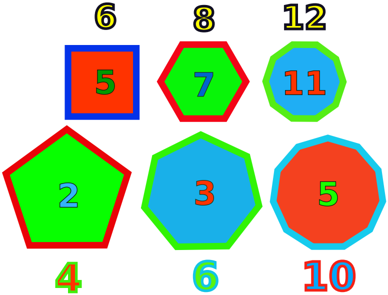
Construction of highly symmetric Riemann surfaces, related manifolds and some exceptional objects, I

Iain Aitchison

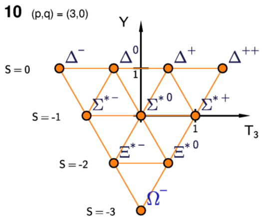


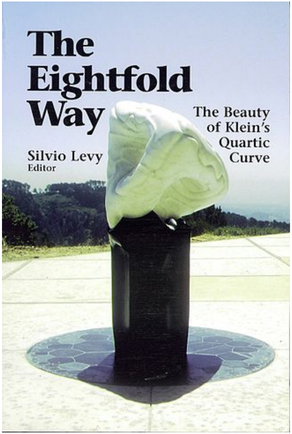
Hiroshima University March 2018

Motivation and context evolution

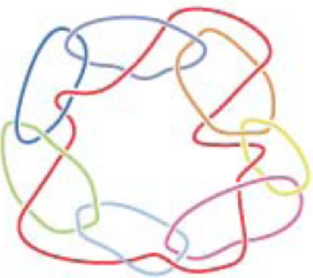
- Immediate: Talk given last year at Hiroshima (originally Caltech 2010)
- 20 years ago: Thurston's highly symmetric 8-component link
- Klein's quartic: sculpture at MSRI Berkeley
- Today: ($p = 5$)
  - Historical context – Antiquity, 19th-20th Centuries
  - Exceptional objects – Last letter of Galois, $p = 5, 7, 11$
  - Arnold Trinities
  - Transcription complexes
  - Construction of some special objects ($p = 5$)
- Tomorrow: ($p = 7, 11$)
  - Construction of some special objects ($p = 7, 11$)
  - WTC = Weeks-Thurston-Christie 8-component link
  - New picture for Mathieu group M_{24}, Steiner $S(5, 8, 24)$ (Golay code, octads)
  - New Arnold Trinity involving Thurston's 8-component link

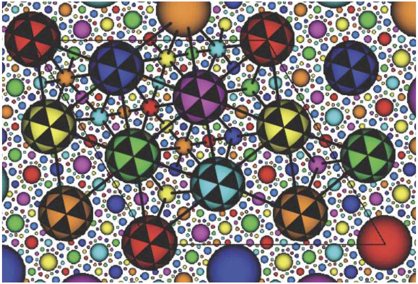
Eightfold Way sculpture at MSRI, Berkeley, commissioned by Thurston





Excerpt of Thurston: 'How to see 3-manifolds'





View of link in S^3 **View from cusp in universal cover $\mathcal{H}^3$**

- Hyperbolic link complement: extraordinary 336-fold symmetry

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Weeks - Thurston -**Joe Christie, seminar talk MSRI 1998**

- Immersed 24-punctured totally geodesic **Klein quartic**
- Complementary regions **28 regular ideal tetrahedra**

Old Babylonian Antiquity – 3,4,5

$3 + 4 + 5 = 12$
$3^2 + 4^2 = 5^2$
$3.4.5 = 60$

Tables of integers, inverses

Right-triangles:

$$\frac{1}{2}(x - \frac{1}{x}) = \frac{x^2 - 1}{2x} \qquad \frac{1}{2}(x + \frac{1}{x}) = \frac{x^2 + 1}{2x}$$

Rescale, give Pythagorean triples when x rational: $x^2 - 1, 2x, x^2 + 1$

** Translated about 70 years ago
** List of Pythagorean triples?
(eg (3,4,5), (5,12,13), ... Integers forming sides of a right-angled triangle.)

$(d/l)^2$	*diagonal* d	*short* s	*row* #	*long* l	$(1-x)/(1+x)$	$x = q/p$	p/q	p	q
				missing	*missing*	*missing*	*missing*	*missing*	*missing*
1.98...	169	119	1	120	0.41	0.417	2.4	12	5
1.95...	4825	3367	2	3456	0.41	0.422	2.37	64	27
1.92...	6649	4601	3	4800	0.4	0.427	2.34	75	32
1.89...	18541	12709	4	13500	0.4	0.432	2.31	125	54
1.82...	97	65	5	72	0.39	0.444	2.25	9	4
1.79...	481	319	6	360	0.38	0.450	2.22	20	9
1.72...	3541	2291	7	2700	0.37	0.463	2.16	54	25
1.69...	1249	799	8	960	0.36	0.469	2.13	32	15
1.64...	769	481	9	600	0.35	0.480	2.08	25	12
1.59...	8161	4961	10	6480	0.34	0.494	2.03	81	40
1.56...	75	45	11	30	1	0.500	2	2	1
1.49...	2929	1679	12	2400	0.32	0.521	1.92	48	25
1.45...	289	161	13	240	0.3	0.533	1.88	15	8
1.43...	3229	1771	14	2700	0.3	0.540	1.85	50	27
1.39...	106	56	15	90	0.29	0.556	1.8	9	5

Ancient Greece

Xenophanes c. 570-475 BC – **all is opinion, no one can know truth**
Pythagoras c. 570-495 BC
Heraclitus c. 535-475 BC – **all existence is change**
Parmenides c. 515BC, pupil of Parmenides, – **all change is illusion**
ZENO, pupil of Xenophanes, c. 490-430 BC – **paradoxes, space, time, motion**
Leucippus, contemporary of Zeno – atomism, mentor of
DEMOCRITUS c. 460-370 BC – **atomic theory of the universe**
Socrates c. 470-399 BC
PLATO c. 427-347 BC – unchanging mathematical reality, **space**, ...

Fire:Tetrahedron Air:Octahedron Earth:Cube Water:Icosahedron
Universe:Dodecahedron

Consequence 2: Democritus and atomism

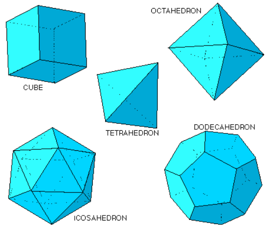
(Aristotle) '... the differences in atoms are the causes of other things. They (Democritus et al) hold that these differences are three – shape, arrangement and position; being, they say, differs only in 'rhythm, touching and turning', of which 'rhythm' is shape, 'touching' is arrangement and 'turning' is position'

rhythm	touching	turning
shape	arrangement	position
$A - N$	$AN - NA$	$Z - N$

(Aristotle) '..these atoms move in the infinite void,separated one from the other and differing in shapes, sizes, position and arrangement; overtaking each other they collide, and some are shaken away in any chance direction, while others, becoming intertwined one with another according to the congruity of their shapes, sizes, positions and arrangements, stay together and so effect the coming into being of compound bodies.'

Consequence 1: Plato's concept of space

Fundamental polyhedra : Platonic solids (dodecahedron problematic)



SPACE: What is enclosed by a surface configuration of special 3-4-5-gons

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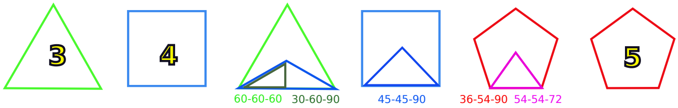
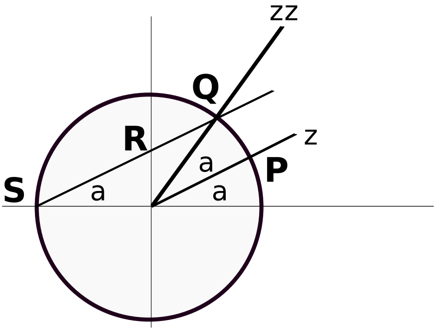
Fundamental triangles



Thales –3,4,5, Pythagorean triples, rational points



$$z = p + iq, \quad z^2 = (p^2 - q^2) + i.2pq$$

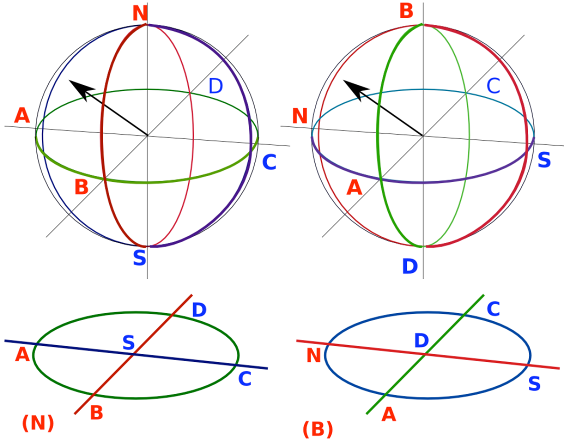
$$Q = \frac{p^2-q^2}{p^2+q^2} + i\frac{2pq}{p^2+q^2} = \frac{(p+iq)^2}{(p+iq)(p-iq)}$$
$$= \frac{1+iq/p}{(1-iq/p)} = \frac{1+i\sigma}{1-i\sigma}, \ \sigma = q/p$$

$$R = i\sigma \to \frac{1+i\sigma}{1-i\sigma} = Q$$

Stereographic projection from S
from Cayley transform,
Half plane to Unit Disc

Pythagorean Triples $(p^2 - q^2, 2pq, p^2 + q^2) \ \leftrightarrow$ Rational points on circle

Stereographic projection is conformal, sends circles on S^2 to circles, lines

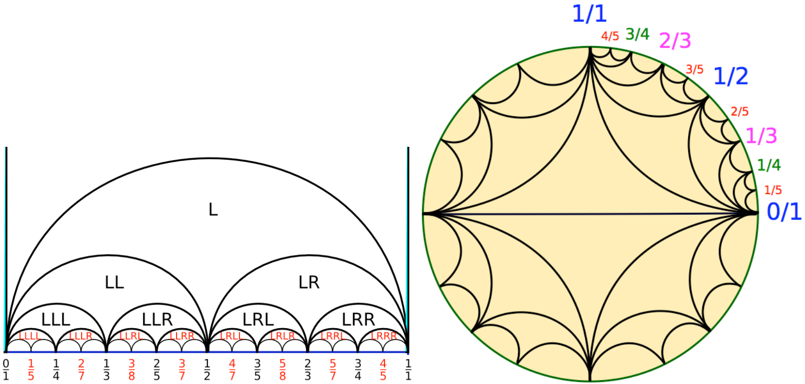


Rotated octahedron

Projection to $\mathbb{C}$

Cayley transform: order-3 rotation of an octahedron around an axis through opposite triangle faces.

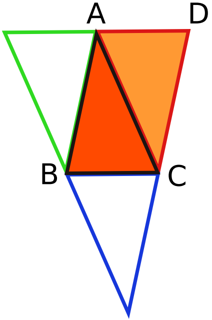
Modular Diagram for $SL_2(\mathbb{Z})$



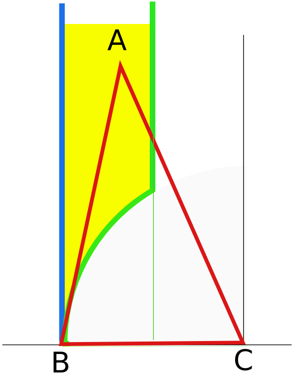
Rational numbers on the real line (as boundary of the upper-half-plane) and as rational points on the circle under the Cayley transform.

Elliptic curves as Euclidean triangles

Torus from parallelogram; Parallelogram gives two Euclidean triangles
Always possible: any Euclidean torus from triangle, angles $\leq \pi/2$

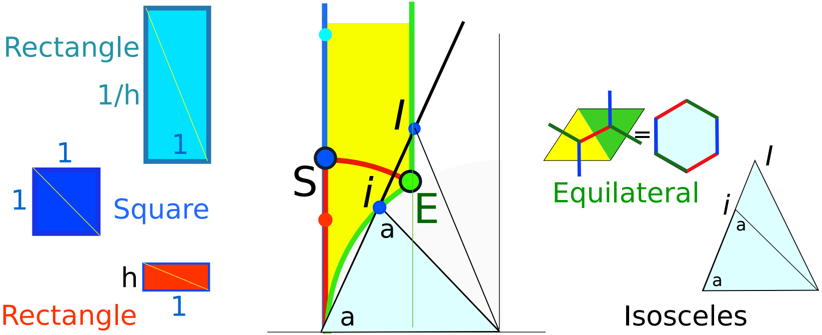


Choice of two triangles



Unit base, apex in fundamental region

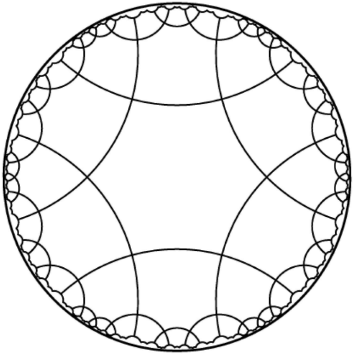
Parallelograms give Euclidean triangles defining moduli space

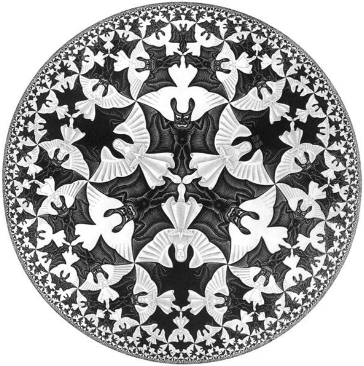


Singular points S, E correspond to Euclidean tori arising from the tessellation of the plane by **SQUARES** or regular **HEXAGONS** (honeycomb lattice).

4, 6: Escher's *Heaven and Hell, Circle Limit IV*

Hyperbolic plane tessellated by hexagons 4 at each vertex; dually, by squares 6 at each vertex.





Remark. (A, 1988): All possible decompositions of all 3-manifolds as a union of generic 3-polyhedra arise from quotient surfaces of this pattern of hexagons. (Heegaard surfaces, handlebodies)

Symmetry : Legacy of Galois

- Non-solvability by radicals for general quintics etc
- simple groups – A_5 simple, series $A_n < S_n,\ n > 4$ simple
- simple groups – $L_2(p) := PSL_2(\mathbb{Z}/p\mathbb{Z})$ simple, p prime
- $p > 11$: $L_2(p)$ has no non-trivial permutation representation on fewer than $p+1$
- Exceptional: $p = 5, 7, 11$, related to Platonic solids

Last letter of Galois

Ainsi pour le cas de $p = 5, 7, 11$, l'equation modulaire s'abaisse au degre p. En toute rigueur, cette reduction n'est pas possible dans les cas plus eleves.

'No subgroup F such that (set-theoretically) $L_2(p) = F \times C_p$'

$L_2(p)$ acts via Möbius transformations on $\mathbb{PF}_p^1 = \mathbb{F}_p \cup \{\infty\}$

$p = 5, 7, 11$, Galois uses involutions

$$\pi_5 = (0, \infty)(1, 4)(2, 3)^*$$

$$\pi_7 = (0, \infty)(1, 3)(2, 6)(4, 5)^*$$

$$\pi_{11} = (0, \infty)(1, 2)(3, 6)(9, 7)(5, 10)(4, 8)^*$$

$L_2(p)$ leaves invariant p involutions $\Pi = \{\alpha^{-i}\pi_p\alpha^i \ : \ 1 \leq i \leq p\}$.

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* See later

Arnold Trinities, Numerology, Mysticism

Arnold (1976, Hilbert Problems) : find a common origin for the ubiquity of ADE occurrences in mathematics
Later, TRINITIES.

	3	4	5	Tetrahedron$_f$, Cube$_f$, Dodecahedron$_f$: Face degrees Pythagorean Triple: right-triangle
	2	4	8	Simply-laced Lie algebras: A_2... D_4... E_8.. Division algebra dimensions: $\mathbb{C}, \mathbb{Q}, \mathbb{C}ay$
/2	1	2	4	Division algebra dimensions: $\mathbb{R}, \mathbb{C}, \mathbb{Q}$
+1	2	3	5	E_8 singularity : Poincare homology sphere
×2	4	6	10	3D symmetric networks : Cubes, Diamond, K4 Laves
+1	5	7	11	Galois : Exceptional actions of $PSL_2(p)$, $p = 5, 7, 11$
+1	6	8	12	Cube$_f$, Cube$_v$, Cube$_e$ Tetrahedron$_e$, Cube$_v$, Dodecahedron$_f$

Arnol'd Trinities: Cube $(f, v, e) = (6, 8, 12) \rightarrow (5, 7, 11)$

$$5 = 6 - 1, \quad 7 = 8 - 1, \quad 11 = 12 = 1$$

Galois: exceptional finite groups for primes $5, 7, 11$:

$$L_2(5) \cong A_5 \sim A_4 \times C_5, \quad L_2(7) \cong L_3(2) \sim S_4 \times C_7, \quad L_2(11) \sim A_5 \times C_{11}$$

Amazing fact: Platonic groups!
TETRAHEDRON (group A_4)
CUBE/octahedron (group S_4) – four body diagonals
DODECAHEDRON/icosahedron (group A_5) – five inscribed tetrahedra

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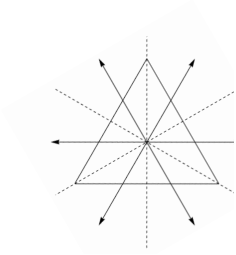
- $L_2(5) \simeq A_5 =$ icosahedron/dodecahedron automorphism group
- $L_2(7) =$ Klein quartic automorphism group
- Is $L_2(11) =$ nice surface??? automorphism group?
    - A surface genus 70 (Martin-Singerman)
    - A surface genus 26 (A-) $=$?
    - a - presumably less studied! - surface studied by Klein

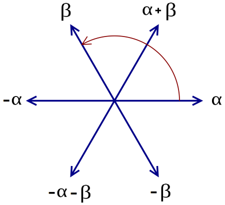
Exceptional groups and symmetric surfaces

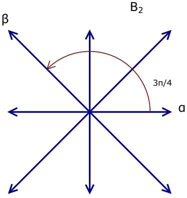
- Three 'exceptional' Galois-groups correspond to the three Platonic groups in $SO(3)$
- Three Platonic groups in $SO(3)$ lift to binary groups in $SU(2)$ (spin)
- (Subgroups of $SU(2)$ correspond to representations of E_8 via the McKay correspondence)
- Three Platonic groups correspond to the three exceptional Lie algebras E_6, E_7, E_8

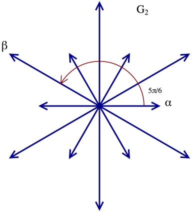
Arnol'd: 'I have heard from John MacKay that the 27 straight lines on a cubical surface, the 28 bitangents of a quartic plane curve and the 120 tritangent planes of a canonic sextic curve of genus 4, form a trinity parallel to E_6, E_7 and E_8.'

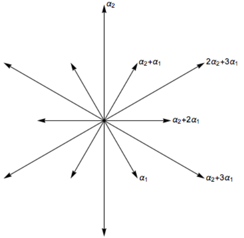
Root systems, semi simple Lie algebras A_2, B_2, G_2, Weyl ...

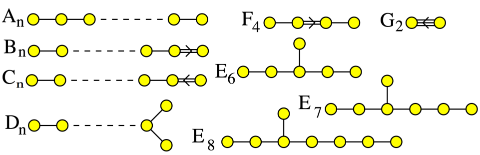












Platonic triangles: angles 90, 60 or 120, 45 or 135, 30 or 150, 0 or 180

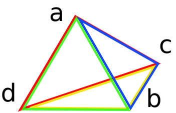
Arnol'd Trinities

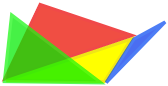
'Referring to the triple of exceptional Galois groups $L_2(5), L_2(7), L_2(11)$ and its connection to the Platonic solids I wrote : It sure seems that surprises often come in triples. Briefly I considered replacing triples by trinities, but then, I didn't want to sound too mystic'
Arnol'd proposed the concept of 'trinities' to unify what appeared to be the ubiquity of exceptional objects in mathematics appearing in threes.

Arnol'd Trinities, from Platonic solids

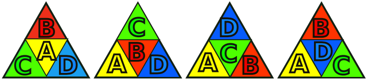
- The exceptional Galois groups $L_2(5)$, $L_2(7)$, $L_2(11)$.
- The associated curves with these groups as symmetry groups: the dodecahedron, Klein's Quartic curve and ? (= mysterious genus 70 curve of Martin and Singerman)
- (algebra) Trinity of: complex numbers, quaternions; octonions
- The trinity of exceptional Lie algebras E_6, E_7 and E_8
- A trinity parallel to that of the exceptional Lie algebra is between the 27 straight lines on a cubic surface, the 28 bitangents on a quartic plane curve and the 120 tritangent planes of a canonic sextic curve of genus 4.
- The three generations of sporadic simple groups: Mathieu (M_{24} giving rise to the automorphisms of the Golay codes), Conway (automorphisms of the Leech lattice), and the Monster (automorphisms of the Griess algebra).

The tetrahedron: 4-punctured universal cover





View from inside -

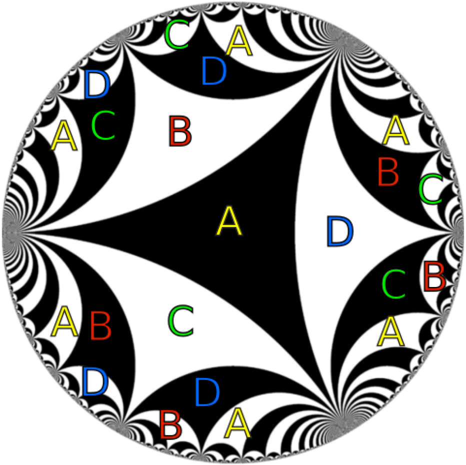


A:BCD	B:CAD	C:DAB	D:BAC
A:CDB	B:ADC	C:ABD	D:ACB
A:DBC	B:DCA	C:BDA	D:CBA

Vertices from inside -

a:bcd	b:cad	c:dab	d:bac

Tetrahedral faces around each enable developing map in the modular diagram. Give each hyperbolic ideal triangle geometry, incircles tangent



Infinite branched covers of triangulated closed surfaces

- Same approach works for any closed triangulated surface – delete vertices, universal cover is universally the same gadget with appropriate geometry on triangles
- Grothendieck, Esquisse d'un programme, Dessins d'enfants; Drawing curves over number fields, by G.B. Shabat and V.A. Voevodsky; Leila Schneps; and of course Degtyarev!
- Equivalently allow Euclidean equilateral triangle geometry on each triangle, obtain all Riemann surfaces defined over algebraic number fields

Genus 0: Congruence subgroups of level n in $\Gamma := SL_2(\mathbb{Z})$

Principal Congruence subgroup of level n:

$$\Gamma(N) := \{G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : G \equiv Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod n\}$$

Modular curves: $Y(n) = \mathcal{H}/\Gamma(n)$. Compactify cusps: $X(n) = \bar{Y}(n)$ genus $0 \iff n = 3, 4, 5$ Tetrahedron, octahedron, icosahedron: Platonic solids
Hecke Congruence subgroup of level n:

$$\Gamma_0(n) := \{G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \mod n\}$$

Modular curve $\bar{X}_0(p) = \mathcal{H}/\Gamma_0(p)$ genus $0 \iff (p-1)|24$
Fricke involution is the involution of the modular curve $X_0(n)$ defined by $\tau \to -\frac{1}{n\tau}$.
Serre-Ogg-Thompson: modular curve $\mathcal{H}/\Gamma_0(p)^+$ gives genus zero if and only if

$$p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71.$$

These are the prime factors of the Monster simple group M

Genus 1: Shimura-Taniyama-Weil Modularity, Fermat

- Hyperbolic uniformization of elliptic curve E – mapping

$$\mathcal{H} - \text{finite union of } \Gamma\text{-orbits} \to E - \text{finite set of points}$$

periodic with respect to a subgroup of finite index in Γ.

- Such a curve is defined by a Weierstrass equation with coefficients A, B which are algebraic numbers (Belyi)
- Uniformization is of arithmetic type if periodic with respect to a congruence subgroup (contains some $\Gamma(n)$)
- Shimura-Taniyama-Weil Conjecture/Theorem: An arithmetic elliptic curve – defining equation with coefficients in $\mathbb{Q}$ – admits a hyperbolic uniformization of arithmetic type

Existence of possibly disguised symmetry in some torus triangulation. (See Mazur: Number Theory as Gadfly)

Observation .. Tetrahedron pattern .. Finally!

$$\begin{array}{llll} A:BCD & B:CAD & C:DAB & D:BAC \\ A:CDB & B:ADC & C:ABD & D:ACB \\ A:DBC & B:DCA & C:BDA & D:CBA \end{array}$$

Start in one face, travel to any other along a path. Sequence of changes

$$A \to B \to D \to C$$

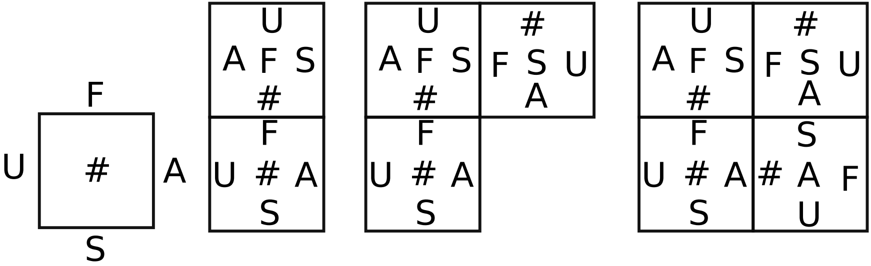
$$A:BCD \to B:ADC \sim B:DCA \to D:BAC \sim D:CBA \to C:DAB$$

$$X:PQR \to P:XRQ$$

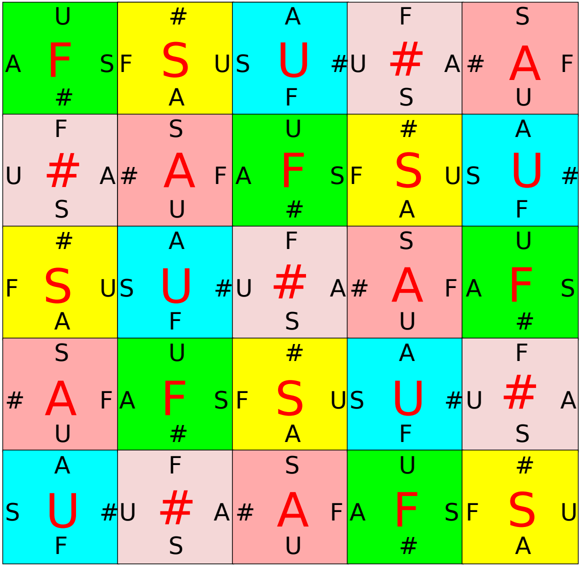
The first transposition is the obvious change of face label. The second represents an involution on the anticlockwise list of adjacent face labels, true for any transition in either direction:

FACE TRANSCRIPTION (definition)!

4-gon example, face transcription rule $X : PQRS \to P : XRQS$



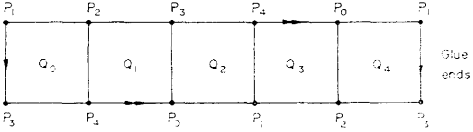
5-square torus. Abstract symbol labels. Meaning?



Horrocks-Mumford bundle torus. Meaning?

Horrocks-Mumford bundle
Stable rank 2 bundle over $\mathbb{C}P^4$, 15000 symmetries, essentially unique

Then a point or line is on a line or a quadric in $\mathbb{P}$ if and only if the corresponding 0-simplex or 1-simplex is on the corresponding 1-simplex or square in Σ. The nice thing is that the Σ you get is homeomorphic to a 2-dimensional real torus:



Glue top and bottom with horizontal shift

Abstract coding, and possible meanings ... Digression

Prokaryotes, Eukaryotes: Bacterial DNA reproduction

- Eukaryota: biological species whose cells have nuclei and organelles
- Prokaryote (Bacteria and archaea): unicellular organisms lacking membrane-bound nucleus (karyon), mitochondria, or any other membrane-bound organelle.
- Typical bacterial chromosomes: a **single circular molecule of DNA**
- **Reproduction may reorder or reverse gene segments**

Rozen: Gene Splicing

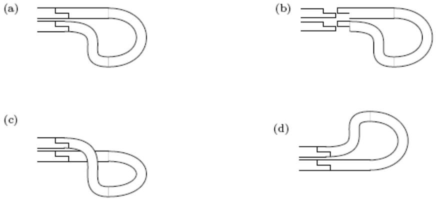


Fig. 4. The operation *hi*

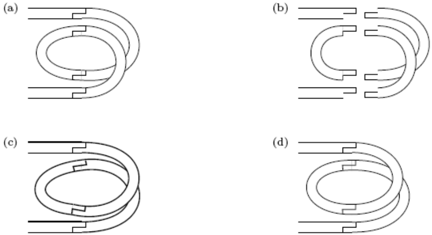
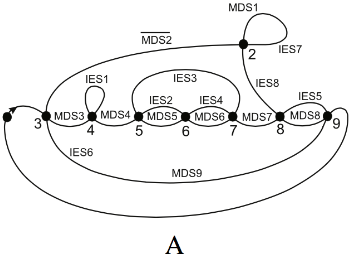


Fig. 5. The operation *dlad*

DNA recombination through assembly graphs



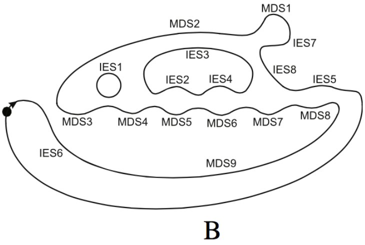


Figure: Schematic representation of simultaneous recombination, at the moment of recombination (left), and resulting molecules after recombination (right).

From: Angela Angeleska, Nataša Jonoska, Masahico Saito, *DNA recombination through assembly graphs*, Discrete Applied Mathematics 157 (2009) 3020-3037, Figure 3

More generally, loops split or combine, segments rearranged and reversed: there is a natural relationship with alternating link constructions

‘MOTIVATION’: DNA, transcription, sequence reversal

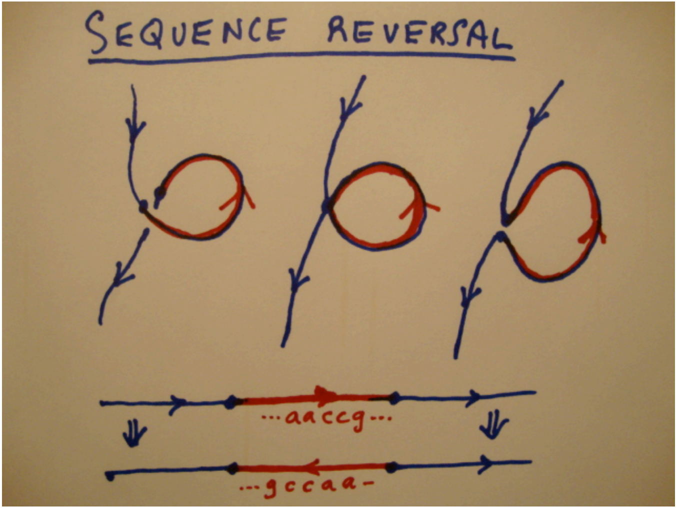


Figure: Simplest sequence reversal – model

Simple circular rearrangement and reversals

(Using A as a reference for position and direction)

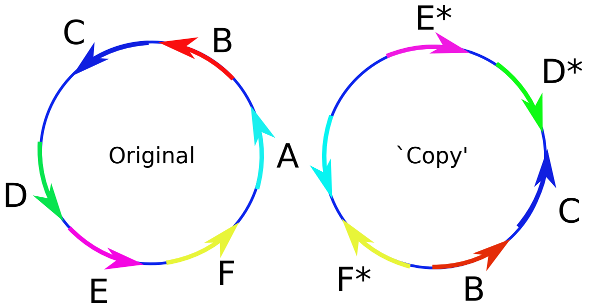


Figure: Original: $(A : BCDEF)$ Copy: $(A : F^{*}BCD^{*}E^{*})$

Suggestive of glueing together two oriented polygons

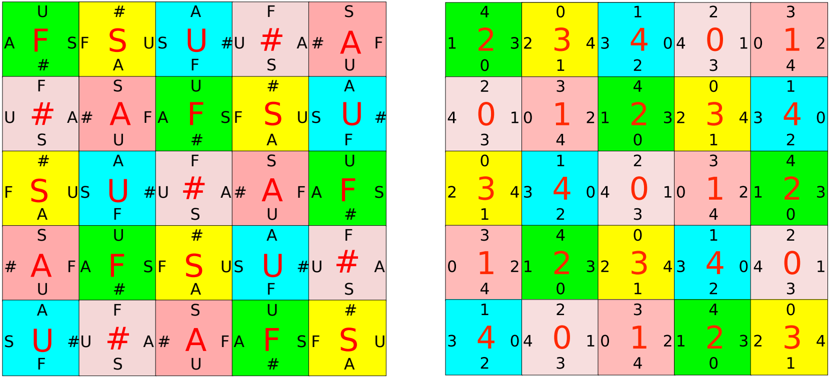
DNA transcription is not abstractly 'symmetric' as a process for permuting and reversing (not an involution: iteration gives other possible outcomes)

Torus: Abstract symbol labels. Meaning?

Two consistent binary operations, relative to $\# = 0,\ A = 1$ (arbitrary):

$\oplus$	$\#$	A	F	S	U
$\#$	$\#$	A	F	S	U
A	A	F	S	U	$\#$
F	F	S	U	$\#$	A
S	S	U	$\#$	A	F
U	U	$\#$	A	F	S

$\otimes$	$\#$	A	F	S	U
$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
A	$\#$	A	F	S	U
F	$\#$	F	U	A	S
S	$\#$	S	A	U	F
U	$\#$	U	S	F	A


F S U # A
A F S U
S U # A F
A F S U #
U # A F S
2 3 4 0 1
0 1 2 3 4
3 4 0 1 2
1 2 3 4 0
4 0 1 2 3

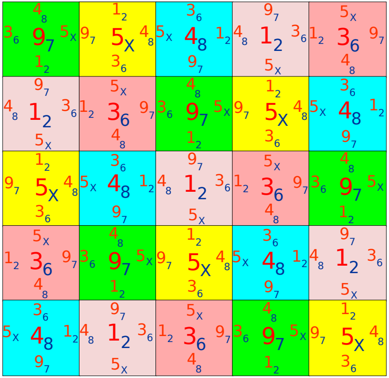
Integers modulo 11, addition becomes multiplication

Correspondence between non-zero integers mod 5 and quadratic residues/non-residues modulo 11: Non-zero integers mod p are multiplicatively cyclic of order $2n = p - 1$; even powers give quadratic residues :

$$x = 5,\ x^2 = 3 \quad x^{2.0} = 1,\ x^{2.1} = 3,\ x^{2.2} = 9,\ x^{2.3} = 5,\ x^{2.4} = 4,\ x^{2.5} = 1$$

∞	0	1	2	3	4	$(\rightarrow +1 \bmod 5)$
0	**1**	**3**	**9**	**5**	**4**	$(\rightarrow \times 3 \bmod 11)$
∞	**2**	**6**	**7**	**X**	**8**	$(\downarrow \times 2 \bmod 11)$

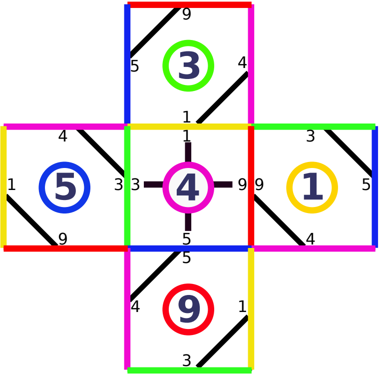
Integers modulo 11 labels, addition becomes multiplication



Outer automorphism of S_6

For $n \neq 6$, automorphisms of the symmetric group S_n are always INNER AUTOMORPHISMS.

S_6 has OUTER AUTOMORPHISMS (Sylvester, 19th centruy)



Sylvester's synthemes for a 6-element set S

- A duad is a 2-element subset of S. Note that there are 15 duads.
- A syntheme is a set of 3 duads forming a partition of S.
- There is a 3:3 correspondence between synthemes and duads.
- There are also 15 synthemes.
- A synthematic total is a set of 5 synthemes partitioning the set of 15 duads.
- There are 6 synthematic totals.
- Any permutation of S gives a permutation of the set T of synthematic totals, so we obtain an action of S_6 on T.
- A bijection between S and T gives an action of S_6 on S.
- This gives a homomorphism from S_6 to itself.

Theorem

(Sylvester) This is an OUTER automorphism of S_6.

Torus synthemes – Outer automorphism of S_6

	A	B	C	D	E	F
A		∞1 59 43	∞3 45 19	∞9 14 35	∞5 31 94	∞4 93 51
B	∞1 59 43		∞5 39 41	∞3 49 15	∞4 91 53	∞9 13 54
C	∞3 45 19	∞5 39 41		∞4 95 13	∞9 14 35	∞1 53 49
D	∞9 14 35	∞3 49 15	∞4 95 13		∞1 54 39	∞5 34 91
E	∞5 31 94	∞4 91 53	∞9 14 35	∞1 54 39		∞3 41 95
F	∞4 93 51	∞9 13 54	∞1 53 49	∞5 34 91	∞3 41 95	

The outer automorphism of S_6

On cycles, it exchanges permutations of type (12) with (12)(34)(56) (class 21 with class 23), and of type (123) with (145)(263) (class 31 with class 32). The outer automorphism also exchanges permutations of type (12)(345) with (123456) (class 2131 with class 61). For each of the other cycle types in S_6, the outer automorphism fixes the class of permutations of the cycle type.
On A_6, it interchanges the 3-cycles (like (123)) with elements of class 32 (like (123)(456)).

Theorem

Any automorphism of S_n, $n \neq 6$ is an INNER automorphism.

Sylvester's synthemes

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f\{\phi(a,b), \phi(c,d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

"To the modulus 6 there are fifteen different synthemes capable of being constructed. At first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families, each will have to be taken twice over; or in other words, the fifteen synthemes to modulus 6 being reduplicated, subdivide into six natural families of five each."

The six families above referred to (in which it is to be understood that $p.q$ and $q.p$ are identical in effect) are the following:—

$$\begin{array}{lll@{\qquad}lll@{\qquad}lll}
a.b & c.d & e.f & a.c & d.e & f.b & a.d & e.f & b.c \\
a.c & b.e & d.f & a.d & c.f & e.b & a.e & d.b & f.c \\
a.d & b.f & c.e & a.e & c.b & d.f & a.f & d.c & e.b \\
a.e & b.d & c.f & a.f & c.e & d.b & a.b & d.f & e.c \\
a.f & b.c & d.e & a.b & c.d & e.f & a.c & d.e & f.b \\[1ex]
a.e & f.b & c.d & a.f & b.c & d.e & a.b & c.d & e.f \\
a.f & e.c & b.d & a.b & f.d & c.e & a.c & b.e & d.f \\
a.b & e.d & f.c & a.c & f.e & b.d & a.d & b.f & c.e \\
a.c & e.b & f.d & a.d & f.c & b.e & a.e & b.d & c.f \\
a.d & e.f & b.c & a.e & f.b & c.d & a.f & b.c & d.e
\end{array}$$

And it will be observed that every two families have one, and only one, syntheme in common between them; and precisely in the same way as in the note above quoted it is especially shown that the one single natural family

$$\left|\begin{array}{ll} a.b & c.d \\ a.c & b.d \\ a.d & b.c \end{array}\right|$$

gives rise to a function of four letters with only one value, so the six functions analogously formed with these six families obviously give rise to six func-

Sylvester's synthemes: Baker, Geometry Vol 2

fifteen things can be identified by naming them after the pairs which can be formed from six arbitrary symbols, we can have a (3, 3) correspondence between any two sets of fifteen things. As has been said, the importance of this for the present purpose was emphasized by Cremona (*loc. cit.* pp. 854, 866, 870). A further remark which is of use is that two synthemes which have no duad in common occur together in only one system, since two systems have only one syntheme in common. Two such synthemes thus serve to identify a system. [Add.]

Of the various possible ways of assigning the names $P, Q, \ldots, R'$ to the systems, one example is given in the following scheme, which can be read either in rows or columns.

	P	Q	R	P'	Q'	R'
P		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
Q	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
R	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
P'	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
Q'	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
R'	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

With this table, the correspondence of the pairs of systems with the synthemes can be enumerated at once; for instance, (Q, R) is associated with (15.26.34), and (R, P') with (14.23.56). Conversely, the duads of the numbers are each associated with three duads of letters; for example, (14) with $(PQ.RP'.Q'R')$, since (14) occurs in the same syntheme in P and Q, in the same, other, syntheme in R and P', and in the same, still other, syntheme in Q' and R'.

Figure: Sylvester, synthemes correspondence

Extended Golay ternary code – first glimpse

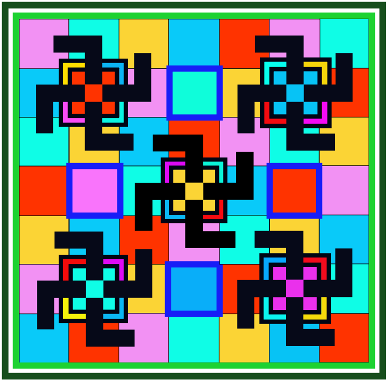
Generating matrix and check matrix

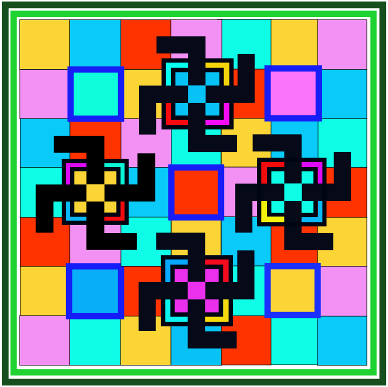
$$G := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

ternary (12,6) code:

$$H := \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Minsa to Manji: Steiner $S(5,6,12)$ first glimpse





Edge transcription example: $X{:}ABC \to X{:}BAC$

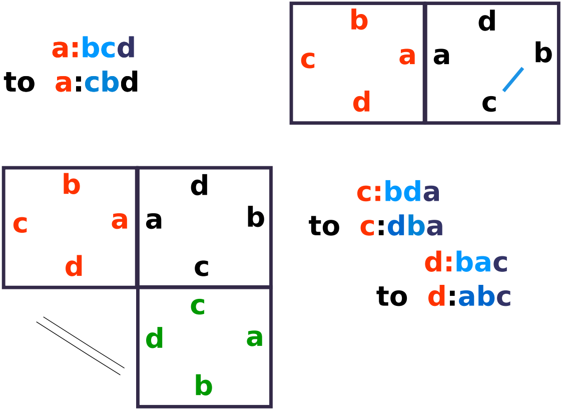


Figure: Degree three vertices: constructs edge-labeled cube (sphere)

Edge transcription: 2-sphere as cube

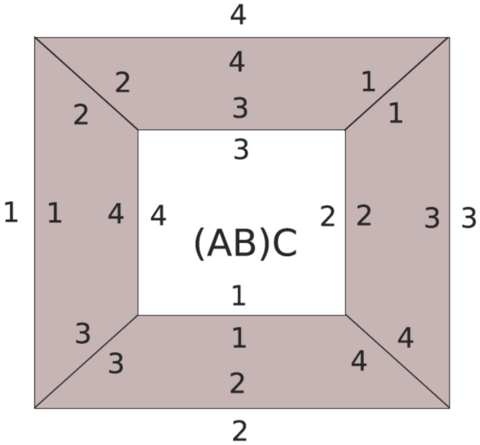


Figure: The 3-dimensional cube, edges labeled: $X{:}ABC \to X{:}BAC$

Alternative transcription: swap last pair instead

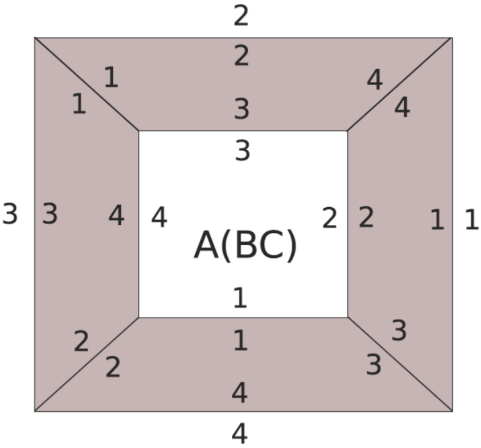


Figure: The 3-dimensional cube, edges labeled. $X{:}ABC \to X{:}ACB$

Implication from polygon orientation

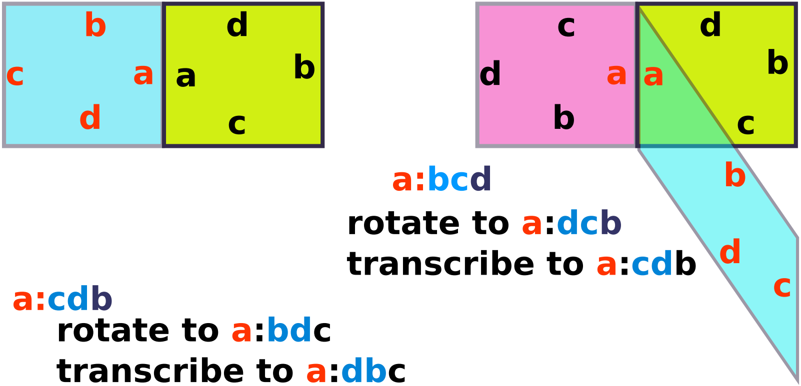


Figure: Degree three edges of cubes in 3-dimensional space

The tesseract, unit quaternion action, Quaternionic space

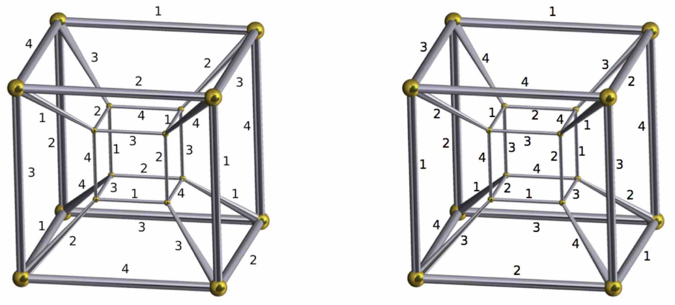


Figure: Edge transcription. Left: $a : bdc = 1 : 243$. Right: $a : cbd = 1 : 324$.

3-sphere as boundary of 4-D cubes: Labeled tesseracts $(\mathbf{4}, \mathbf{3}, \mathbf{3})$
From front to back, left figure labels rotate clockwise: the right, anti-clockwise. (Left and right unit quaternion action)
Quotient of S^3 under this action is Quaternionic space.

Quaternionic space: opposite-face identifications

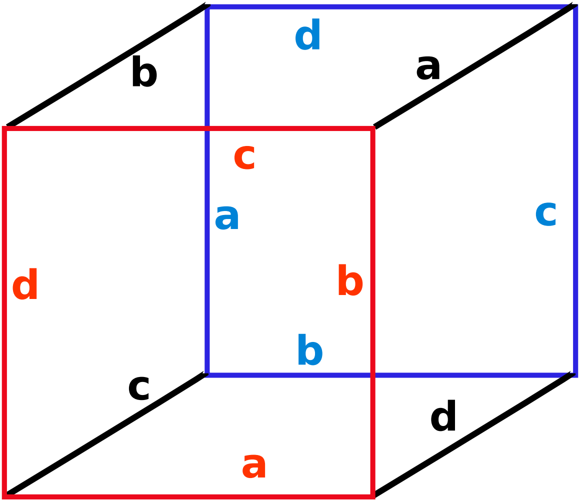


Figure: The 3-dimensional cube: translate with $\pi/2$ rotation clockwise

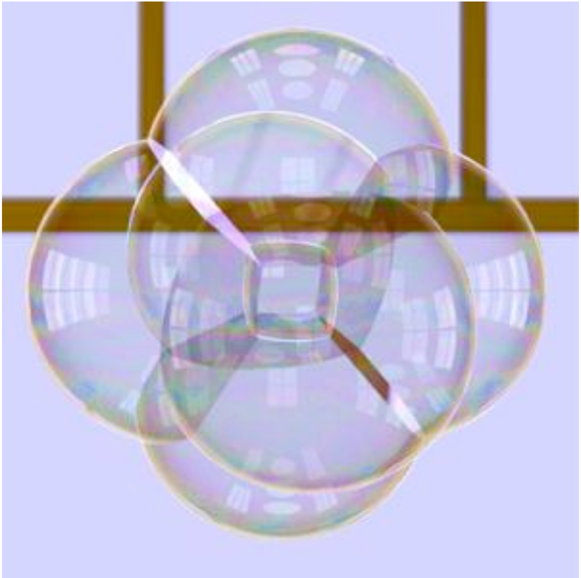
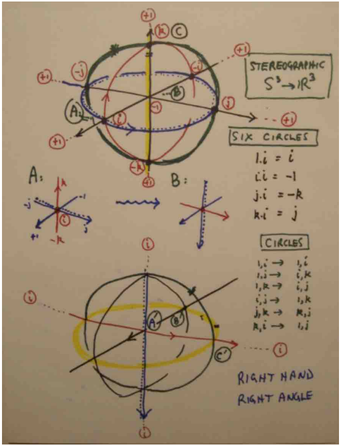


Figure: John Sullivan's soap bubble rendering of the tesseract projected to $\mathbb{R}^3$

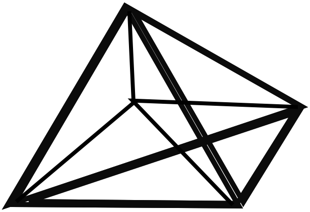


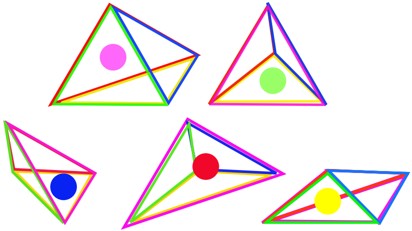
Edge transcription yields the tesseract, the tessellation of the 3-sphere by eight 3-dimensional cubes, naturally labeled revealing the left and right action of the unit quaternions, with quaternionic space as quotient.

Space: tetra transcription complex can give a 4-simplex

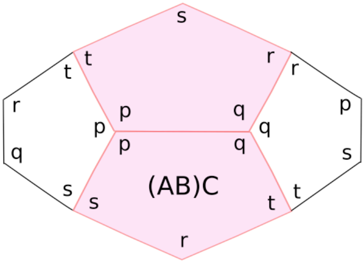
- Start with a labelled triangle, transcribe labels to adjacent triangles. Develop around each vertex (finite order, here 3). Obtain a tetrahedron
- Seen from the other side, order is reversed around any face. Choose a new colour for the face, and develop the reversed-order adjacent faces. Obtain a new tetrahedron.

Put a colored light at the center of a tetrahedron
Different colors for adjacent tetrahedra along a common face triangle





Vertex transcription: $ab{:}cde \to ba{:}dce$; $ab{:}cde \to ba{:}ced$



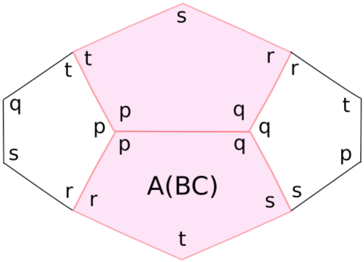


Figure: Left: (AB)C $ab{:}cde \to ba{:}dce$; Right: A(BC) $ab{:}cde \to ba{:}ced$ and

Both cases: pentagon vertices have degree 3.

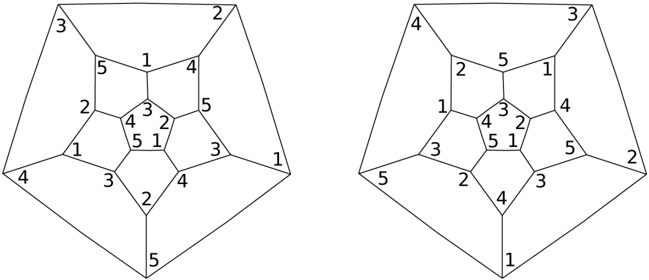


Figure: (L) **(AB)C** : 12:345 → 21:435 (R) **A(BC)** : 12:345 → 21:354.

Attaching further 5-gons, using the symbols 12345 for *abcde*: the process terminates with the construction of a labeled dodecahedron from the innermost pentagon.

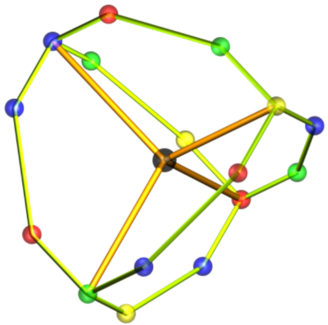


Figure: Tetrahedral links for A(BC) and (AB)C.

Degree 3 edges in 3-space, as for Quaternionic space

- Grow a 2-complex in 3-space, with each edge common to three pentagons.
- Each pair of pentagons at an edge naturally grows to create a dodecahedron containing/defining a 3-dimensional cell.
- At vertices, we see six pentagons in a tetrahedral configuration, creating four dodecahedra with common vertex.
- Each coloured vertex is closest to four differently coloured vertices.
- Each tetrahedral configuration naturally grows to give four dodecahedra meeting there
- Result: the tessellation $\{5, 3, 3\}$ in standard notation.

- The complex is a union of 119 dodecahedral bubbles in 3-space, contained within a single outermost dodecahedron.
- Add a point at infinity to obtain a final solid dodecahedron, creating both the 3-sphere and its decomposition as the 120-cell $\{5, 3, 3\}$.
- Every vertex-labeled oriented dodecahedron has the same combinatorial structure, and there is a unique way to identify any dodecahedral 3-cell with any face-adjacent 3-cell, by translating along a symmetry axis through the common face, and rotating so that vertex labels match.

  This defines the combinatorial action of the binary icosahedral group.

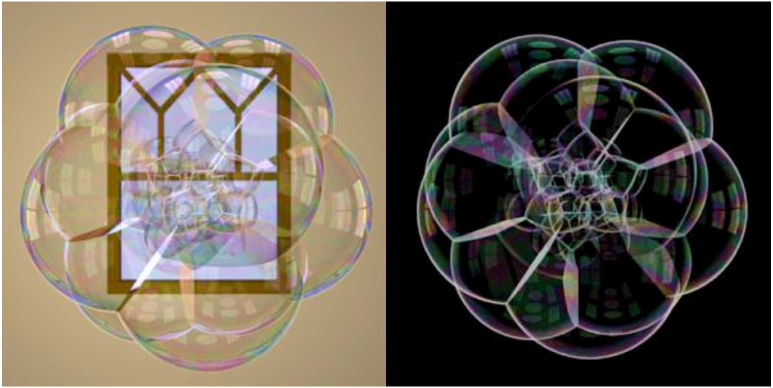


Figure: John Sullivan's rendering of the 120-cell as a 'foam' of 119 bubbles in 3-space

A new construction (#9) for Poincaré's homology sphere

Theorem

Both vertex transcriptions $xy{:}abc = yx{:}bac$, and $xy{:}abc = yx{:}acb$ canonically define Poincaré's homology sphere, by giving each dodecahedron the geometric structure of a regular spherical dodecahdron (all dihedral angles $2\pi/3$)
Vertex labels canonically define the quaternion action of the binary icosahedral group on the 3-sphere.

Remark: This is the 'ninth' construction: Cf Kirby & Scharlemann, *Eight faces of the Poincaré homology 3-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 113 - 146, Academic Press, New York-London, 1979.

Curtis' symmetric generators for M_{12}

8 *The Mathieu group* M_{12}

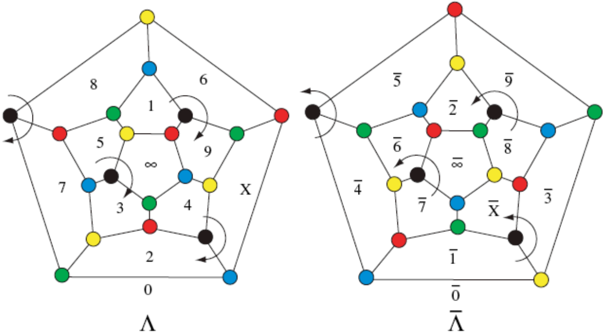
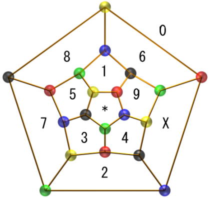
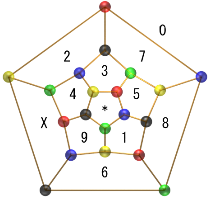


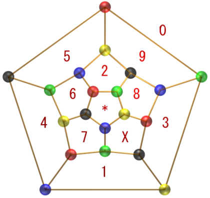
Figure 1.1. Two dodecahedra, each with its 20 vertices labelled using five colours.



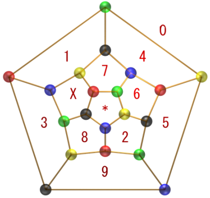
$D^e_{a(bc)}$



$D^e_{(ab)c}$



$D^o_{a(bc)}$



$D^o_{(ab)c}$

Figure: Four homology spheres, using five colours to label vertices: face labels following Curtis using elements of the projective line $PL(\mathbb{Z}_{11})$. Curtis uses *black* $\leftrightarrow 1$, *green* $\leftrightarrow 2$, *blue* $\leftrightarrow 3$, *red* $\leftrightarrow 4$, *yellow* $\leftrightarrow 5$.

Bring's curve, 1786 (Klein, Icosahedron lectures)

Arose from attempts to solve quintics by radicals, using transformations as with reduction of cubic polynomials

- $t^5 - t + A = 0$: three vanishing coefficients leads to
- $v+w+x+y+z = v^2+w^2+x^2+y^2+z^2 = v^3+w^3+x^3+y^3+z^3 = 0.$
- Intersection of a cubic and a quadric
- Sextic plane curve of genus 4. 120 tritangent planes
- Symmetry group S_5. Largest possible automorphism group of a genus 4 complex curve.
- Triple cover of the sphere branched in 12 points.
- 12 regular hyperbolic pentagons, degree 5 vertices.
- Weber (2002): 24 regular hyperbolic pentagons, degree 4 vertices.
- Apery-Yoshida: quotient gives configuration space of 5 points on S^1.
- Piovan (2013): Tonnetz model for 'cycles of fifths' in music

Edge transcription: 1:2345 $\to$ 1:4523

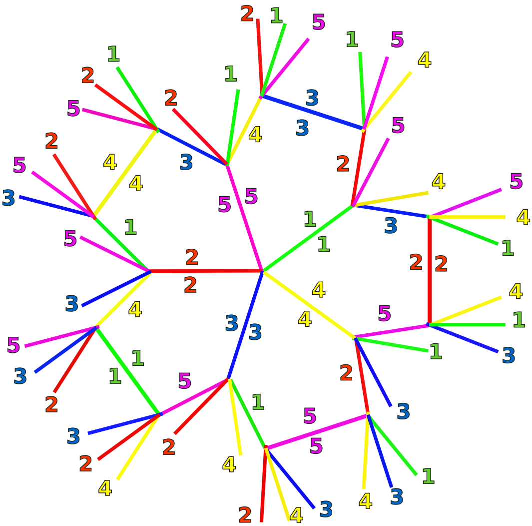


Figure: 12 regular pentagons, degree 5 vertices. Face pentagons even.

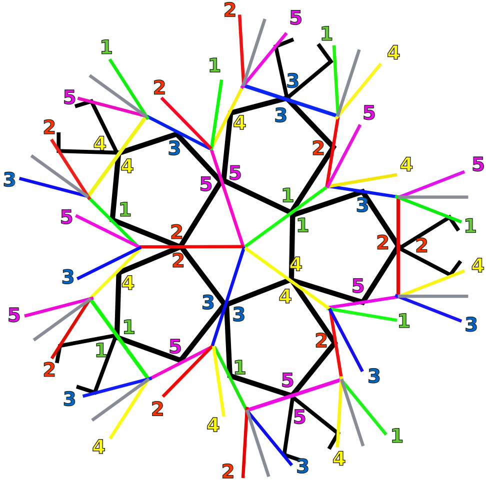


Figure: 24 regular pentagons, degree 4 vertices. Weber showed that the medial pentagon of a regular $2\pi/5$ pentagon is right-angled. Vertex-pentagons are odd.

Example - Vertex transcription: 12:345 → 21:345

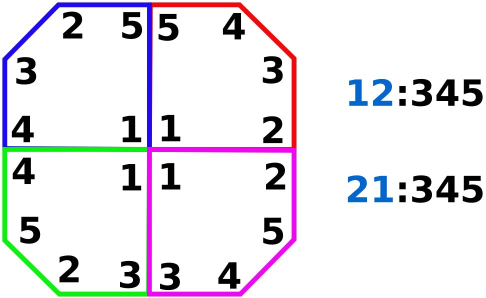


Figure: The symmetric group S_5 is generated by a transposition and a 5-cycle. There are $5! = 120$ permutations, and $24 = 120/5$ up to cyclic reordering.

Example - Face transcription: 1:2 345 $\to$ 2:1 345

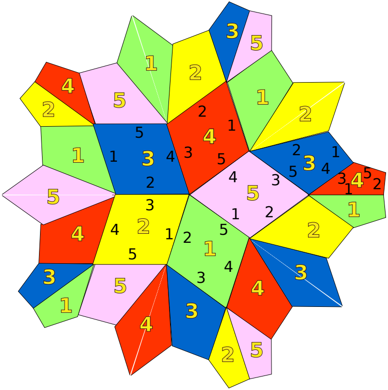
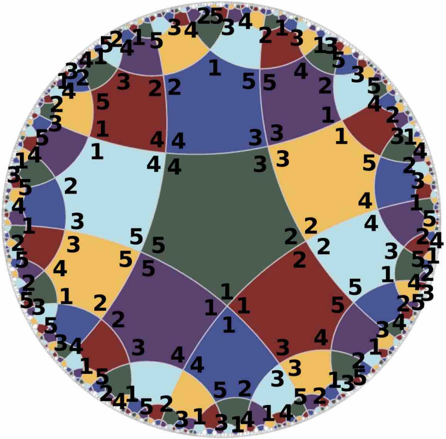


Figure: The symmetric group S_5 is generated by a transposition and a 5-cycle. There are $5! = 120$ permutations, and $24 = 120/5$ up to cyclic reordering. There are 24 degree-5 vertices for the surface: dually, 24 pentagons meeting 4 at each vertex.

Bring's surface of genus 4: labels in the universal cover



Bring's Surface: vertex transcription

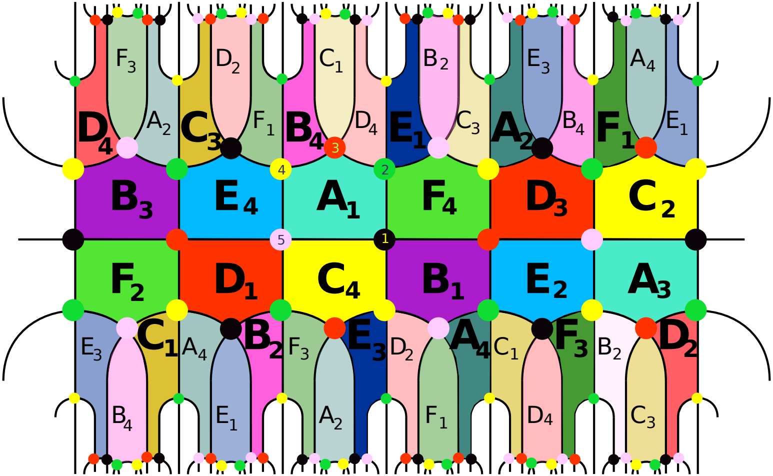


Figure: Labelled Bring's surface: 24 pentagons labelled by S_4. Six colours, four of each. 30 vertices,10 geodesics length six, each give three pairs eg $(AC)(BF)(DE)$

Bring's Surface: vertex transcription

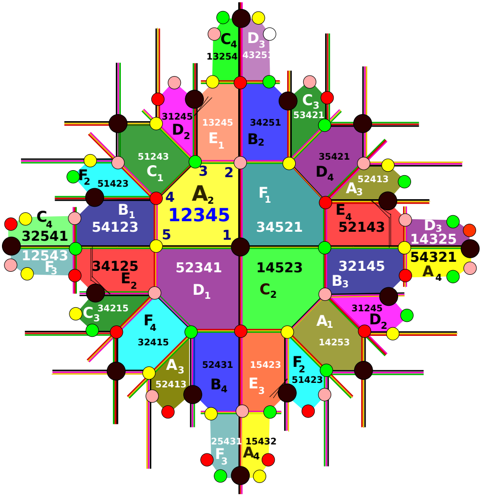


Figure: Another view, vertex-centered.

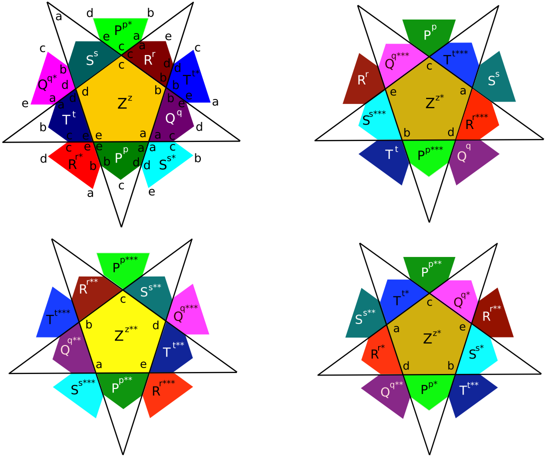


Figure: Six face types, P, Q, R, S, T, Z: each four times, different cyclic adjacency

Bring's Surface: vertex transcription

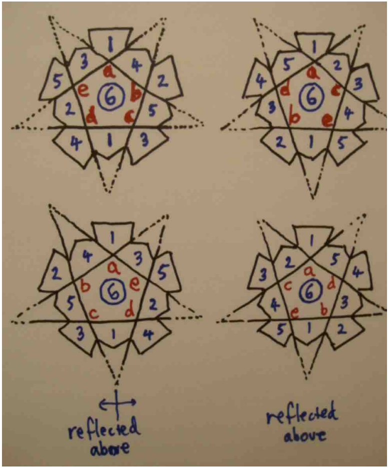


Figure: Pentagons of same colour as orbits of a squared 5-cycle: $(12345) \to (13524) \to (15432) \to (14253) \to (12345)$. These are naturally in two pairs. Note the triple of pairs $(16)(34)(25)$ for face $6(F)$ at a

The outer automorphism of S_6: face pairs from 5 vertices and 10 geodesics

Table from Coxeter's $PG(5,3)$ collineation group of order 95,040

	F	A	B	C	D	E
F	$*$	$52,34,16$	$13,45,26$	$24,51,36$	$35,12,46$	$41,23,56$
A	$16,34,52$	$*$	$23,64,15$	$35,62,14$	$42,65,13$	$54,63,12$
B	$26,45,13$	$15,64,23$	$*$	$34,65,21$	$41,63,25$	$53,61,24$
C	$36,51,24$	$14,62,35$	$21,65,34$	$*$	$45,61,32$	$52,64,31$
D	$46,12,35$	$13,65,42$	$25,63,41$	$32,61,45$	$*$	$51,62,43$
E	$56,23,41$	$12,63,54$	$24,61,53$	$31,64,52$	$43,62,51$	$*$

Surface face labels $\Leftrightarrow \{1,2,3,4,5,6\}$. 'The symmetric group permutes the numbers and letters in corresponding but different ways'

Weber-Zimmerman-Bring

Proposition 2 (Zimmerman).
The hyperbolic surface F of genus 4 associated to the small stellated dodecahedron is the totally geodesic boundary of the compact hyperbolic 3-manifold M obtained from the truncated regular hyperbolic icosahedron with dihedral angles $2\pi/5$ by identifying opposite faces after a twist of $2\pi/6$. The isometric A_5-action on F extends to the isometric A_5-action on M, where A_5 denotes the dodecahedral group of order 60 (the orientation preserving symmetry group of both the small stellated dodecahedron and the icosahedron).

Bring's surface, configurations, Horrocks-Mumford bundle

(Melliez:) There exists an isomorphism between
1. an open set of the moduli space of $(1,5)$polarized abelian surfaces up to duality;
2. the moduli space of 6 distinct points in $P_1(\mathbb{C})$ up to $PSL(2,\mathbb{C})$.
In the first section (Brings curve), we consider the sextic K itself making essentially use of $SL(2,F_5)$modules (especially McKays correspondence). The graph of the $(3,3)$ correspondence between the curve K and its tritangent planes is a genus 4 curve whose automorphisms group contains A_5. This last property characterizes Brings curve – here it is the graph of the incidence correspondence between a twisted cubic curve and a dual twisted cubic curve with apolar nets of quadrics.
We see the generic genus 4 curve as the graph of the incidence correspondence between a twisted cubic curve and a dual twisted cubic curve, which allows us to construct its canonical model in $\mathbb{C}P^3$ as the complete intersection of two surfaces (degrees 2, 3) determined by the datum of the curve.
The cubic surface so constructed comes with a marked double-six.

Today and Tomorrow

- Today: $p \leq 5$. TRIANGLES, SQUARES, PENTAGONS: face (squares), vertex (pentagons), edge (squares) $(AB)C$
- Tomorrow: $p = 7, 11$. The HEXAGON. Klein's quartic, Mathieu groups, Golay codes, Steiner systems, octonions, triality