Certain right-angled Artin groups in mapping class groups

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Hiroshima University, March 9, 2018

Contents

- Embeddings of RAAGs into MCGs (Main Theorem)
- Embeddings between finite index subgroups of MCGs (applications)

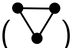
Right-angled Artin groups

Γ: a finite (simplicial) graph
$V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

Definition

The right-angled Artin group (RAAG) $G(\Gamma)$ on Γ is the group given by the following presentation:

$$G(\Gamma) = \langle v_1, v_2, \dots, v_n \mid [v_i, v_j] = 1 \textit{ if } \{v_i, v_j\} \notin E(\Gamma) \rangle.$$

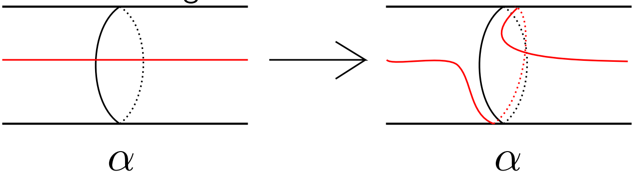
$G(\Gamma_1) \cong G(\Gamma_2)$ if and only if $\Gamma_1 \cong \Gamma_2$.
e.g.
$G($  $) \cong \mathbb{Z}^3$
$G($  $) \cong \mathbb{Z} \times F_2$
$G($  $) \cong \mathbb{Z} * \mathbb{Z}^2$
$G($  $) \cong F_3$

The mapping class groups of surfaces

$\Sigma := \Sigma_{g,p}$: the orientable surface of genus g with p punctures
The mapping class group of Σ is defined as follows.

$$\mathrm{Mod}(\Sigma) := \mathrm{Homeo}_+(\Sigma)/\text{isotopy}$$

Ori. pres. homeomorphisms can interchange punctures.
α: an essential closed curves on Σ
The Dehn twist along α:

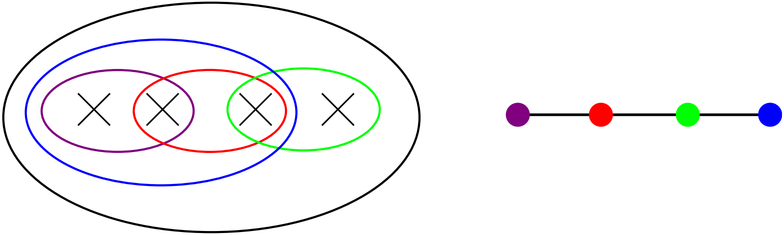


The co-curve graphs of surfaces

$\Sigma := \Sigma_{g,p}$: the orientable surface of genus g with p punctures
The co-curve graph $\bar{\mathcal{C}}(\Sigma)$ is a graph such that

- $V(\bar{\mathcal{C}}(\Sigma)) = \{$isotopy classes of escc on $\Sigma\}$
- escc α, β span an edge iff $i(\alpha, \beta) > 0$.

e.g.



Note: the co-curve graph is the complement graph of $\mathcal{C}(\Sigma)$ which is the 1-skeleton of the curve complex.

Fact (Subgroup generated by two Dehn twists)

Let α and β be non-isotopic escc on $\Sigma_{g,p}$.

(1) If $i(\alpha, \beta) = 0$, then the Dehn twists T_α and T_β generate $\mathbb{Z}^2 \cong G(\bullet \quad \bullet)$ in $\mathrm{Mod}(\Sigma_{g,p})$.

(2) If $i(\alpha, \beta) = 1$, then T_α and T_β generate $\mathrm{SL}(2, \mathbb{Z})$ or B_3 (the braid group on 3 strands).

(3) If $i(\alpha, \beta) \geq 2$, then T_α and T_β generate $F_2 \cong G(\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet)$ (Ishida, 1996).

Mostly the subgroup generated by two Dehn twists is a right-angled Artin group.

Theorem (Koberda, 2012)

Γ: a finite graph, $\chi(\Sigma_{g,p}) < 0$.
If $\Gamma \leq \bar{\mathcal{C}}(\Sigma_{g,p})$, then sufficiently high powers of “the Dehn twists corresponding to $V(\Gamma)$” generate $G(\Gamma)$ in $\mathrm{Mod}(\Sigma_{g,p})$.

Theorem (Koberda, 2012)

Λ: a finite graph, $\chi(\Sigma_{g,p}) < 0$.
If $\Lambda \leq \bar{\mathcal{C}}(\Sigma_{g,p})$, then $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$.

Here, an injective map $\iota \colon V(\Lambda) \to V(\Gamma)$ is called a full embedding if $\{u, v\} \in E(\Lambda) \Leftrightarrow \{\iota(u), \iota(v)\} \in E(\Gamma)$ for all $u, v \in V(\Lambda)$.
A fully embedded image $\iota(\Lambda)$ is called a full subgraph.
We denote by $\Lambda \leq \Gamma$ if Λ is a full subgraph of Γ.

e.g.

 is a subgraph but not full...

Motivation

Problem (Kim–Koberda, 2014)

Decide whether $G(\Gamma)$ is embedded into $\mathrm{Mod}(\Sigma_{g,p})$.

Theorem (Birman–Lubotzky–McCarthy, 1983)

$\mathbb{Z}^n \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if $n \leq 3g - 3 + p$.

Theorem (McCarthy, 1985)

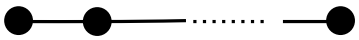
$F_2 \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if $(g, p) \neq (0, \leq 3)$.

Theorem (Koberda, Bering IV–Conant–Gaster, K, 2017)

$F_2 \times F_2 \times \cdots \times F_2 \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if the number of the direct factors F_2 is at most $g - 1 + \lfloor \frac{g+p}{2} \rfloor$.
Here, $F_2 \times F_2 \times \cdots \times F_2 \cong G(\bullet\!\!-\!\!\!-\!\!\bullet \sqcup \bullet\!\!-\!\!\!-\!\!\bullet \sqcup \cdots \sqcup \bullet\!\!-\!\!\!-\!\!\bullet)$.

Main Theorem

P_m: the path graph on m vertices



P_n

Main Theorem (K.–Kuno)

$G(P_m) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0),(0,1),(0,2),(0,3)) \\ 2 & ((g,p) = (0,4),(1,0),(1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Application

Let g be a positive integer ≥ 2.

Theorem (Birman–Hilden 1973 and Farb–Margalit 2011)

If $p \leq 2g+2$, then $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of $\mathrm{Mod}(\Sigma_{0,p})$.

Main Theorem implies the following.

Corollary A (K.–Kuno)

Suppose that $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of $\mathrm{Mod}(\Sigma_{0,p})$.
Then, $p \leq 2g+2$.

Note: residual finiteness of the mapping class groups guarantees that a large supply of finite index subgroups of the mapping class groups.
$\cap H$ ($H \leq \mathrm{Mod}(\Sigma_{g,p})$: finite index) $= 1$.

Theorem (Birman–Hilden 1973 and Perron–Vannier 1999)

If $n \leq 2g$, then $\mathrm{Mod}(\Sigma_g)$ contains the braid group B_n on n strands.

Theorem (K.)

Suppose that $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of B_n.
Then $n \leq 2g$.

This generalizes the following theorem.

Theorem (Castel, 2016)

Suppose that $\mathrm{Mod}(\Sigma_g)$ contains B_n.
Then $n \leq 2g$.

Summary

The following hold.

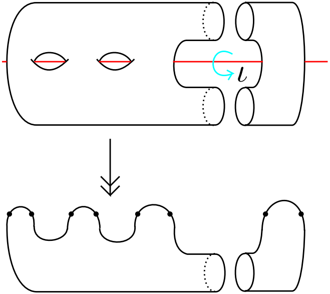
(1) $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of $\mathrm{Mod}(\Sigma_{0,p})$ if and only if $p \leq 2g + 2$.

(2) $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of B_n if and only if $n \leq 2g$.

Quick Review: Birman–Hilden double branched cover (1/3)

Theorem

$B_{2g} \hookrightarrow \mathrm{Mod}(\Sigma_g)$.

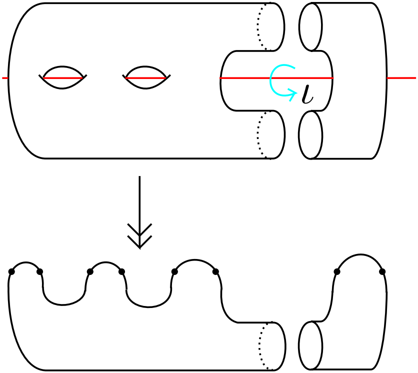


$B_{2g} := \mathrm{Mod}(\Sigma^1_{2g}) \cong \mathrm{SMod}(\Sigma^2_{g-1}) \hookrightarrow \mathrm{Mod}(\Sigma_g)$. SMod: fib. pres.
$PB_{2g} \cong \mathrm{PMod}(\Sigma_{0,2g+1}) \times \mathbb{Z}$ (Clay–Leininger–Margalit).

Corollary

$\mathrm{PMod}(\Sigma_{0,p}) \hookrightarrow \mathrm{Mod}(\Sigma_g)$ for $\forall p \leq 2g+1$.

Quick Review: Birman–Hilden double branched cover (2/3)



Theorem

$\mathrm{SMod}(\Sigma_g)/\langle\iota\rangle \cong \mathrm{Mod}(\Sigma_{0,2g+2})$.

Pick a finite index subgroup H of $\mathrm{SMod}(\Sigma_g)$ avoiding ι.
Then H is embedded in $\mathrm{Mod}(\Sigma_{0,2g+2})$ as a finite index subgroup.
Natural inclusion $H \subset \mathrm{Mod}(\Sigma_g)$ is a desired embedding.

Quick Review: Birman–Hilden double branched cover (3/3)

Hence, the conditions arise from topological context.

Summary

The following hold.

(1) $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of $\mathrm{Mod}(\Sigma_{0,p})$ if $p \leq 2g + 2$.

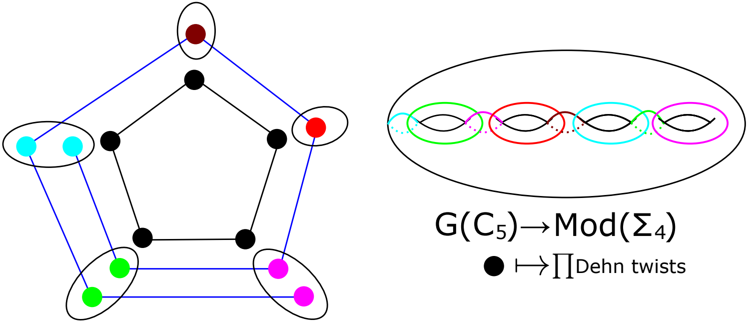
(2) $\mathrm{Mod}(\Sigma_g)$ contains the braid group B_n on n strands if $n \leq 2g$.

Proof of Main Theorem and Corollary A

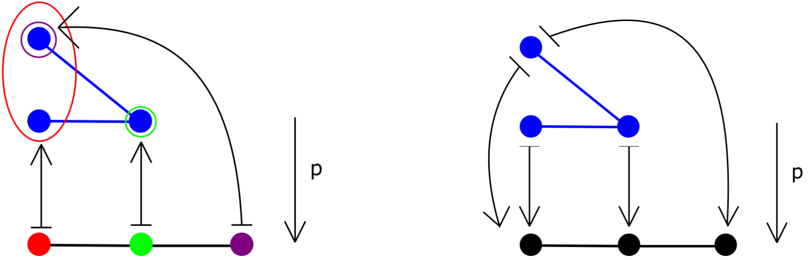
Theorem (Kim–Koberda)

Suppose that $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ with $\chi(\Sigma_{g,p}) < 0$.
Then there is an embedding $\psi\colon G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ such that $\forall v \in V(\Lambda)$, $\exists$ Dehn twists $T_{v,1}, \ldots, T_{v,m_v}$; $\psi(v) = T_{v,1}^{e_{v,1}} \cdots T_{v,m_v}^{e_{v,m_v}}$, where $T_{v,i}$ and $T_{v,j}$ are commutative.

Note: $\{T_{v,i} | v \in V(\Lambda)\}$ induces a full subgraph $\Gamma \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.



Multi-valued projection of graphs



Definition

Let Λ and Γ be graphs.
A multi-valued projection $p\colon \Gamma \rightrightarrows \Lambda$ is a correspondence from $V(\Gamma)$ to $V(\Lambda)$ satisfying the following.

(0) The vertex-images $p(v)$ are non-empty sets of vertices.

(1) If $v_1, v_2 \in V(\Gamma)$ are adjacent, then any pair of vertices u_1 and u_2, where $u_1 \in p(v_1)$ and $u_2 \in p(v_2)$, are adjacent.

(2) The correspondence p is surjective.

KK embedding induces an MV projection

Theorem (Kim–Koberda, recall)

Suppose that $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$.
Then there is an embedding $\psi\colon G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ such that $\exists \Gamma \leq \bar{\mathcal{C}}(\Sigma_{g,p})$; $\psi(v)$ is a product of non-adjacent vertices of Γ.

Observation

The above embedding ψ induces an MV projection $p\colon \Gamma \rightrightarrows \Lambda$ by setting $T_{v,i} \overset{p}{\mapsto} v$.

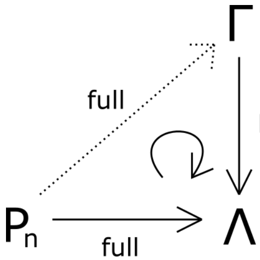
Proof.

Pick $u_1 \in p(T_{v_1,i})$ and $u_2 \in p(T_{v_2,j})$ with $\{T_{v_1,i}, T_{v_2,j}\} \in E(\Gamma)$.
Since the vertices $T_{v_1,i}$ and $T_{v_2,j}$ are non-commutative and since ψ is injective, the vertices u_1, u_2 must be non-commutative (adjacent).
Hence, p satisfies the axiom (1).
Moreover, p is surjective (2), because $\psi(v)$ is non-trivial. □

Lemma (K.)

Let $p\colon \Gamma \rightrightarrows \Lambda$ be an MV projection associated to a KK embedding $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$.
For any full embedding $\iota\colon P_n \to \Lambda$, there is a full embedding $\tilde{\iota}\colon P_n \to \Gamma$ such that $p \circ \tilde{\iota} = \iota$.
In particular, $G(P_n) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ implies $P_n \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

Recall: $\Gamma \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.



Path-lifting Lemma (2/5)

Lemma (K.)

Let $p\colon \Gamma \rightrightarrows \Lambda$ be an MV projection associated to a KK embedding $\psi\colon G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$.
For any full embedding $\iota\colon P_n \to \Lambda$, there is a full embedding $\tilde{\iota}\colon P_n \to \Gamma$ such that $p \circ \tilde{\iota} = \iota$.
In particular, $G(P_n) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ implies $P_n \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

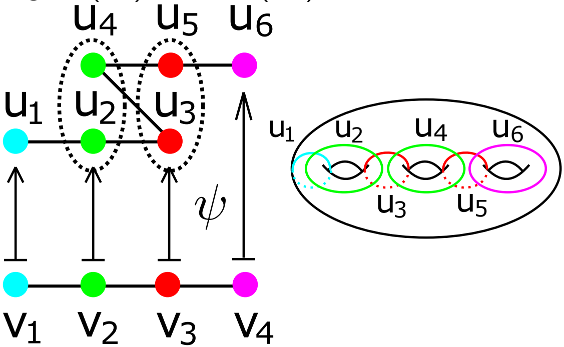
- $n = 1$ case: obvious.
- $n = 2$ case: a full embedding $P_2 \to \Lambda$

"$=$" a pair of non-commutative vertices
It must have a lift (if not, $\mathrm{Ker}\psi$ contains the commutator of the vertices).

- $n = 3$ case: essentially due to Kim–Koberda

Path-lifting Lemma (3/5)

$n = 4$ case: e.g. $G(P_4) \to \mathrm{Mod}(\Sigma_3)$



$\psi(v_1) = u_1,\ \psi(v_2) = u_2 u_4,\ \psi(v_3) = u_3 u_5,\ \psi(v_4) = u_6\ (u_i := T_{u_i})$.

Claim. $\mathrm{Ker}\psi \neq 1$.

$[v_1^{v_2 v_3}, v_4] = (v_3^{-1} v_2^{-1} v_1 v_2 v_3) v_4 (v_3^{-1} v_2^{-1} v_1^{-1} v_2 v_3) v_4^{-1} \neq 1$.

This is a shortest word representing $[v_1^{v_2 v_3}, v_4]$.

We now prove that $[\psi(v_1)^{\psi(v_2)\psi(v_3)}, \psi(v_4)] = 1$.

We first obtain a good representative of $\psi(v_1)^{\psi(v_2)}$.

Path-lifting Lemma (4/5)

$\psi(v_1) = u_1$, $\psi(v_2) = u_2 u_4$, $\psi(v_3) = u_3 u_5$, $\psi(v_4) = u_6$.

Representative of $\psi(v_1)^{\psi(v_2)}$:

$$\begin{aligned}\psi(v_1)^{\psi(v_2)} &= (u_4^{-1} u_2^{-1}) u_1 (u_2 u_4) \\ &= u_2^{-1} u_1 u_2\end{aligned}$$

$$\begin{aligned}(\psi(v_1)^{\psi(v_2)})^{\psi(v_3)} &= (u_5^{-1} u_3^{-1}) u_2^{-1} u_1 u_2 (u_3 u_5) \\ &= u_3^{-1} u_2^{-1} u_1 u_2 u_3\end{aligned}$$

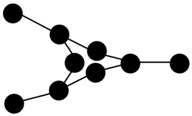
Thus, $(\psi(v_1)^{\psi(v_2)})^{\psi(v_3)}$ is commutative with $\psi(v_4) = u_6$.
i.e. $[(\psi(v_1)^{\psi(v_2)})^{\psi(v_3)}, \psi(v_4)] = 1$.
The projection is not induced by an embedding!

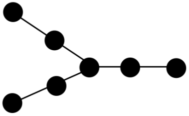
Path-lifting Lemma (5/5)

General case: given a full embedding $\iota\colon P_n \hookrightarrow \Lambda$, consider the commutator $[\psi(\iota(v_1))^{\psi(\iota(v_2))\cdots\psi(\iota(v_{n-1}))}, \psi(\iota(v_n))]$.
Then we can prove that the following;
if there is no lift of ι, then $\psi(\iota(v_1))^{\psi(\iota(v_2))\cdots\psi(\iota(v_{n-1}))}$ has a representative consisting of vertices in Γ commutative with $\psi(\iota(v_n))$.
This implies that ι has a lift for any projection associated to an embedding ψ.

Theorem (Lee–Lee, 2017)

There is a pair of deg 3 tree T and a graph $\Gamma \leq \bar{\mathcal{C}}(\Sigma_{g,p})$ such that $G(T) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ and T has no lift w.r.t. the projection.





Theorem (Kim–Koberda, 2015)

If $3g - 3 + p \geq 4$, then there is a finite graph Λ such that $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ but $\Lambda \not\leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

Proof of Main Theorem (1/6)

Main Theorem (recall)

$G(P_m) \leq \mathrm{Mod}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0),(0,1),(0,2),(0,3)) \\ 2 & ((g,p) = (0,4),(1,0),(1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof of Main Theorem (2/6)

Lemma

Suppose that $\chi(\Sigma_{g,p}) < 0$.
If $G(P_m) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$, then $P_m \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

Problem

Decide whether $G(P_m)$ is embedded into $\mathrm{Mod}(\Sigma_{g,p})$.

By Koberda's embedding theorem and the above lemma, the above problem is reduced into the following problem when $\chi < 0$:

Problem

Decide whether $P_m \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

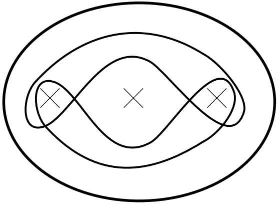
Problem (recall)

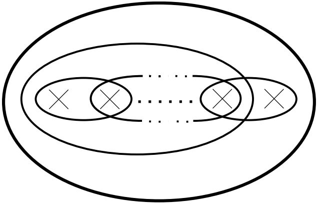
Decide whether $P_m \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

A sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of closed curves on $\Sigma_{g,p}$ is called a linear chain if this sequence satisfies the following.

- Any two distinct curves α_i and α_j are non-isotopic.
- Any two consecutive curves α_i and α_{i+1} intersect non-trivially and minimally.
- Any two non-consecutive curves are disjoint.

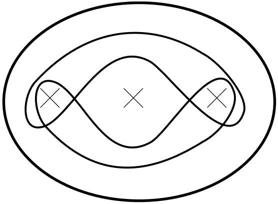
If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a linear chain, we call m its length.



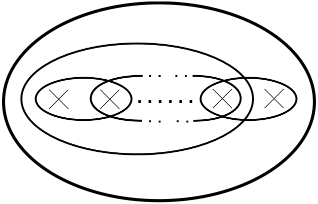


Proof of main Theorem (4/6)

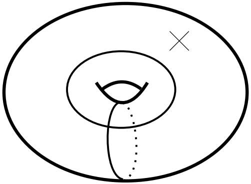
Note that if $|\chi(\Sigma_{g,p})| < 0$ and $\Sigma_{g,p}$ is not homeomorphic to neither $\Sigma_{0,4}$ nor $\Sigma_{1,1}$, then there is a linear chain of length m on $\Sigma_{g,p}$ if and only if $P_m \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.



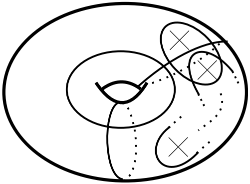
length 2



length $p - 1$

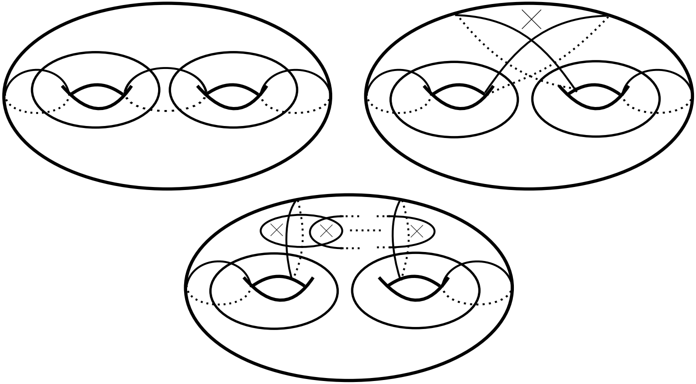


length 2



length $p + 2$

Proof of main Theorem (5/6)



length $2g + p + 1$

$\to P_{2g+p+1} \leq \bar{\mathcal{C}}(S_{g,p})$

Proof of Main Theorem (6/6)

Main Theorem*

$P_m \leq \bar{\mathcal{C}}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0),(0,1),(0,2),(0,3)) \\ 2 & ((g,p) = (0,4),(1,0),(1,1)) \\ p-1 & (g=0,\ p \geq 5) \\ p+2 & (g=1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof) Double induction on the ordered pair (g,p).
$(g,p) = (0,5)$ case:
Suppose that $\alpha_1, \ldots, \alpha_m$ is a linear chain on $\Sigma_{0,5}$.
Then the last curve α_m is separating and $\Sigma_{0,5} \cong \Sigma_{0,3} \cup_{\alpha_m} \Sigma_{0,4}$.
Either $\Sigma_{0,3}$ or $\Sigma_{0,4}$ contains a linear chain of length $m-2$.
Hence, we have $m-2 \leq 2$ i.e. $m \leq 4$.

Thus we have Main Thm.

Main Theorem (recall)

$G(P_m) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if

$$m \leq \begin{cases} 0 & ((g,p) = (0,0), (0,1), (0,2), (0,3)) \\ 2 & ((g,p) = (0,4), (1,0), (1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof of Corollary

Lemma

Let H be a group and K a finite index subgroup of H.
If a RAAG G is embedded in H, then G is also embedded in K.

Proof.

Suppose that G is embedded in H.
For all $n > 0$, the RAAG G has property that the "n-th power homomorphism" $v \mapsto v^n$ is injective.
Since K is of finite index, n-th power homomorphism is an embedding of G into K. □

Corollary

Let g be an integer ≥ 2.
Suppose that $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup H of $\mathrm{Mod}(\Sigma_{0,p})$.
Then, $p \leq 2g + 2$.

Proof.

Main Theorem implies $G(P_{p-1}) \hookrightarrow \mathrm{Mod}(\Sigma_{0,p})$.
By previous lemma, we have $G(P_{p-1}) \hookrightarrow H$.
By Main Theorem, the maximum m such that $G(P_m) \hookrightarrow \mathrm{Mod}(\Sigma_g)$ is $2g + 1$.
Thus we have $p - 1 \leq 2g + 1$. □

If we use the rank of free abelian subgroup, then we have a non-sharp inequality, $p \leq 3g$.

We also obtain the following result as a corollary of Main Theorem.

Corollary

Let g and g' be integers ≥ 2. Suppose that $\mathrm{Mod}(\Sigma_{g,p})$ is virtually embedded into $\mathrm{Mod}(\Sigma_{g',p'})$. Then the following inequalities hold:

(1) $3g + p \leq 3g' + p'$,

(2) $2g + p \leq 2g' + p'$.

It is easy to observe that, if $(3g + p, 2g + p) = (3g' + p', 2g' + p')$, then $(g, p) = (g', p')$. Namely, we have;

Corollary

Let g and g' be integers ≥ 2.
If $\mathrm{Mod}(\Sigma_{g,p}) \underset{virtual}{\hookrightarrow} \mathrm{Mod}(\Sigma_{g',p'})$ and $\mathrm{Mod}(\Sigma_{g',p'}) \underset{virtual}{\hookrightarrow} \mathrm{Mod}(\Sigma_{g,p})$, then $(g, p) = (g', p')$.

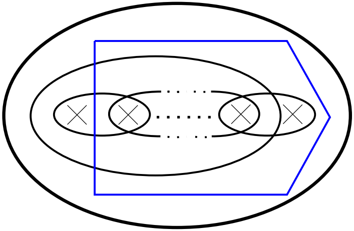
Braid groups into closed surface MCGs

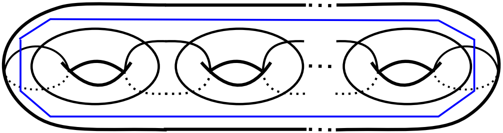
Theorem (K.)

Suppose that $\mathrm{Mod}(\Sigma_g)$ contains a finite index subgroup of the braid group B_n on n strands. Then $n \leq 2g$.

Idea) If we try to use free abelian subgroups and $G(P_n)$ in order to deduce the conclusion;
Free abelian: $n \leq 3g - 2$
$G(P_n)$: $n \leq 2g + 1$
Hence, we use the right-angled Artin groups of the form $G(C_n) \times \mathbb{Z}$.

<u>Claim.</u> $G(C_p) \leq \mathrm{PMod}(\Sigma_{0,p})$.





Hence, $G(C_{n+1}) \times \mathbb{Z} \hookrightarrow B_n$.
On the other hand, $C_{2g+2} \leq \bar{\mathcal{C}}(\Sigma_g)$.

Braid groups into closed surface MCGs

Claim. $C_{2g+2} \sqcup \{\mathrm{pt}\} \not\leq \bar{\mathcal{C}}(\Sigma_g)$.

Claim. $G(C_{2g+2}) \times \mathbb{Z} \not\hookrightarrow \mathrm{Mod}(\Sigma_g)$.

Thus, $B_n \underset{virtual}{\hookrightarrow} \mathrm{Mod}(\Sigma_g)$ implies $n+1 \leq 2g+1$.

Future work

Today we discussed embeddablitiy between finite index subgroups of specific MCGs.

Corollary

$\mathrm{Mod}(\Sigma_{0,p}) \underset{virtual}{\hookrightarrow} \mathrm{Mod}(\Sigma_g)$ if and only if $p \leq 2g+2$.

Theorem (Ivanov–McCarthy, 1999)

Suppose that $g \geq 2$ and $(g', p') \neq (2, 0)$.
If $|(3g'+p') - (3g+p)| \leq 1$, then every embedding $\mathrm{Mod}(\Sigma_{g,p})$ into $\mathrm{Mod}(\Sigma_{g',p'})$ is an isomorphism induced by a homeomorphism.

Theorem (Bell–Margalit, 2004)

Let p be an integer ≥ 5.
Then $\mathrm{Mod}(\Sigma_{0,p})$ is not embedded in $\mathrm{Mod}(\Sigma_{0,p+1})$.

Theorem (Aramayona–Souto, 2012)

Suppose that $g \geq 6$ and $g' \leq 2g - 1$;
if $g' = 2g - 1$, we further assume that $p' = 0$.
Then every embedding $\mathrm{PMod}(\Sigma_{g,p}) \to \mathrm{PMod}(\Sigma_{g',p'})$ is an isomorphism.

Question

What about the other cases?

Problem (Kim–Koberda, 2014)

Decide whether $G(\Lambda)$ is embedded into $\mathrm{Mod}(\Sigma_{g,p})$.

Theorem (Aougab–Biringer–Gaster, 2017)

There is an algorithm that determines, given a graph Λ and a pair (g, p), whether $\Lambda \leq \bar{\mathcal{C}}(\Sigma_{g,p})$.

Method: give a bound for self-intersection number of the curve systems representing Λ, and check through the bounded complexity triangulations of $\Sigma_{g,p}$ for curve systems embedded in their 1-skeleta.

Question

Algorithm that determines given a graph Λ has the following property; $G(\Lambda) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ iff $\Lambda \leq \bar{\mathcal{C}}(\Sigma_{g,p})$?

Thank you for your attention.