Branched Coverings, Degenerations, and related topics

2018/03/06+07. Hiroshima University

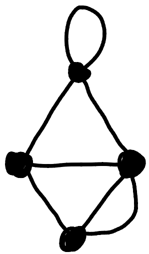
Tutte Polynomials in Geometry and Combinatorics

Masahiko Yoshinaga (吉永正彦), Hokkaido U.

What is a Tutte polynomial?

It is a polynomial invariant for a finite graph.

Example

G:  , then the Tutte polynomial is

$$T_G(x,y) = x^3y + x^2y^2 + xy^3 + y^4 + 2x^2y + 3xy^2$$
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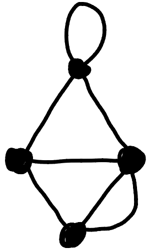
G:  , then the Tutte polynomial is

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$T_G(x,y)$ carries lots of information. e.g. $b_1(G) = \deg$ in y.

Plan of this talk

1. Definition and basic properties of $T_G(x,y)$.
2. Specializations of $T_G(x,y)$.
3. Generalizations and recent works.
   - Arithmetic Tutte by L. Moci,
   - log-concavity of chromatic poly.

     after June Huh.
4. G-Tutte polynomial.

   (j.w/ Ye Liu & Tan Nhat Tran)

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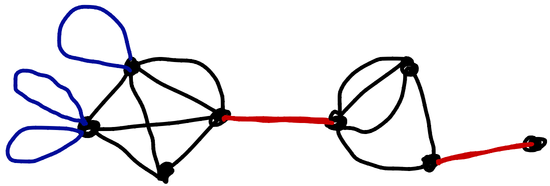
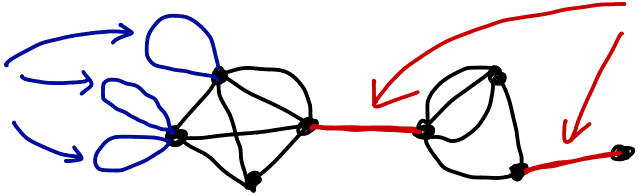
Terminology & operations on graphs.



1. Definition of the Tutte polynomial.

Terminology & operations on graphs.

loops



bridges (coloop/isthmus) 地峡

($\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ deletion increases $b_0 :=$ # of conn. comp.)

$G \setminus e$: deletion G/e : contraction

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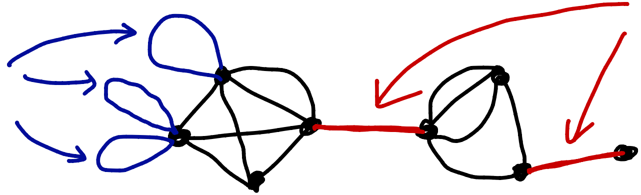

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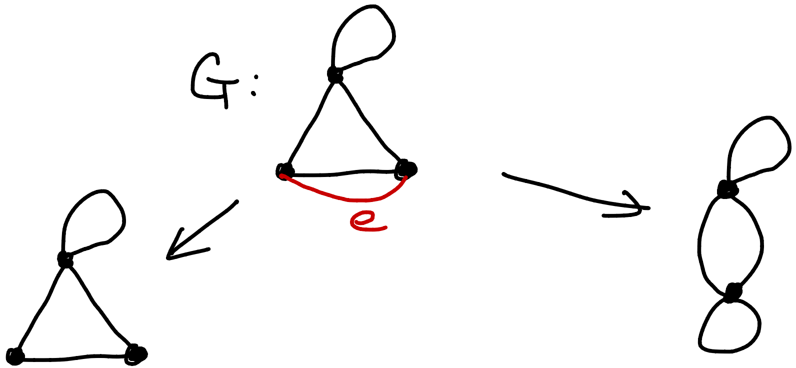

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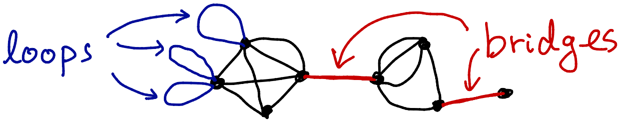

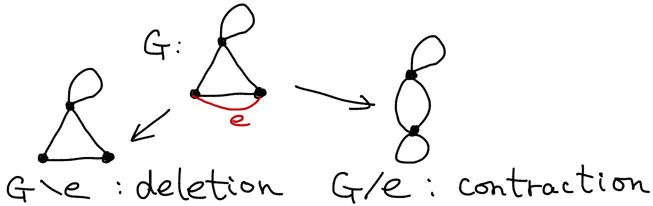

<u>Def.</u> Let $G=(V,E)$ be a finite graph.

Define $T_G(x,y) \in \mathbb{Z}[x,y]$ by

- If $E=\phi$, $T_G(x,y)=1$
- If $e \in E$ is a loop, $T_G(x,y) = y \cdot T_{G\setminus e}(x,y)$
- If $e \in E$ is a bridge, $T_G(x,y) = x \cdot T_{G/e}(x,y)$
- If $e \in E$ is neither loop nor bridge,

$$T_G = T_{G/e} + T_{G\setminus e}.$$

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  $\Rightarrow T_G = T_{G/e} + T_{G\setminus e}$.

Thm. $T_G(x,y)\in\mathbb{Z}[x,y]$ is well-defined.

(explained later). ("Tutte poly.")

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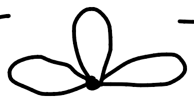
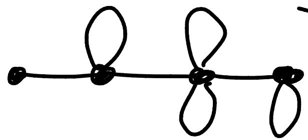
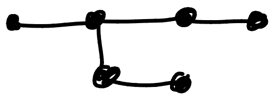
Examples (Notation $T_G =: [G]$)

- $\left[\right.$  $\left.\right] = y^3$
- $\left[\right.$  $\left.\right] = x^3y^4$
- $\left[\right.$  $\left.\right] = x^5$

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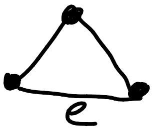
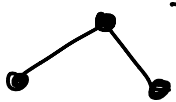
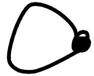
[] = [] + []

= [] + [] + x^2

$= y+x+x^2.$

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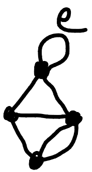
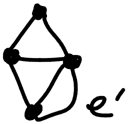
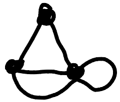
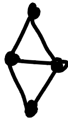
[] $=$ y [] $=$ y [] $+$ y [] $= \cdots$

$$= x^3y + x^2y^2 + xy^3 + y^4 + 2x^2y + 3xy^2$$
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1. Definition of the Tutte polynomial.

Towards another expression of $T_G(x, y)$,

<u>Def.</u> $G = (V, E)$. For $S \subseteq E$, define

$$r_S := |V| - b_0(S)$$

\# of conn. comp. of $G' := (V, S)$.

Other interpretations are

submodule of $\mathbb{Z}^{\oplus V} = \bigoplus_{v \in V} \mathbb{Z} \cdot v$

$$r_S = \operatorname{rank}\left(\underbrace{\langle v_1 - v_2 \mid (v_1, v_2) \in S \rangle}_{\mathbb{Z}} \right)$$

$$= \text{\# of edges of spanning forest of } G' = (V, S)$$

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<u>Thm.</u>
$$T_G(x,y) = \sum_{S \subseteq E} (x-1)^{r_E - r_S} \cdot (y-1)^{\#S - r_S}.$$

(proof):

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$=$ # of edges of spanning forest of $G'=(V,S)$

Thm. $$T_G(x,y)=\sum_{S\subseteq E}(x-1)^{r_E-r_S}\cdot(y-1)^{\#S-r_S}.$$

(Proof): Check the recursive relations

- $E=\phi \Rightarrow T_G(x,y)=1$
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<u>Prop.</u> Write $T_G(x,y) = \sum_{S \subseteq E} (x-1)^{r_E - r_S} \cdot (y-1)^{\#S - r_S}$

as $\sum t_{ij} x^i y^j$. Then $t_{ij} \geq 0$.

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Prop. Write $T_G(x,y) = \sum_{S \subseteq E} (x-1)^{r_E - r_S} \cdot (y-1)^{\#S - r_S}$ as $\sum t_{ij} x^i y^j$. Then $t_{ij} \geq 0$.

(Proof) Use induction with

$$\begin{cases} \bullet\ E = \phi \Rightarrow T_G(x,y) = 1 \\ \bullet\ e \in E : \text{loop} \Rightarrow T_G(x,y) = y \cdot T_{G \setminus e}(x,y) \\ \bullet\ e \in E : \text{bridge} \Rightarrow T_G(x,y) = x \cdot T_{G/e}(x,y) \\ \bullet\ e \in E : \text{not loop, not bridge.} \\ \quad \Rightarrow T_G = T_{G/e} + T_{G \setminus e}. \end{cases}$$

(Q.E.D.)

2. Specializations of $T_G(x, y)$.

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① Chromatic polynomial $C_G(t)$ of G.

② Poincaré poly. of the graph configuration sp.

③ Expectation of chromatic poly. of random subgraphs.

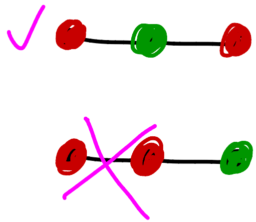
④ Bracket poly. of an alternating link.

⑤ Partition function of Ising model.

Ref. D.J.A. Welsh.
Complexity: Knots, Colourings and Counting. (1993)

2. Specializations of $T_G(x,y)$.

① Chromatic polynomial $C_G(t) \in \mathbb{Z}[t]$.



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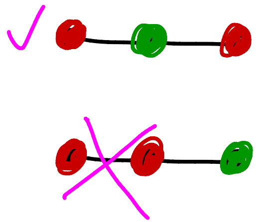

Let $G = (V, E)$ be a finite graph.

$[n] := \{1, 2, \cdots, n\}$.

Def. $f : V \to [n]$ is a n-coloring if for each $e = (v_1, v_2) \in E$, we have $f(v_1) \neq f(v_2)$.

$$\mathrm{Col}_G(n) := \{ f : V \to [n] \mid n\text{-coloring} \}.$$

Rem. If G has a loop, $\mathrm{Col}_G(n) = \phi$ for $^\forall n \geq 1$.

Thm-Def $\exists\, C_G(t) \in \mathbb{Z}[t]$ s.t. $|\mathrm{Col}_G(n)| = C_G(n)$ $(n \geq 1)$.

("Chromatic poly.")

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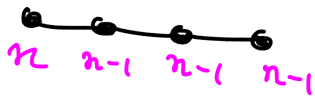
<u>Example</u> (i) If $E = \phi$, $C_G(t) = t^{|V|}$.

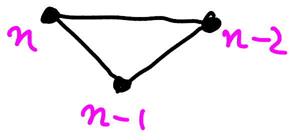
(ii) If G is a connected tree,

$C_G(t) = t \cdot (t-1)^{|V|-1}$.



(iii) If $G = K_\ell$ (complete graph with ℓ vertices),

$C_G(t) = t(t-1)\cdots(t-\ell+1)$.



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<u>Thm-Def</u> $\exists\, C_G(t) \in \mathbb{Z}[t]$ s.t. $|Col_G(n)| = C_G(n)$

<u>Thm.</u> (Deletion-Contraction formula) Let $e \in E$.

Then $$C_G(t) = C_{G\setminus e}(t) - C_{G/e}(t).$$

(Proof) $Col_{G\setminus e}(n) = Col_G(n) \sqcup Col_{G/e}(n)$

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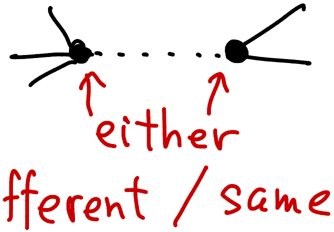

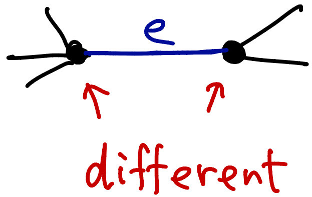

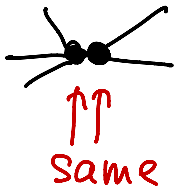

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Thm-Def $\exists C_G(t) \in \mathbb{Z}[t]$ s.t. $|\mathcal{C}ol_G(n)| = C_G(n)$

Thm. (Deletion-Contraction formula) $C_G(t) = C_{G \setminus e}(t) - C_{G/e}(t)$

Thm Let $G=(V,E)$. Then

$$C_G(t) = (-1)^{r_E} \cdot t^{|V|-r_E} \cdot T_G(1-t, 0).$$

(Proof) Induction on $|E|$. //

2. Specializations of $T_G(x, y)$.

② Graph configuration space.

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② Graph configuration space.

$\nexists$ loops, multiedges.

(For simplicity, assume G is a simple graph.)

<u>Def.</u> $\mathrm{Col}_G(\mathbb{C}) := \{ f : V \to \mathbb{C} \mid f(v_1) \neq f(v_2) \text{ if } (v_1, v_2) \in E \}$.

(" $\mathbb{C}$-valued points " of colorings).

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② Graph configuration space.

<u>Def.</u> $\mathcal{C}ol_G(\mathbb{C}) := \{ f : V \to \mathbb{C} \mid f(v_1) \neq f(v_2) \text{ if } (v_1, v_2) \in E \}$

Another description of $\mathcal{C}ol_G(\mathbb{C})$:

Let $V = \{1, 2, \cdots, \ell\}$. For an edge $e = (i,j) \in E$,

$$H_e := \{ (x_1, \cdots, x_\ell) \in \mathbb{C}^\ell \mid x_i = x_j \} \subset \mathbb{C}^\ell.$$

Then

$$\mathcal{C}ol_G(\mathbb{C}) = \mathbb{C}^\ell \setminus \bigcup_{e \in E} H_e.$$

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$V = \{1,2,\cdots,\ell\}$. $H_e := \{(x_1,\cdots,x_\ell) \in \mathbb{C}^\ell \mid x_i = x_j\}$

$$\mathcal{C}ol_G(\mathbb{C}) = \mathbb{C}^\ell \setminus \bigcup_{e\in E} H_e.$$

Translation

$(x_1,\cdots,x_\ell) \mapsto (x_1+t,\cdots,x_\ell+t)$

<u>Def.</u> (Projective graph config. sp.)

$$\mathbb{P}\mathcal{C}ol_G(\mathbb{C}) := \mathbb{P}\left(\mathcal{C}ol_G(\mathbb{C}) / \mathbb{C}\cdot(1,1,\cdots,1) \right)$$

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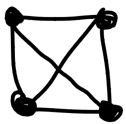
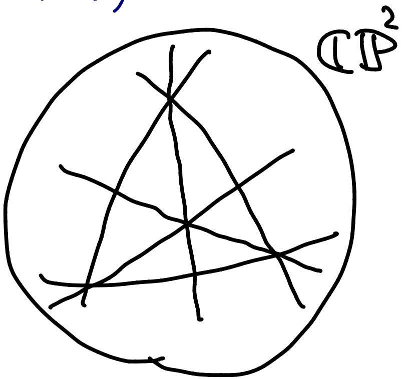
Example

$G = K_4$  Then $\mathbb{P}\mathcal{C}ol_G(\mathbb{C}) =$ 

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$V = \{1, 2, \cdots, \ell\}$. $H_e := \{ (x_1, \cdots, x_\ell) \in \mathbb{C}^\ell \mid x_i = x_j \}$

$$\mathcal{C}ol_G(\mathbb{C}) = \mathbb{C}^\ell \setminus \bigcup_{e \in E} H_e .$$

$$\mathbb{P}\mathcal{C}ol_G(\mathbb{C}) := \mathbb{P}\left(\mathcal{C}ol_G(\mathbb{C}) / \mathbb{C}\cdot(1,1,\cdots,1) \right)$$

Thm. The Poincaré poly of $\mathcal{C}ol_G(\mathbb{C})$ is

$$\mathrm{Poin}(\mathcal{C}ol_G(\mathbb{C}), t) = (-t)^{|V|} \cdot c_G\left(-\frac{1}{t}\right).$$

2. Specializations of $T_G(x, y)$.

③ Expectation of $C_{G'}(t)$ of random $G' \subseteq G$

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③ Expectation of $C_{G'}(t)$ of random $G' \subseteq G$

Let $G = (V, E)$. Fix $0 < p < 1$.

Denote by G_p the random subgraph of G constructed by choose each $e \in E$ independently with probability p ($\Longleftrightarrow$ delete $e \in E$ with prob. $1-p$.)

<u>Thm</u>. G : connected. Then the expectation of $C_{G_p}(t)$ is

$$\mathbb{E}\left[C_{G_p}(t)\right] = (-p)^{|V|-1} \cdot t \cdot T_G\left(1 - \frac{t}{p}, 1 - p\right)$$

2. Specializations of $T_G(x, y)$.

④ Bracket polynomial for alternating link.

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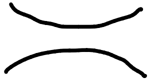
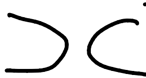
④ Bracket polynomial for alternating link.

<u>Def.-Thm.</u> (Kauffman bracket polynomial)

For any link diagram L, $\exists\, [L] \in \mathbb{Z}[A^{\pm 1}]$ s.t.

- [] $= A \cdot$ [] $+ A^{-1} \cdot$ []
- $[L \sqcup \bigcirc] = -(A^{-2} + A^{2}) \cdot [L]$
- $[\bigcirc] = 1$

unknot.

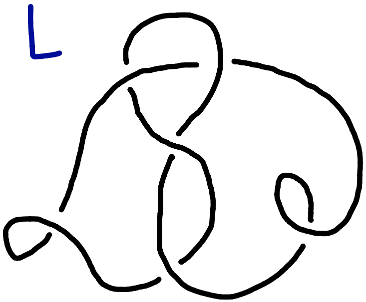
<u>Rem.</u> $[L]$ is invariant under Reidemeister II, III.

Jones poly $\fallingdotseq$ modified $[L]$.

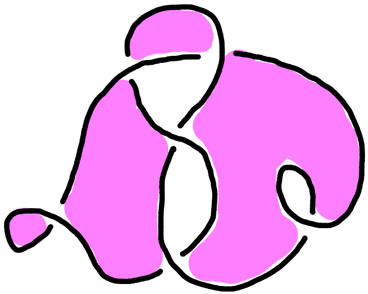
2. Specializations of $T_G(x,y)$.

④ Bracket polynomial for alternating link.

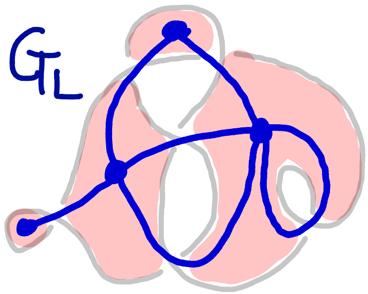
Obtaining a graph from an alternating link.



alternating link L



Checkerboard Coloring

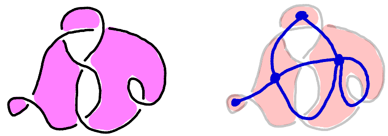


Adjacency graph G_L

2. Specializations of $T_G(x,y)$.

④ Bracket polynomial for alternating link.

- $[\times] = A \cdot [\asymp] + A^{-1} \cdot [\supset\subset]$
- $[L \cup \bigcirc] = -(A^{-2}+A^{2}) \cdot [L]$
- $[\bigcirc] = 1$



<u>Thm.</u> $[L] = A^{2|V(G)|-|E(G)|-2} \cdot T_G(-A^{-4}, -A^{4})$, $G = G_L$

(proof).

2. Specializations of $T_G(x,y)$.

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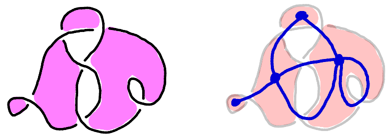

- $[\times] = A\cdot[\asymp] + A^{-1}\cdot[)(]$
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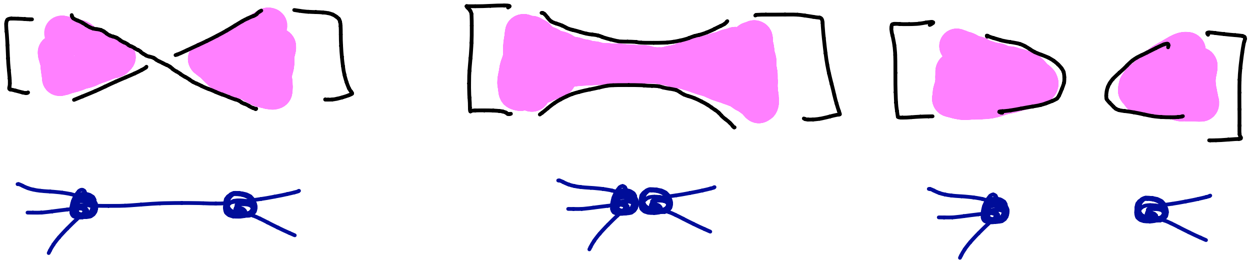
<u>Thm.</u> $[L] = A^{2|V(G)|-|E(G)|-2}\cdot T_G(-A^{-4}, -A^{4})$, $G=G_L$

(proof). Compare the recursions.



//

2. Specializations of $T_G(x, y)$.

5 Partition function of Ising model

2. Specializations of $T_G(x,y)$.

⑤ Partition function of Ising model

Let $G=(V,E)$ be a simple graph.

For each "states" σ ($\underset{\text{def}}{\Longleftrightarrow}$ a map $\sigma: V \to \{\pm 1\}$),

the "energy" of σ is measured by the

"Hamiltonian" $H(\sigma) = \sum_{(i,j)\in E} J\cdot\sigma(i)\cdot\sigma(j)$, where

where J is the "interaction energy" (i.e. $J \in \mathbb{R}_{>0}$).

The "partition function" $Z = Z(G, \beta, J)$ is defined by

$$Z = \sum_{\sigma} e^{-\beta \cdot H(\sigma)},$$ where β is $\cdots$

2. Specializations of $T_G(x,y)$.

⑤ Partition function of Ising model

Let $G=(V,E)$ be a simple graph. For each "states" $\sigma: V \to \{\pm 1\}$ the "energy" of σ is measured by the "Hamiltonian"

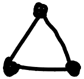
$$H(\sigma) = \sum_{(i,j)\in E} J\cdot \sigma(i)\cdot\sigma(j),$$ where $J \in \mathbb{R}_{>0}$ is the "interaction energy"

The "partition function" $Z = Z(G,\beta,J)$ is defined by $Z = \sum_{\sigma} e^{-\beta\cdot H(\sigma)}$

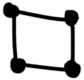
Thm. $$Z = \left(2e^{-\beta J}\right)^{|E|-r_E}\cdot\left(4\sinh\beta J\right)^{r_E}\cdot T_G\left(\coth\beta J,\ e^{2\beta J}\right).$$

3. Recent works

(1) log-concavity of $C_G(t)$.

Examples • $G =$  $\Longrightarrow$ $C_G(t) = t(t-1)(t-2)$

$$= t^3 - 3t^2 + 2t$$

• $G =$  $\Longrightarrow$ $C_G(t) = t(t-1)(t^2 - 3t + 3)$

$$= t^4 - 4t^3 + 6t^2 - 3t$$

Conjectures Let $C_G(t) = t^{\ell} - c_1 \cdot t^{\ell-1} + c_2 \cdot t^{\ell-2} \cdots + (-1)^{\ell} c_{\ell}$.

(a) (Read 1968) $\{c_i\}$ is unimodal, i.e. $c_1 \le c_2 \le \cdots \le c_k \ge \cdots \ge c_{\ell}$.

(b) (Hoggar 1974) $\{c_i\}$ is log-concave, i.e. $c_i^2 \ge c_{i-1} \cdot c_{i+1}$

Rem. (b) is stronger than (a).

Thm. (Huh. 2012) (b) is true.

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(1) log-concavity of $C_G(t)$.

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Thm. (Huh. 2012) (b) is true.

Huh's proof used Hodge theory (Lefschetz dec.) of some compactification of $\mathbb{P}\mathrm{Col}_G(\mathbb{C})$.

The "Wonderful compactification" (De Concini - Procesi)

Recall: $\mathbb{P}\mathrm{Col}_G(\mathbb{C}) = \mathbb{P}^{|V|-2} \setminus \bigcup_{e \in E} \bar{H}_e$.

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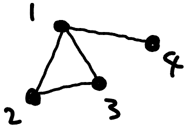
(b) (Hoggar 1974) $\{c_i\}$ is log-concave, i.e. $c_i^2 \ge c_{i-1} \cdot c_{i+1}$ $\rightsquigarrow$ <u>Thm.</u> (Huh. 2012)

Recall: $\mathbb{P}\mathrm{Col}_G(\mathbb{C}) = \mathbb{P}^{|V|-1} \setminus \bigcup_{e \in E} \bar{H}_e$.

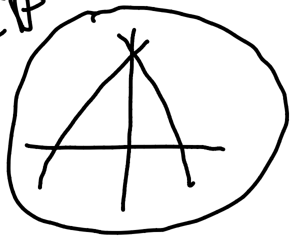


$: G$

$\Downarrow$

$: \mathbb{P}\mathrm{Col}_G(\mathbb{C})$

$\mathbb{CP}^2$



<u>Sketch of the proof.</u>

<u>Step 1</u> Poincaré poly. of $\mathbb{P}\mathrm{Col}_G(\mathbb{C})$ is

$$\frac{1 + c_1 t + \cdots + c_\ell t^\ell}{1+t} =: 1 + \mu_1 t + \cdots + \mu_{\ell-1} t^{\ell-1}.$$

$$\mu_i^2 \ge \mu_{i-1} \cdot \mu_i \; (\forall i) \Longrightarrow c_i^2 \ge c_{i-1} \cdot c_{i+1}$$

(elementary)

Plan of this talk

Relations of several "Tutte polynomials" (T.P.)

T.P. for (arithm.) matroids

T.P. for a list in $\mathbb{Z}^\ell$

(Arithmetic Tutte)

$(G = \mathbb{C}^*)$

G-T.P. for list in $\mathbb{Z}^\ell$ and Abelian Lie group G

$(G = \{0\})$

T.P. for a graph

T.P. for a list of vectors

chromatic poly.
(log-concavity)
bracket poly.
partition function
. . .

characteristic poly. of hyp. arr.
Poincaré poly.
. . .

Poincaré poly. of toric arr.
Ehrhart poly. of zonotopes

NEW Plan of this talk

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§4

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3 List of vectors and log-concavity

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Let V be a vector space / $\mathbb{K}$.

$\mathcal{A} = \{v_1, v_2, \cdots, v_n\}$: a list of vectors.

($v_i = 0$, $v_i = v_j$ allowed).

<u>Question</u> How to define Tutte polynomial

$T_{\mathcal{A}}(x, y) \in \mathbb{Z}[x, y]$?

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<u>Def.</u> Let $G = (V, E)$ be a finite graph.

- $E = \phi \Rightarrow T_G(x, y) = 1$
- $e \in E$: loop $\Rightarrow T_G(x, y) = y \cdot T_{G \setminus e}(x, y)$
- $e \in E$: bridge $\Rightarrow T_G(x, y) = x \cdot T_{G/e}(x, y)$
- $e \in E$: not loop, not bridge.

  $\Rightarrow T_G = T_{G/e} + T_{G \setminus e}$.

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{r_E - r_S} \cdot (y-1)^{\#S - r_S}.$$

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$$\underbrace{T_G(x, y) = \sum_{S \subseteq E} (x-1)^{r_E - r_S} \cdot (y-1)^{\# S - r_S}}_{\text{This works !!}}.$$

3 List of vectors and log-concavity

$\mathcal{A} = \{v_1, v_2, \cdots, v_n\} \subset V$

subspace of V.

<u>Def.</u> For a subset $S \subset \mathcal{A}$, $r_S := \dim \langle S \rangle$.

Define the Tutte polynomial of $\mathcal{A}$ by

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To settle basic properties (recursive relations), we need to formulate "loop", "bridge", "deletion", "contraction".

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This plays role later.

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- $v \in \mathcal{A}$ is a loop if $v = 0$ ($\Leftrightarrow r_{\{v\}} = 0$.)
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<u>Thm.</u>
- $\mathcal{A} = \phi \Longrightarrow T_{\mathcal{A}}(x,y) = 1$
- $v \in \mathcal{A}$: loop $\Longrightarrow T_{\mathcal{A}}(x,y) = y \cdot T_{\mathcal{A}\setminus v}(x,y)$
- $v \in \mathcal{A}$: bridge $\Longrightarrow T_{\mathcal{A}}(x,y) = x \cdot T_{\mathcal{A}/v}(x,y)$
- $v \in \mathcal{A}$: neither loop nor bridge, then

$$T_{\mathcal{A}} = T_{\mathcal{A}\setminus v} + T_{\mathcal{A}/v}.$$

3 List of vectors and log-concavity

$$T_A(x,y) = \sum_{S \subseteq A} (x-1)^{r_A - r_S} \cdot (y-1)^{\# S - r_S}.$$

<u>Thm.</u>
- $A = \phi \Rightarrow T_A(x,y) = 1$
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- $v \in A$: neither loop nor bridge, then $T_A = T_{A \setminus v} + T_{A/v}$.

<u>Example</u> Let $G = (V,E)$ be a graph.

Define $A_G := \{ v - v' \text{ (or } v' - v) \mid (v,v') \in E \} \subset \mathbb{K}^V$.

List of vectors in $\mathbb{K}^V$.

Then $T_G(x,y) = T_{A_G}(x,y)$.

3 List of vectors and log-concavity

<u>Example</u> Let $G=(V,E)$ be a graph. $\mathcal{A}_G := \{ v-v' \ (\text{or } v'-v) \mid (v,v')\in E \} \subset \mathbb{K}^V$.
Then $T_G(x,y) = T_{\mathcal{A}_G}(x,y)$.

vector space

<u>Def.</u> (characteristic poly.) Let $\mathcal{A} \subset V$.

$$\chi_{\mathcal{A}}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{\dim V - r_{\mathcal{A}}} \cdot T_{\mathcal{A}}(1-t, 0),$$

which is characterized by

$$\begin{cases} \bullet \ \chi_{\phi}(t) = t^{\dim V}. \\ \bullet \ \chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}\setminus v}(t) - \chi_{\mathcal{A}/v}(t), \ (v\in\mathcal{A}). \end{cases}$$

<u>Example</u> $\chi_{\mathcal{A}_G}(t) = C_G(t)$ (chromatic poly.)

3 List of vectors and log-concavity

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Now we consider $V = \mathbb{C}^{\ell}$, $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset V^*$.

$$M := M(\mathcal{A}) := \mathbb{C}^{\ell} \setminus \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha).$$

<u>Example</u> For a graph G, $M(\mathcal{A}_G) = \mathcal{C}ol_G(\mathbb{C})$.

3 List of vectors and log-concavity

$\mathcal{A} \subset V$. $\chi_{\mathcal{A}}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{\dim V - r_{\mathcal{A}}} \cdot T_{\mathcal{A}}(1-t, 0)$,

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Now we consider $V = \mathbb{C}^{\ell}$, $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset V^*$. $M := M(\mathcal{A}) := \mathbb{C}^{\ell} \setminus \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha)$

Set $\chi_{\mathcal{A}}(t) = t^{\ell} - b_1 t^{\ell-1} + b_2 t^{\ell-2} - \cdots + (-1)^{\ell} b_{\ell}$.

<u>Thm.</u> (Orlik-Solomon) $b_i = b_i(M(\mathcal{A}))$.

<u>Thm.</u> (J. Huh 2012) $b_i^2 \geq b_{i-1} \cdot b_{i+1}$.

We sketch the proof following the strategy of Adiprasito-Huh-Katz (arXiv:1511.02888)

3 List of vectors and log-concavity

$\chi_{\mathcal{A}}(t)$ is characterized by $\begin{cases} \bullet \ \chi_{\phi}(t) = t^{\dim V}. \\ \bullet \ \chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}\setminus v}(t) - \chi_{\mathcal{A}/v}(t) \end{cases}$

Now we consider $V = \mathbb{C}^{\ell}$, $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset V^*$. $M := M(\mathcal{A}) := \mathbb{C}^{\ell} \setminus \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha)$

<u>Thm.</u> (J. Huh 2012) $\chi_{\mathcal{A}}(t) = t^{\ell} - b_1 t^{\ell-1} + b_2 t^{\ell-2} - \cdots + (-1)^{\ell} b_{\ell}$. Then

$$b_i^2 \geq b_{i-1} \cdot b_{i+1}.$$

<u>Step 1</u> (elementary)

Set $$\frac{\chi_{\mathcal{A}}(t)}{t-1} = t^{\ell-1} - \mu_1 t^{\ell-2} + \cdots + (-1)^{\ell-1} \mu_{\ell-1}.$$

$$\mu_i^2 \geq \mu_{i-1} \cdot \mu_i \ (\forall i) \implies b_i^2 \geq b_{i-1} b_{i+1} \ (\forall i).$$

The goal is to prove $\boxed{\mu_i^2 \geq \mu_{i-1} \cdot \mu_{i+1}}$

3 List of vectors and log-concavity

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Now we consider $V = \mathbb{C}^{\ell}$, $\mathcal{A} = \{\alpha_1, \cdots, \alpha_m\} \subset V^*$. $M := M(\mathcal{A}) := \mathbb{C}^{\ell} \setminus \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha)$

$$\frac{\chi_{\mathcal{A}}(t)}{t-1} = t^{\ell-1} - \mu_1 t^{\ell-2} + \cdots + (-1)^{\ell-1}\mu_{\ell-1}.$$

$$\boxed{\text{Goal} : \mu_i^2 \geq \mu_{i-1} \cdot \mu_{i+1}}$$

Step2 (long construction)

Express μ_i as intersection numbers.

(2-1) $\mathbb{P}M(\mathcal{A}) := M(\mathcal{A})/\mathbb{C}^* \hookrightarrow Y_{\mathcal{A}}$ "Wonderful-compactification"

(2-2) $\alpha, \beta \in H^2(Y_{\mathcal{A}}, \mathbb{Z})$

(2-3) $\mu_k = \alpha^{\ell-1-k} \cdot \beta^{k}$.

3 List of vectors and log-concavity

Now we consider $V = \mathbb{C}^\ell$, $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset V^*$. $M := M(\mathcal{A}) := \mathbb{C}^\ell \setminus \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha)$

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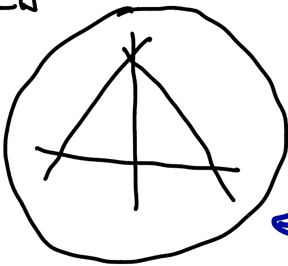
$\mathbb{CP}^{\ell-1}$

$\mathbb{P}M(\mathcal{A}) \hookrightarrow$



Complement to hyperplanes $H_\alpha = \overline{\ker \alpha}$

(0) blow up at 0-dim strata

(1) — strict tr. of 1-dim —

(2) — 2 —

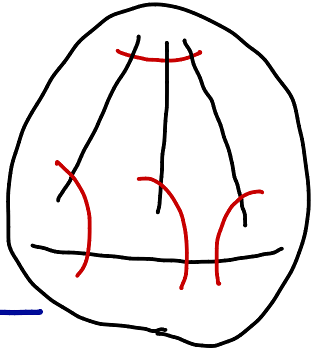
⋮

(ℓ-3) — (ℓ-3) —

(= codim 2)

$\xleftarrow{\ \pi\ }$

$Y_{\mathcal{A}}$.



3 List of vectors and log-concavity

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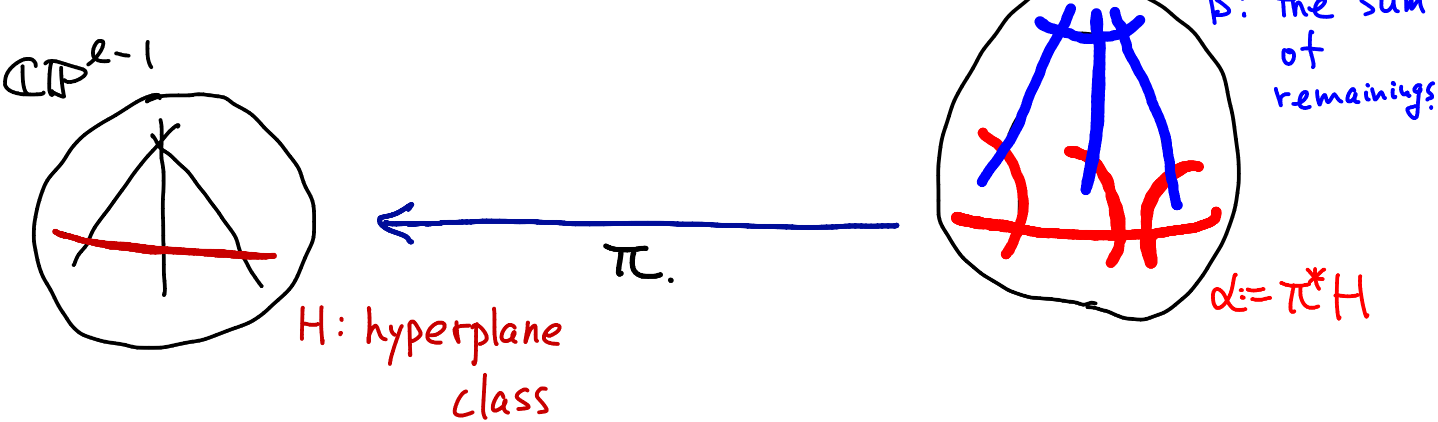

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(2-2) $\alpha, \beta \in H^2(Y_{\mathcal{A}}, \mathbb{Z})$

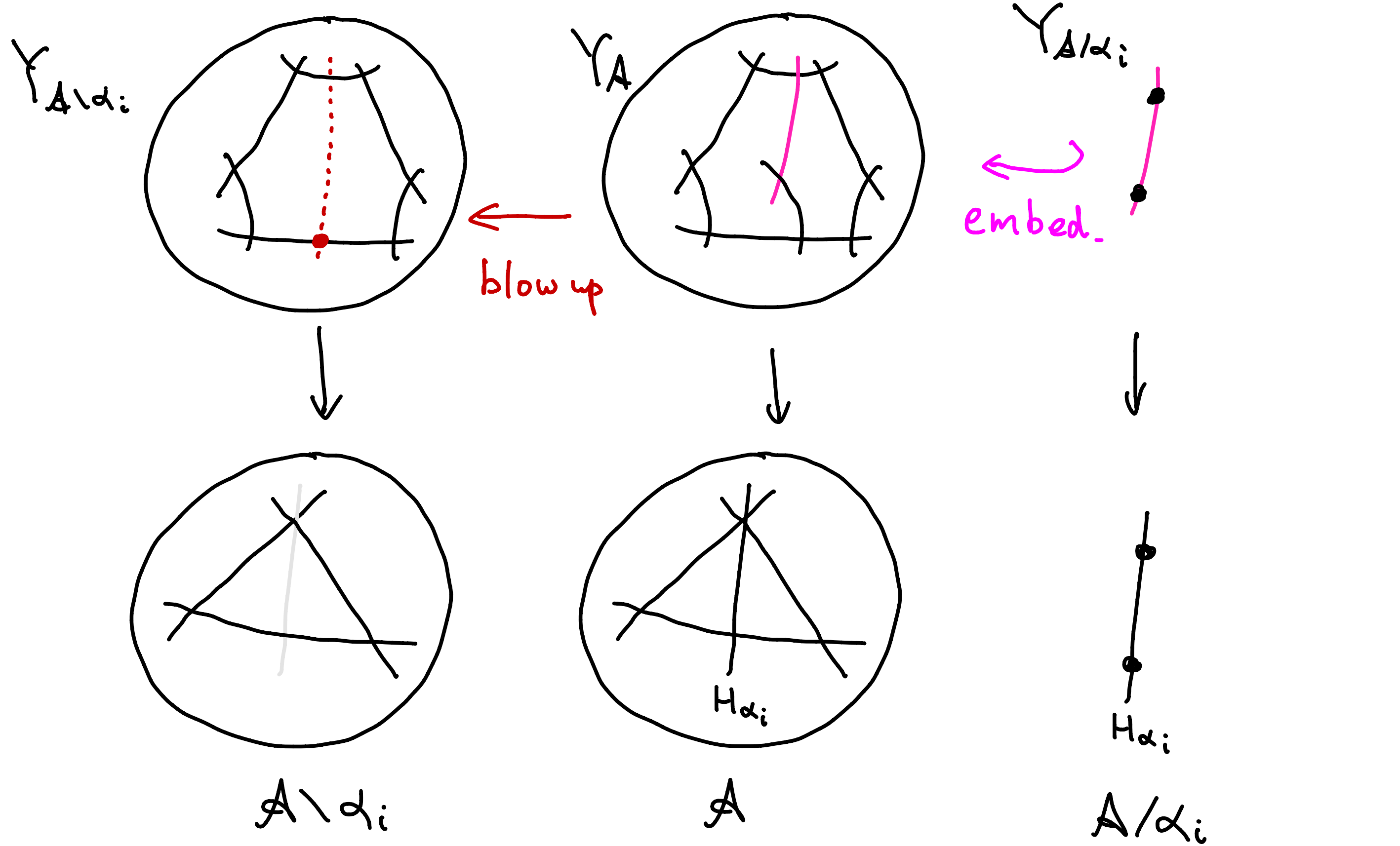
(2-3) $\mu_k = \alpha^{\ell-1-k} \cdot \beta^k.$

Induction using $\dfrac{\chi_{\mathcal{A}}}{t-1} = \dfrac{\chi_{\mathcal{A}\setminus\alpha_i}}{t-1} - \dfrac{\chi_{\mathcal{A}/\alpha_i}}{t-1}.$

Geometric idea is to compare $Y_{\mathcal{A}}, Y_{\mathcal{A}/\alpha_i}, Y_{\mathcal{A}\setminus\alpha_i}$

3 List of vectors and log-concavity

(2-3) $\mu_k = \alpha^{\ell-1-k} \cdot \beta^{k}$.



3 List of vectors and log-concavity

$\boxed{\text{Goal}: \mu_i^2 \geq \mu_{i-1} \cdot \mu_{i+1}}$

<u>Step2</u> (2-1) $\mathbb{P}M(A) := M(A)/\mathbb{C}^* \hookrightarrow Y_A$

(2-2) $\alpha, \beta \in H^2(Y_A, \mathbb{Z})$

(2-3) $\mu_k = \alpha^{\ell-1-k} \cdot \beta^k$.

<u>Step3</u> (Positivity arguments : (probably) routine for experts)

(3-1) α, β are nef.

(3-2) Fix a Kähler class $\omega \in H^2(Y_A, \mathbb{R})$. $\alpha_t := \alpha + t\omega$ is Kähler for $t > 0$. Apply Lefschetz decomp. and Hodge-Riemann ineq. to get $\mu_{\ell-2}^2 \geq \mu_{\ell-3} \cdot \mu_{\ell-1}$.

(3-3) For other $i < \ell - 2$, use Lefschetz hyperplane Thm.

(See Adiprasito-Huh-Katz §9.2 for details of (3-2).)

4 Arithmetic / G-Tutte polynomial

4 Arithmetic / G-Tutte polynomial

Let Γ be a finitely generated abelian group.

$(\Gamma \cong \mathbb{Z}^{r_\Gamma} \oplus \underbrace{\Gamma_{tor}}_{\text{torsion part.}})$

$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset \Gamma.$

For $S \subset \mathcal{A}$, $r_S := \operatorname{rank}\langle S \rangle$ ← subgroup of Γ generated by S.

Def. (Moci's Arithmetic Tutte poly.)

$$T_{\mathcal{A}}^{arith}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S)(x-1)^{r_{\mathcal{A}} - r_S}(y-1)^{|S| - r_S}.$$

Def. (G-Tutte poly.) Let G be an abelian Lie group.

$$T_{\mathcal{A}}^{G}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S;G)(x-1)^{r_{\mathcal{A}} - r_S}(y-1)^{|S| - r_S}.$$

4 Arithmetic / G-Tutte polynomial

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<u>Def.</u> (Moci's Arithmetic Tutte poly.)

$$T_A^{arith}(x,y) = \sum_{S \subseteq A} m(S)\,(x-1)^{r_A - r_S}\,(y-1)^{|S| - r_S}.$$

<u>Def.</u> (G-Tutte poly.) Let G be an abelian Lie group.

$$T_A^G(x,y) = \sum_{S \subseteq A} m(S:G)\,(x-1)^{r_A - r_S}\,(y-1)^{|S| - r_S}.$$

4 Arithmetic / G-Tutte polynomial

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$$T_{\mathcal{A}}^{G}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S:G)(x-1)^{r_{\mathcal{A}} - r_S}(y-1)^{|S| - r_S}.$$

__Def.__ The multiplicities $m(S)$ and $m(S:G)$ are

$$m(S) := \#\left(\Gamma / \langle S \rangle\right)_{tor}$$

$$m(S:G) := \#\mathrm{Hom}\left(\left(\Gamma / \langle S \rangle\right)_{tor}, G\right).$$

__Rem.__ $m(S) = m(S, \mathbb{C}^\times) = m(S, S^1)$ because if F is finite abelian, $F \cong \mathrm{Hom}(F, S^1) \cong \mathrm{Hom}(F, \mathbb{C}^\times)$ as (abstract) abelian groups.

4 Arithmetic / G-Tutte polynomial

$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset \Gamma.$

$m(S) := \#(\Gamma/\langle S\rangle)_{tor}$

$$T_{\mathcal{A}}^{arith}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S)\,(x-1)^{r_{\mathcal{A}}-r_S}\,(y-1)^{|S|-r_S}.$$

$m(S:G) := \#\mathrm{Hom}\big((\Gamma/\langle S\rangle)_{tor}, G\big).$

$$T_{\mathcal{A}}^{G}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S:G)\,(x-1)^{r_{\mathcal{A}}-r_S}\,(y-1)^{|S|-r_S}.$$

Results on $T_{\mathcal{A}}^{G}(x,y)$

- Recursion holds (need a modification.)
- In particular, $\chi_{\mathcal{A}}^{G}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T_{\mathcal{A}}^{G}(1-t, 0)$

  satisfies

$$\chi_{\mathcal{A}}^{G}(t) = \chi_{\mathcal{A}\setminus\alpha}^{G}(t) - \chi_{\mathcal{A}/\alpha}^{G}(t).$$

4 Arithmetic / G-Tutte polynomial

$\mathcal{A}=\{\alpha_1,\cdots,\alpha_n\}\subset\Gamma.$

$m(S):=\#(\Gamma/\langle S\rangle)_{tor}$

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$$T_{\mathcal{A}}^{G}(x,y)=\sum_{S\subseteq\mathcal{A}} m(S:G)\,(x-1)^{r_{\mathcal{A}}-r_S}(y-1)^{|S|-r_S}.$$

$\chi_{\mathcal{A}}^{G}(t):=(-1)^{r_{\mathcal{A}}}\cdot t^{r_\Gamma-r_{\mathcal{A}}}\cdot T_{\mathcal{A}}^{G}(1-t,0)$

Specializations
- $T_{\mathcal{A}}^{S^1}=T_{\mathcal{A}}^{\mathbb{C}^\times}=T_{\mathcal{A}}^{arith}$, $\quad T_{\mathcal{A}}^{\{0\}}=T_{\mathcal{A}}.$

- Ehrhart polynomial of zonotopes is a specialization of arithmetic Tutte poly. (D'Adderio–Moci)

- The constituent of characteristic quasi-poly is $\chi_{\mathcal{A}}^{\mathbb{Z}/k\mathbb{Z}}(t).$
  (≒ mod k counting, defined by Kamiya–Takemura–Terao)

4 Arithmetic / G-Tutte polynomial

$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset \Gamma$. $\qquad m(S:G) := \#\mathrm{Hom}\left((\Gamma/\langle S \rangle)_{\mathrm{tor}}, G \right)$.

$\chi_{\mathcal{A}}^{G}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T_{\mathcal{A}}^{G}(1-t, 0)$ $\qquad T_{\mathcal{A}}^{G}(x,y) = \sum_{S \subseteq \mathcal{A}} m(S:G)\,(x-1)^{r_{\mathcal{A}} - r_S} (y-1)^{|S| - r_S}$.

Specializations

For $\alpha \in \Gamma$, $H_\alpha := \{ \varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = 0 \} \subset \mathrm{Hom}(\Gamma, G)$.

Define $M := \mathrm{Hom}(\Gamma, G) \setminus \bigcup_{\alpha \in \mathcal{A}} H_\alpha$. $\leftarrow$ a generalization of $M(\mathcal{A})$, $\mathrm{Col}_G(\mathbb{C})$.

Euler char. $e(M)$ and Poincaré poly $P_M(t)$ can be expressed by $\chi_{\mathcal{A}}^{G}(t)$.

4 Arithmetic / G-Tutte polynomial

$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset \Gamma$.

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$\chi_{\mathcal{A}}^{G}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T_{\mathcal{A}}^{G}(1-t, 0)$

$T_{\mathcal{A}}^{G}(x, y) = \sum_{S \subseteq \mathcal{A}} m(S:G)(x-1)^{r_{\mathcal{A}} - r_S}(y-1)^{|S| - r_S}$.

$H_\alpha := \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = 0\} \subset \mathrm{Hom}(\Gamma, G)$.

$M := \mathrm{Hom}(\Gamma, G) \setminus \bigcup_{\alpha \in \mathcal{A}} H_\alpha$

<u>Thm</u> (Liu-Tran-Y.) Let $G = (S^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group. Let $g = p + q = \dim G$.

(1) (Euler char.) $e(M) = (-1)^{g \cdot r_\Gamma} \cdot \chi\left((-1)^g \cdot e(G)\right)$.

(2) (Poincaré poly). If $q > 0$, $P_M(t) = (-t^{g-1})^{r_\Gamma} \cdot \chi_{\mathcal{A}}^{G}\left(-\frac{P_G(t)}{t^{g-1}}\right)$

<u>Rem.</u> When $q = 0$ (G: compact), this does not hold.

4 Arithmetic / G-Tutte polynomial

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Rem. When $q=0$ (G: compact), this does not hold.

Question Let $G=(V,E)$ be a graph,

M: a d-dim. manifold.

$\mathrm{Col}_G(M):=\{f:V\to M \mid f(v)\neq f(v') \text{ for } (v,v')\in E\}$

When $P_{\mathrm{Col}_G(M)}(t)=\left(-t^{d-1}\right)^{|V|}\cdot C_G\left(-\dfrac{P_M(t)}{t^{d-1}}\right)$ holds?

4 Arithmetic / G-Tutte polynomial

(Meta-) Problem If one learnes something on classical Tutte polynomial, try to generalize to G-Tutte.

Example (expectation of # of hom's)

Γ : fin. gen. Abelian. $\mathcal{A} \subset \Gamma$. ($r_{\mathcal{A}} = r_{\Gamma}$).

G : finite abelian group. Let $0 < p < 1$.

Choose each $\alpha \in \mathcal{A}$ independently with probability p to have a random subset $S_p \subseteq \mathcal{A}$, and a group $\Gamma_{S_p} = \Gamma / \langle S_p \rangle$.

Then

$$\mathbb{E}[\#\mathrm{Hom}(\Gamma_{S_p}, G)] = p^{r_\Gamma} \cdot (1-p)^{\#\mathcal{A} - r_\Gamma} \cdot T_{\mathcal{A}}^{G}\left(1 + \#G \cdot \frac{1-p}{p}, \frac{1}{1-p}\right).$$

Ref. Ye Liu, Tan Nhat Tran, Y.
G-Tutte polynomials and Abelian Lie
group arrangements. arXiv:1707.04551.