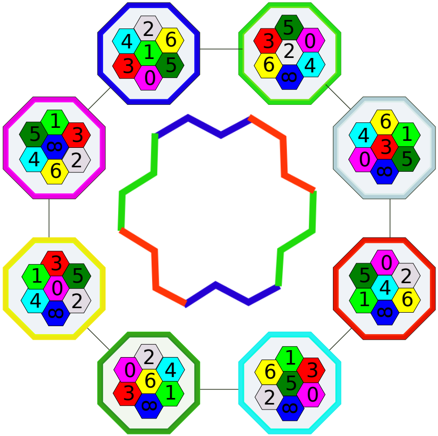
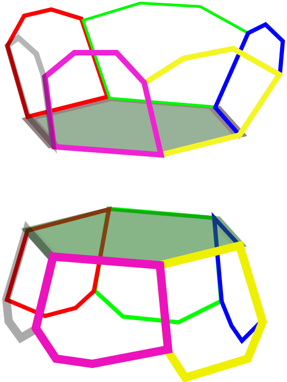
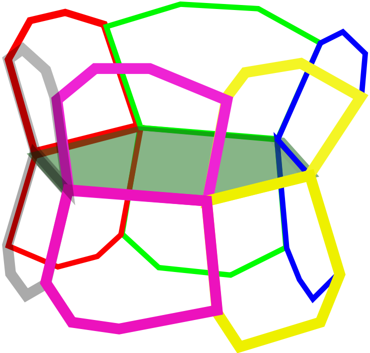


Figure: Choose any coloured octagon A, and one of its coloured hexagons B. Its ring of hexagons, cyclically reversed, is the ring of hexagons surrounding hexagon coloured A in the coloured octagon B

Hyperbolic 3-space – Identifying two half-spaces along a hexagon





Each hexagon in a torus is surrounded by six others: degree-3 edges require reversed cyclic order of neighbouring faces.

Four half spaces meet at ‘tetrahedral’ vertices

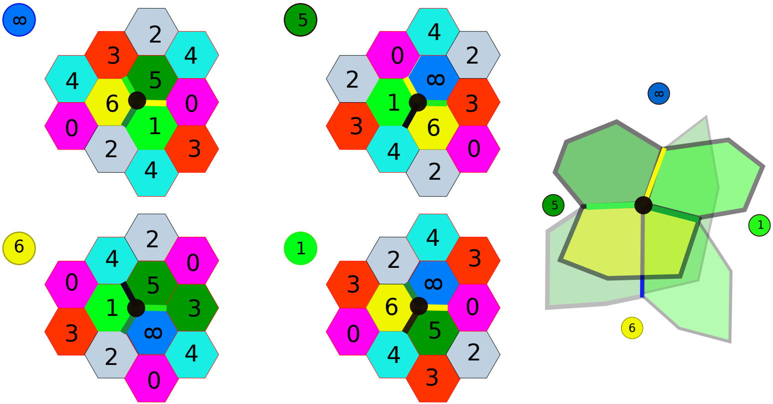


Figure: Four horoballs meet at each vertex. The 2-complex from 8 tori has $(8 \times 14)/4 = 28$ vertices

Tetrahedra give the dual decomposition

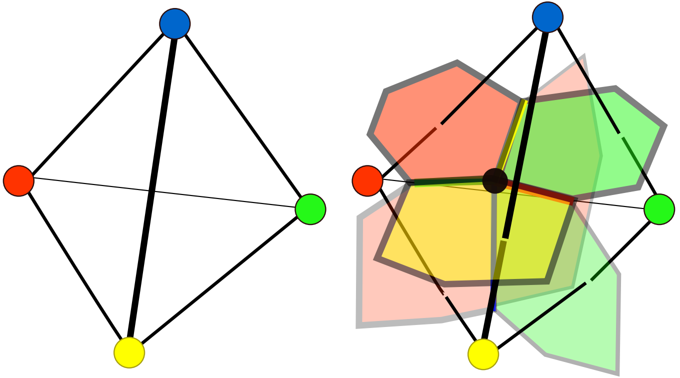


Figure: Each vertex is dual to an ideal tetrahedron: The link complement with 8 cusps is decomposed into $(8 \times 14)/4 = 28$ tetrahedra, meeting six around each edge. Triangle faces of tetrahedra intersect hexagons in line segments dual to hexagon edges. At each tetrahedron edge we can continue a triangle to an opposite triangle, to create an immersed punctured surface.

Identifying 28 tetrahedra

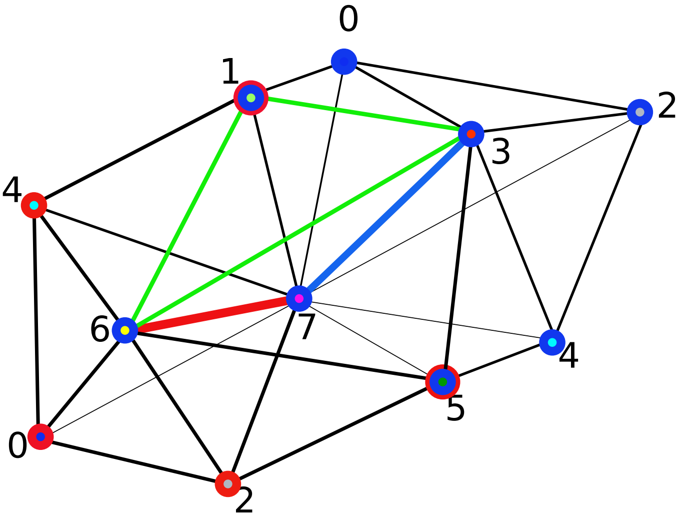


Figure: tetrahedraglued

28 two-coloured tetrahedra: 4 families of 7

56 ordered pairs from 8-set = 56 unordered triples from 8-set . Black numbers indicate k-blocks from Conway-Smith octonions: (their notation uses ∞ for 0, 0 for 7)

$Fano - 0$	$NFano$	$Fano$	$NFano - 0$
$026\infty : \infty\emptyset$	$0625 : 5\emptyset$	$3415 : \infty\Omega$	$1430 : 5\Omega$
$615\infty : \infty\emptyset$	$6514 : 4\emptyset$	$2304 : \infty\Omega$	$032\infty : 4\Omega$
$504\infty : \infty\emptyset$	$5403 : 3\emptyset$	$1263 : \infty\Omega$	$621\infty : 3\Omega$
$463\infty : \infty\emptyset$	$4362 : 2\emptyset$	$0152 : \infty\Omega$	$510\infty : 2\Omega$
$352\infty : \infty\emptyset$	$3251 : 1\emptyset$	$6041 : \infty\Omega$	$406\infty : 1\Omega$
$241\infty : \infty\emptyset$	$2140 : 0\emptyset$	$5630 : \infty\Omega$	$365\infty : 0\Omega$
$130\infty : \infty\emptyset$	$1036 : 6\emptyset$	$4526 : \infty\Omega$	$254\infty : 6\Omega$

Two copies - red, blue - of the **Steiner triple-system** $S(3, 4, 8)$
Every 3-tuple occurs in one of the blocks of 4 of the 8-tuple $\{\infty, 1, 2, 3, 4, 5, 6, 00\}$. Eg 560 in 0625 and 5630

Six tetrahedra about each ideal edge

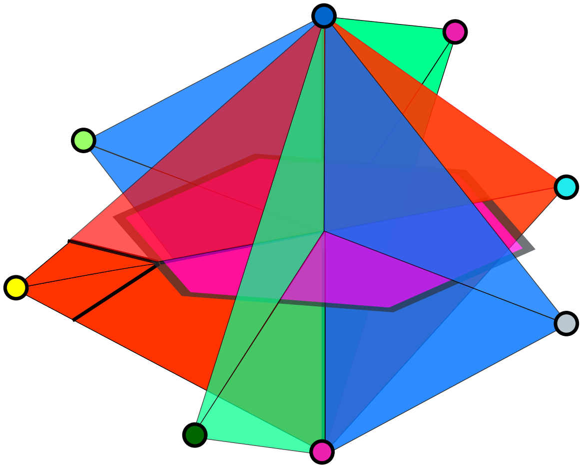
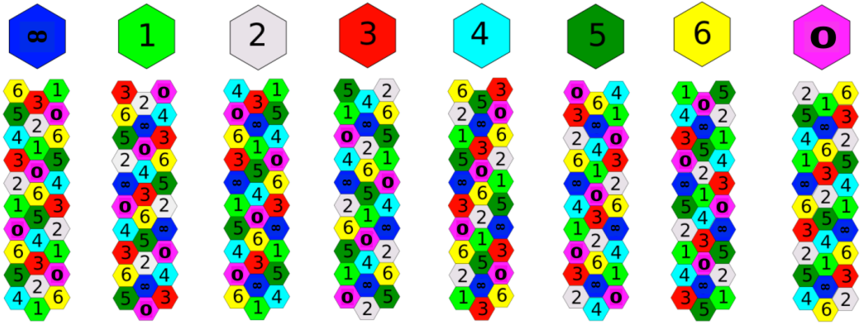
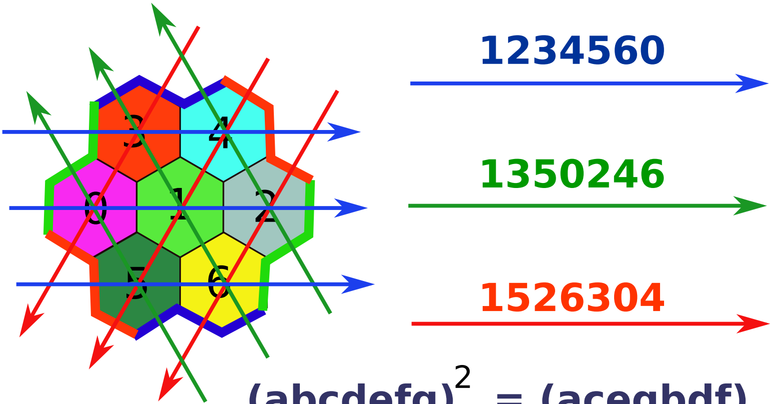


Figure: At each tetrahedron edge we can continue a triangle to an opposite triangle, to create an immersed (totally geodesic) punctured surface.

14 vertices in each of 8 cusps: 28 tetrahedra, 56 ideal triangles.
COMPACTIFIED, THESE GIVE THE KLEIN QUARTIC



Each cusp defines 3 seven-cycles : Labelled 7-gons



$$(\mathbf{abcdefg})^2 = (\mathbf{acegbdf})$$

Figure: Three directions give 3 7-cycles: rotation by $2\pi/3$ corresponds to squaring a 7-cycle (period 3)

Three versions of 7 triangles meet at each ideal vertex

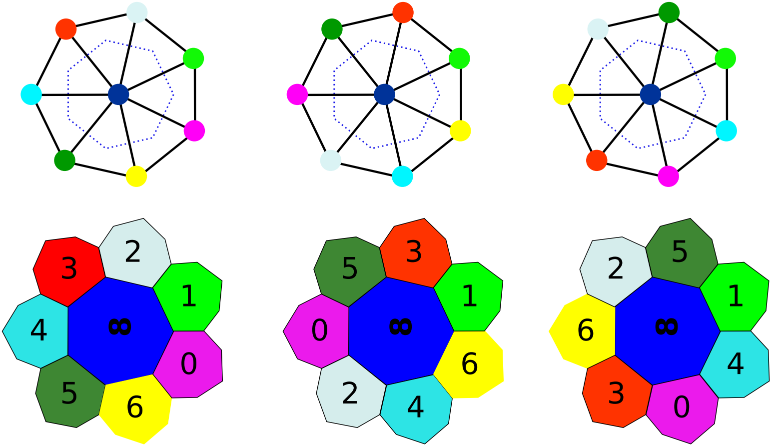


Figure: Three copies at each cusp of 7 triangles: Dually, three coloured 7-gons for each of 8 coloured cusps: 24 heptagons glued together, degree three vertices. (All 24 are visible at any chosen cusp! – But how do they fit together?)

24 coloured 7-gons: adjacent 7-gon labels

Cusp	Right	Up/Left : τ	Down/Left : τ^2
∞	$\cdots 01234560 \cdots$	$\cdots 02461350 \cdots$	$\cdots 04152630 \cdots$
1	$\cdots \infty 256340 \infty \cdots$	$\cdots \infty 530264 \infty \cdots$	$\cdots \infty 324506 \infty \cdots$
2	$\cdots \infty 360451 \infty \cdots$	$\cdots \infty 641305 \infty \cdots$	$\cdots \infty 435610 \infty \cdots$
3	$\cdots \infty 401562 \infty \cdots$	$\cdots \infty 052416 \infty \cdots$	$\cdots \infty 546021 \infty \cdots$
4	$\cdots \infty 512603 \infty \cdots$	$\cdots \infty 163520 \infty \cdots$	$\cdots \infty 650132 \infty \cdots$
5	$\cdots \infty 623014 \infty \cdots$	$\cdots \infty 204631 \infty \cdots$	$\cdots \infty 061243 \infty \cdots$
6	$\cdots \infty 034125 \infty \cdots$	$\cdots \infty 315042 \infty \cdots$	$\cdots \infty 102354 \infty \cdots$
0	$\cdots \infty 145236 \infty \cdots$	$\cdots \infty 426153 \infty \cdots$	$\cdots \infty 213465 \infty \cdots$

Domain from any cusp: three 7-gons, and neighbours

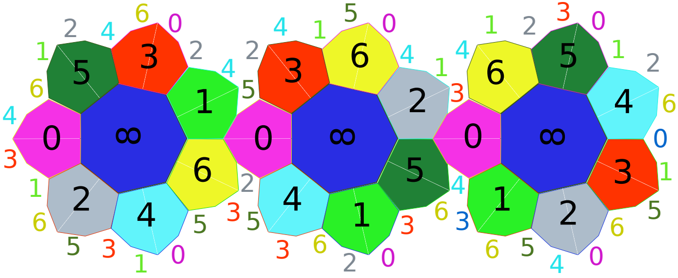
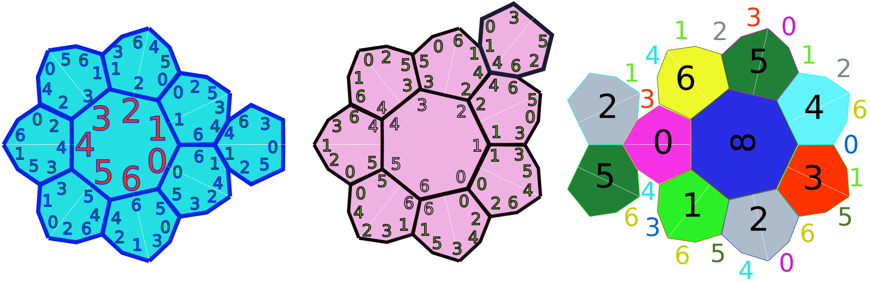


Figure: A fundamental domain for Klein's quartic: each colour appears three times, and the neighbouring faces differ by the squaring of a 7-cycle. For each ring, the region adjoining a vertex opposite has colour the antipode. Each outward vertex on fills an inward on the next to the left, **From any vertex, move along any edge, turn left: at the next vertex you face the same coloured face**. Similarly for a right turn.

Thus 025 gives a triangle in tetrahedra 0251 and 0256. Every vertex has a label by one of the 56 ordered pairs of distinct numbers chosen from $\{\infty, \ldots, 0\}$, giving 56 pairs corresponding to 56 5-tuples defining a complementary 3-tuple.

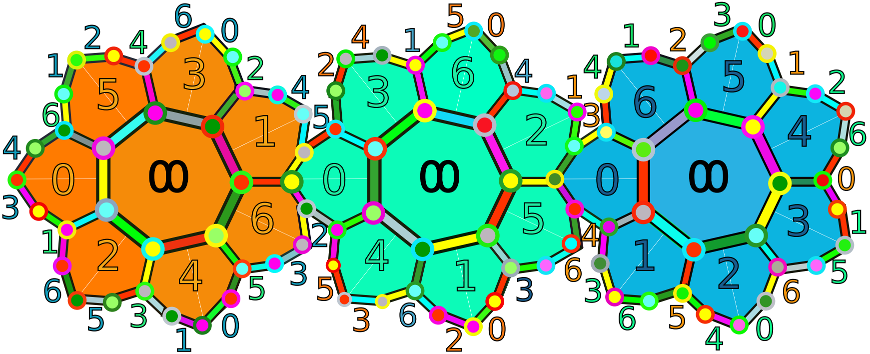
Klein's quartic from vertex, edge, face transcription. Meanings?



Edge	Vertex	Face
p:qrstuv	pq:rstuv	P:Q RSTUVW
→ p:usrtqv	→ qp:rvtus	→ Q:P WTSVUR

Figure: Three ways to generate the Klein quartic: Edge transcription, vertex transcription, face transcription. A more useful vertex transcription is: *pq*:*rstuv* → *qp*:*ustrv*; a less useful edge transcription is *p*:*qrstuv* → *p*:*trsqvu*.

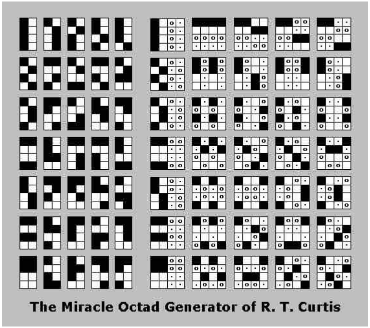
Edges can be 7-coloured using the octonion structure. Curtis used colouring to describe symmetric generators for M_{24} – but with completely different face labels. Here, three sets of 8, and cyclic labelling suggests triality underlying

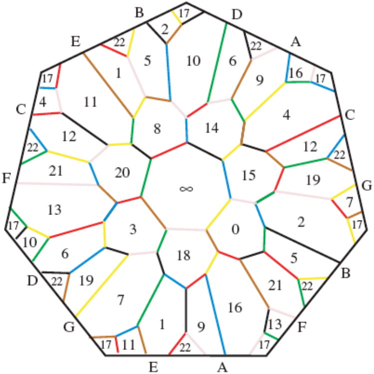


Three 7-gon edge colour sequences for each cusp form 3-cycles under7-cycle squaring, the infinity cusp in reverse direction

Cusp	Blue/Right	Red/Left	Green/Center
∞	$\cdots 0123456 \cdots$	$\cdots 0246135 \cdots$	$\cdots 0415263 \cdots$
1	$\cdots 1534206 \cdots$	$\cdots 1250364 \cdots$	$\cdots 1326540 \cdots$
2	$\cdots 2645310 \cdots$	$\cdots 2361405 \cdots$	$\cdots 2430651 \cdots$
3	$\cdots 3056421 \cdots$	$\cdots 3402516 \cdots$	$\cdots 3541062 \cdots$
4	$\cdots 4160532 \cdots$	$\cdots 4513620 \cdots$	$\cdots 4652103 \cdots$
5	$\cdots 5201643 \cdots$	$\cdots 5624031 \cdots$	$\cdots 5063214 \cdots$
6	$\cdots 6312054 \cdots$	$\cdots 6035142 \cdots$	$\cdots 6104325 \cdots$
0	$\cdots 0423165 \cdots$	$\cdots 0146253 \cdots$	$\cdots 0215436 \cdots$

Curtis' MOG = Miracle Octad Generator. Labels from integers mod 23





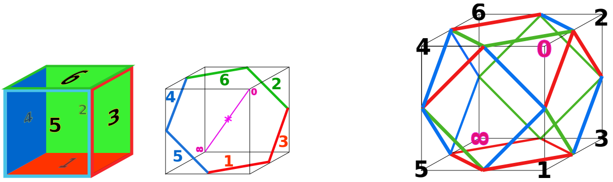
Curtis' edge-colouring of the Klein Quartic corresponds to the octonion hexagon-edge-bisector colouring described naturally earlier. From any coloured edge, go left, then right to find the same colour.

Triality is a non-linear outer automorphism of $\mathit{Spin}(8)$, whose Lie algebra has Coxeter-Dynkin diagram D_4. The root lattice of D_4 corresponds to the tessellation of $\mathbb{R}^4$ by 24-cells, and defines an optimal sphere packing. The 24-cell can be constructed from a cuboctahedron. The three 8-dimensional representations – vector, Spin$+$ and Spin- – correspond to the highest weight vectors on the legs of the Dynkin diagram, and are permuted by symmetry of the 24-cell and cuboctahedron

Symmetric generation of the Mathieu group M_{24}

Lemma

A dice-labelled cuboctahedron can be edge 3-coloured invariantly under diagonal rotation, corresponding to multiplication by 2 mod 7.



Lemma

The 3-coloured dice-labelled cuboctahedron defines a 12-transposition involution of 24 edges, swapping opposite coloured, labelled edges of six squares.

$$\mu_0 = (\infty 1)(24)(35)(60)(\infty 2)(41)(63)(50)(\infty 4)(12)(56)(30)$$

Symmetric generation of the Mathieu group M_{24}

Theorem (Aitchison)

The Mathieu group M_{24} has 7 involutary generators defined by opposite-edge swaps of the six squares of a coloured cuboctahedron. The seven generators are obtained from a single μ_0, defined by the labelling of a standard dice, by adding integers mod 7.

$$\mu_0 = \begin{matrix} (\infty 1)(24)(35)(60) \\ (\infty 2)(41)(63)(50) \\ (\infty 4)(12)(56)(30) \end{matrix} , \quad \mu_1 = \begin{matrix} (\infty 2)(35)(46)(01) \\ (\infty 3)(52)(04)(61) \\ (\infty 5)(23)(60)(41) \end{matrix} , \quad \text{etcetera}$$

Curtis gave seven generators, each an involution of 12 transpositions, defined in terms of reflections of faces of Klein's quartic as 24 heptagons. The proof of the above is a relabelling of the Klein quartic, enabling a cuboctahedron interpretation of Curtis' generators, and as such should be considered a corollary of his work.

Conway-Sloane, ‘Sphere packings, lattices and groups’

‘ . . . in two dimensions the following familiar hexagonal lattice solves the packing, kissing, covering and quantizing problems.’

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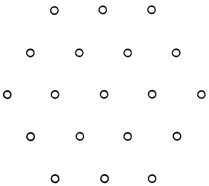
We remark that in two dimensions the familiar hexagonal lattice



solves the packing, kissing, covering and quantizing problems. In a sense this whole book is simply a search for similar nice patterns in higher dimensions.

‘In a sense this whole book is simply a search for similar nice patterns in higher dimensions’ – eg 24 dimensional Leech lattice, etc

Quotes from The Atlas of Finite Groups

- For about a hundred years, the only sporadic simple groups were the Mathieu groups $M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24}$
- *The group M_{24} is one of the most remarkable of all finite groups.*
- Many properties of the larger sporadic groups reduce on examination to properties of M_{24}. This centenarian group can still startle us with its youthful acrobatics.
- The Leech lattice is a 24-dimensional Euclidean lattice which is easily defined in terms of the Mathieu group M_{24}.
- **It hardly needs to be said that the Mathieu group M_{24} plays a vital role in the structure of the Monster.**

Some simple groups, timeline, orders

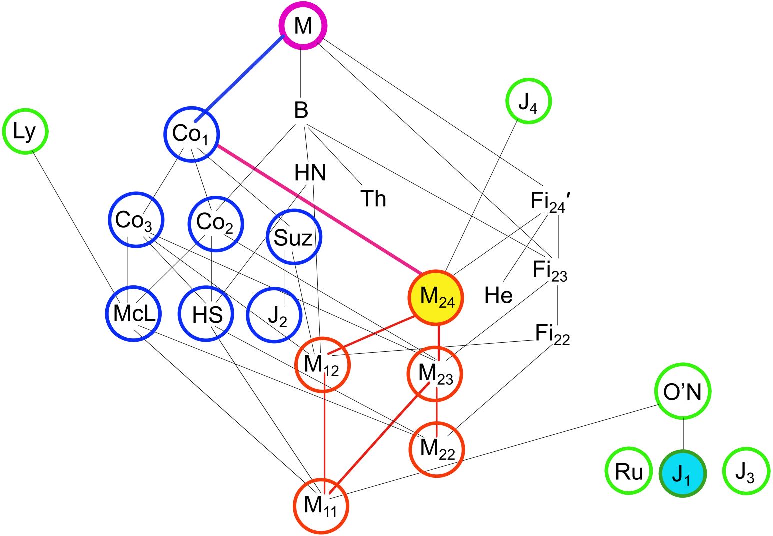
Mathieu group M_{24}, Janko group J_1, Monster group M

1861 – 73	$M_{24}: \ 2^{10}.3^3.5.7.11.23$
1961	$J_1: \ 2^3.3^1.5.7.11.19$
(1967	Leech Lattice – Witt 1940?)
1980 – 82	$M: \ 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$

Conway et al., The Atlas of Finite Groups

Six pariah groups :	Ly	$O'N$	J_1	J_3	J_4	Ru
Associated fields :	$\mathbb{F}_5$	$\mathbb{F}_7$	$\mathbb{F}_{11}$	$\mathbb{F}_4$	$\mathbb{F}_2$	$\mathbb{F}_2$

Generations: Mathieu – from Leech lattice – from Monster – Pariahs



Mathieu groups

- M_{11} is the automorphism group of Steiner system $S(4,5,11)$
- M_{12} is the automorphism group of Steiner system $S(5,6,12)$
  The automorphism group of the extended ternary Golay code is $2.M_{12}$
  Codewords with six nonzero digits – positions at which these nonzero digits occur form the Steiner system $S(5,6,12)$.
- M_{22} is the unique index 2 subgroup of the automorphism group of Steiner system $S(3,6,22)$
- M_{23} is the automorphism group of Steiner system $S(4,7,23)$
- M_{24} is the automorphism group of Steiner system $S(5,8,24)$. The 8-element sets of $S(5,8,24)$ correspond to octads in the extended 24-dimensional Golay code over $\mathbb{Z}/2\mathbb{Z}$
- **"Looking carefully at Golay's code is like staring into the sun."**
  – Richard Evan Schwartz.

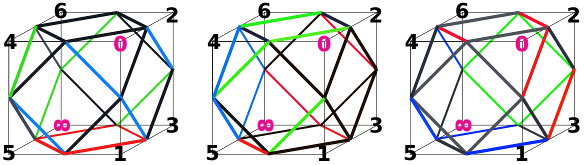
Steiner systems

- Steiner system $S(p, q, r)$ is an r-element set S together with a set of q-element subsets of S (called blocks) such that each p-element subset of S is contained in exactly one block.
- $S(5, 6, 12)$: 12-point set as the projective line over $F_{11} = \mathbb{Z}/11\mathbb{Z} \cup \infty$
  Quadratic residues $\{0, 1, 3, 4, 5, 9\}$ form a block.
  Other blocks by applying fractional linear transformations: These blocks then form a (5,6,12) Steiner system.
- $S(5, 8, 24)$– Particularly remarkable – connected with many of the sporadic simple groups and with the exceptional 24-dimensional Leech lattice.
  The automorphism group of S(5, 8, 24) is the Mathieu group M_{24}

Octads of Golay G_{24} defining $S(5, 8, 24)$

Theorem (Aitchison)

These 759 octads for $S(5, 8, 24)$ can be described by a naturally coloured cuboctahedron, defined by the labelling of a standard dice, and by adding integers mod 7. (24 =8+8+8)



$$\begin{array}{lll} L & (315\infty, 05, 21), & (315\infty, 4\infty, 63) \\ C & (546\infty, 06, 14), & (546\infty, 2\infty, 35) \\ R & (623\infty, 03, 42), & (623\infty, 1\infty, 56) \end{array}$$

(4,2,2) elements are of four kinds – three more complicated can be read off from this picture. Similarly for the other Golay octad types

Linear codes – Hamming, extended Golay

- Golay code can be constructed from three copies of the Hamming 8-code (Tyurin)

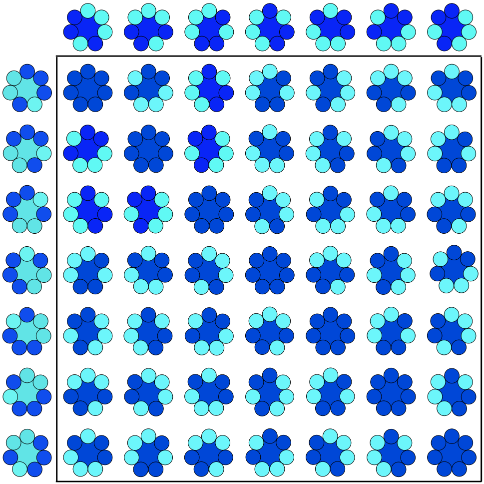
$$(A+X, B+X, A+B+X), \qquad A, B \ Hamming, \quad X \ \bar{Hamming}$$

  There are $16 = 2^4$ choices for each of A, B, X resulting in $(2^4)^3 = 2^{12}$ code words as a subset of the possible 2^{24} length-24 words
- "The Golay code is probably the most important of all codes for both practical and theoretical reasons." – (F. MacWilliams, N. Sloane, The Theory of Error Correcting Codes, p. 64)
- Robert Gallagers tribute: Marcel Golays one-page paper, Notes on Digital Coding (Proc. IRE, vol. 37, p. 657, 1949) is surely the most remarkable paper on coding theory ever written
- The Golay extended binary code is an error-correcting code capable of correcting up to three errors in each 24-bit word, and detecting a fourth. It was used to communicate with the Voyager probes – more compact than the previously-used Hadamard code.

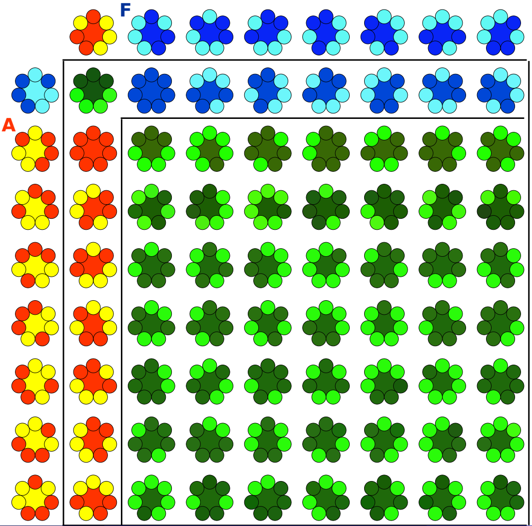
The Mathieu groups: Golay, Steiner, Leech, Lie

- The automorphism group of the octads of the Golay code is M_{24}.
- The Golay code, together with the related Witt design $S(5, 8, 24)$, features in a construction for the 196560 minimal vectors in the Leech lattice.
- The automorphism group of the Leech lattice is Conway's group Co_1.
- Conway's group is used in the construction of the Monster group.
- The transcription-labelled hexagonal tiling and oriented Fano plane encodes octonion multiplication, as well as for the exceptional Lie algebra E_8
- The Leech lattice can be constructed by using three copies of the E_8 lattice, in the same way that the binary Golay code can be constructed using three copies of the extended Hamming code H_8.

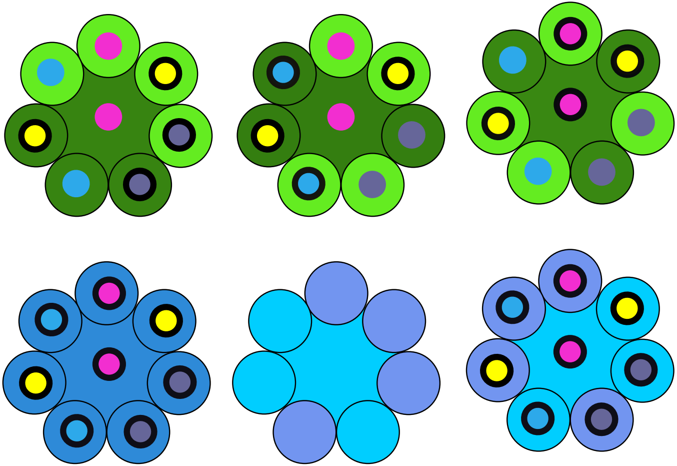
Hamming $\mathbb{Z}/2\mathbb{Z}$ 8-code can be considered as Fano lines $+\infty$, and complements. Symmetric difference of set addition $= \mathbb{Z}/2\mathbb{Z}$ arithmetic



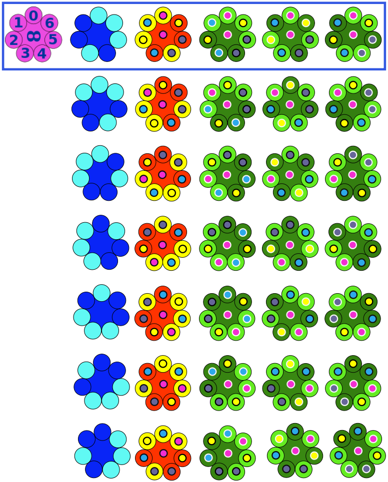
Adding Hamming F and Anti-Hamming X=A to understand Golay codewords. Given A,B, C = A+B from above (hence Fano pattern for adding X)



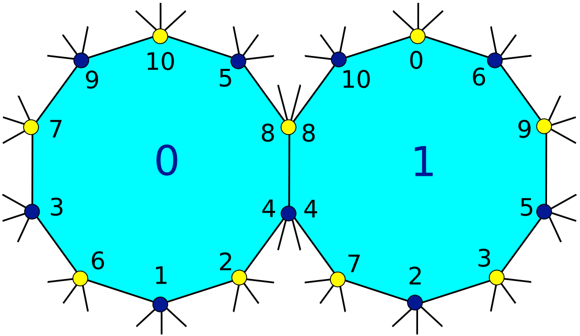
Possible octads are of type (8,0,0), (4,4,0), (4,2,2) permuted, rotated



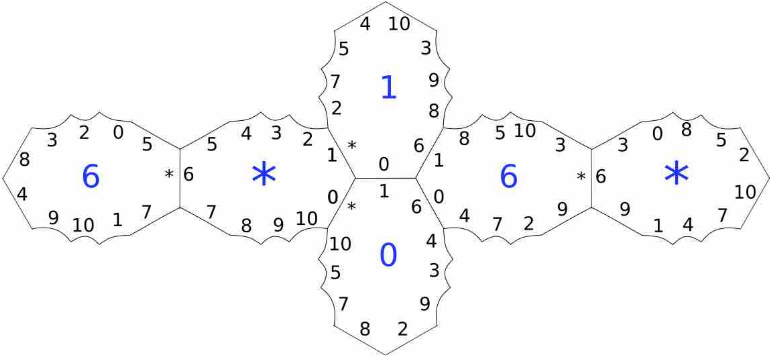
MOG and MAGOG. Analogue of 35 pictures in Curtis MOG : Graphically Augmented Miracle Octad Generator.



Arnold Trinity from $p = 11$: Integers modulo 11 labels, arithmetic defined as before. No longer a torus, but genus 12 from 11 decagons, 5 at each vertex. Link duals are ideal icosahedra, 3-manifold ends are no longer cusps



Analogue of Klein: genus 26, 60 11-gons degree 3, 660 automorphisms. Already studied by Klein? Transcription rule



Theorem (Aitchison)

This construction gives a 3-manifold for $p = 5, 7, 11$, but not for larger primes p

Concluding remarks I

- The results stated on the Mathieu group and Golay code would not surprise the authors of SPLAG, where 'a kind of triality' is mentioned, and Conway's Hexacode as an interpretation of Curtis' MOG is described. Our version can be construed as an alternative geometric view of these constructions
- In Fulton and Harris' book on representation-theory of Lie algebras, they excuse the reader for possibly not understanding their presentation of Triality, stating that they themselves do not understand it.
- In Hitchin's interviews with Sir Michael Atiyah, available on youtube, Atiyah confesses to having no real idea of what a spinor actually is, despite being able to algebraically/formally manipulate them. He says 'Perhaps Dirac understood, but Dirac is dead'.

Concluding remarks II

- Until 2014, $S(5,8,24)$ realized the largest n for an $S(n,k,l)$. It is now known that there are infinitely many larger ones, but the proof is not constructive, but probabilistic.
- In a similar vein, the construction leading to the third of the Trinity presented applies equally to the finite fields $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for $p > 11$. But for general primes p, it is known that $\mathbb{F}_p^*$ is cyclic, but Artin's Conjecture remains unsolved: for any given prime q, there are infinitely many primes p such that q generates $\mathbb{F}_p$.
  The best, and curious, result in this direction is that for any three distinct primes r, s, t, Artin's Conjecture is true for one of them.