Mapping class groupoids and Thompson's groups

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March 8, 2018

Branched Coverings, Degenerations and Related Topics 2018
Hiroshima University

TÜBITAK GRANT NO: 115F412 GSU GRANT NO: 17.504.001

Foreword

Aim:

- To present a unified picture for the mapping class groups of punctured surfaces together with Thompson's groups T & F; and
- To define a simultaneous generalization of them.

These groups will be presented as the isotropy groups of a (disconnected) groupoid **ΠMCG**.

ΠMCG is a subgroupoid of a bigger (disconnected) groupoid **OMG**.

- Thompson's group V appears as an isotropy group of **OMG**
- Outer automorphism groups of free groups also appear as an isotropy group of **OMG**.

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Basic object

I will talk a lot about an oriented surface of genus g and with at least one puncture, denoted as

$$S = S_n^g \quad (n \geq 1)$$

and about its mapping class group, denoted as

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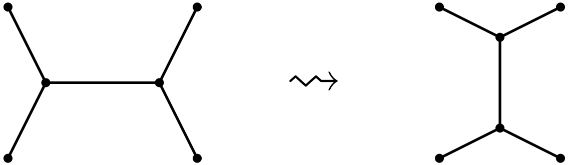
Rough Idea

bu noktadan sonra bir daha geri donmene gerek yok artik. Dogru gidecek sekilde ayarla.. (A quick tour of my talk)

A **flip** is an operation which replaces an H-shaped part of a trivalent fatgraph spine $\mathcal{G}$ of S by an I-shaped graph, producing a new spine $\mathcal{G}'$, as below:



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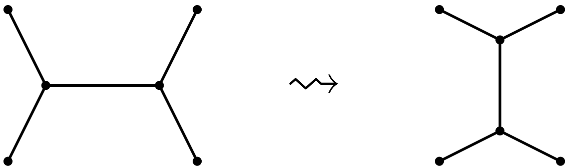

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Main Observation. Flips induce isomorphisms of fundamental groupoids of the trivalent fatgraphs they relate (viewed as abstract fatgraphs, i.e. graphs without embeddings)

Warning. Flips do not induce homeomorphisms of the fatgraphs they relate. In fact they are "atomic" discontinuous modifications of fatgraphs.

Hence we have a groupoid **ΠMCG** whose objects are trivalent fatgraphs and morphisms are flip-induced isomorphisms between their fundamental groupoids. The idea is to prove that these two groupoids are isomorphic:

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- If $\mathcal{G} \hookrightarrow S$ is a finite trivalent fatgraph spine, then

$$\text{(isotropy group) } \mathrm{Aut}_{\mathbf{\Pi MCG}}(\mathcal{G}) \simeq \mathrm{Mod}(S)$$

- If $\mathcal{F}$ is the infinite trivalent planar tree, then

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- If $\mathcal{T}$ is the infinite rooted trivalent planar tree, then

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- For other infinite graphs, this construction permits us to define their Mapping Class/Thomson hybrid groups.

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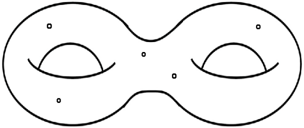

$S = S_n^g$: An **oriented surface** (real 2-manifold) of genus g and with $n > 0$ punctures

$\mathrm{Mod}(S) = \mathrm{Homeo}^*(S)/\sim$: The **mapping class group** of S, i.e. the group of isotopy classes of orientation-preserving homeomorphisms of S preserving the free homotopy classes of loops around the punctures.

$\mathrm{Out}(\mathsf{F}_d) = \mathrm{Aut}(\mathsf{F}_d)/\mathrm{Inn}(\mathsf{F}_d)$: the group of **outer automorphisms** of a free group F_d of rank d.

$(\pi_1(S_n^g) \simeq \mathsf{F}_d$ with $d = 2g + n - 1.)$

Thompson's groups T, V, F**:**

- $T \simeq \mathrm{PPSL}_2(\mathbb{Z})$: Thompson's group of piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ homeomorphisms of the circle with break points at rationals.
- V: Thompson's group of piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ bijections of the circle with break points at rationals.
- F: Thompson's group of piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ homeomorphisms of the unit interval with break points at rationals.

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Groupoid: A small category in which every morphism is an isomorphism.

If $\mathbf{X}$ is a groupoid and a is an object, then $\mathrm{Mor}_{\mathbf{X}}(a, a)$ is always a group, called the **isotropy group of X at** a.

One usually assumes that $\mathbf{X}$ is **connected**: between any two objects, there is a morphism.

Fact. Isotropy groups of a connected groupoid are all isomorphic, i.e.

$$\mathbf{X} \text{ connected} \implies \mathrm{Mor}_{\mathbf{X}}(a, a) \simeq \mathrm{Mor}_{\mathbf{X}}(b, b) \quad \forall a, b \in \mathrm{Obj}(\mathbf{X})$$

(these isomorphisms are not canonical)

Groupoid Example I.

Let S be a (nice) topological space. Its **fundamental groupoid** $\Pi_1(S)$ admits the points of S as its objects. Morphisms from x to y are homotopy classes of paths from x to y.

If S is connected then so is $\Pi_1(S)$.

The isotropy group of $\Pi_1(S)$ at $x \in S$ is just $\pi_1(S, x)$.

Groupoid Example II.

If G is any group acting freely on a set X from the left, then there is the **associated groupoid** $[G\backslash X]$ whose object set is the set of orbits $G\backslash X$ and such that

$$\mathrm{Mor}(Gx, Gy) = \{G(a,b) \,:\, a \in Gx, \quad b \in Gy\}$$

The composition of $G(a,b)$ and $G(b,c)$ is defined to be $G(a,c)$.

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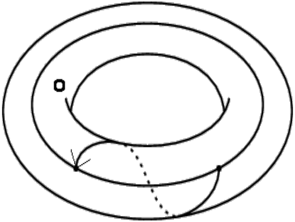
Groupoids-Example III

Fact. The mapping class group Mod(S) acts freely on the set of isotopy classes of trivalent fatgraph spines of S with a doe (distinguished oriented edge).

The associated groupoid is called the Mapping Class Groupoid and denoted **MCG**(S).

$\Longrightarrow$ The isotropy groups of **MCG**(S) are all isomorphic to Mod(S)

But what is a trivalent fatgraph spine of S with a doe ?



Something like this. To be more precise...

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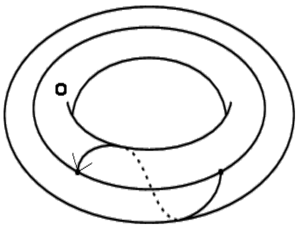

Something like this. To be more precise...

- A **(topological) graph** is a one-dimensional CW-complex comprised of vertices and edges;
- a **fatgraph** or **ribbon graph** is a topological graph together with a cyclic ordering of edges emanating from each vertex.

Definition

- An **ideal arc** of S is an embedded arc connecting punctures in S, which is not homotopic to a point relative to punctures.
- An **ideal cell decomposition of** S is a collection of ideal arcs so that each region complementary to arcs is a polygon with vertices among the punctures.
- A maximal ideal cell decomposition is called an **ideal triangulation**.

What is a maximal ideal cell decomposition?

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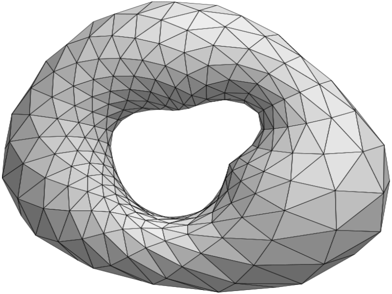
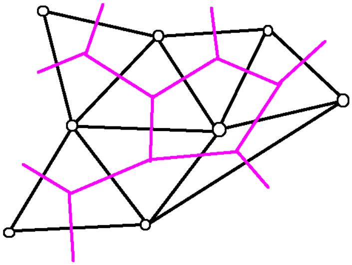
Triangulation-Example

Here is a triangulation of the torus



(the vertices of the triangulation are viewed as punctures-or ideal points.)

Spines



Let $\mathcal{G} \hookrightarrow S$ be an embedding of a topological graph $\mathcal{G}$.

Definition

$\mathcal{G} \hookrightarrow S$ is called a **spine** of S if it is dual to an ideal cell decomposition.

How the dual graph is constructed ?

Facts

- Every spine $\mathcal{G} \hookrightarrow S$ is a strong deformation retract of S.
- Every spine $\mathcal{G} \hookrightarrow S$ acquires a natural fatgraph structure from the orientation of S.
- A spine dual to an ideal triangulation is a trivalent fatgraph.
- Isotopies of S acts on the set of all trivalent fatgraph spines of S.

Denote the set of spines modulo isotopy as

$$\mathrm{SPINE}(S) := \{\varphi : \mathcal{G} \hookrightarrow S \ : \text{ is a spine}\}/\text{isotopy}. \qquad (1)$$

The isotopy class of a spine $\mathcal{G} \hookrightarrow S$ is denoted as $[\mathcal{G} \hookrightarrow S]$ and is again called a spine.

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Spines

Further Facts...

- Mod(S) acts by post-composition on $\mathrm{SPINE}(S)$, though not freely.
- Every automorphism of $\mathcal{G}$ extends to an element of Mod(S), which in turn fixes $[\mathcal{G} \hookrightarrow S]$.
- The Mod(S)-orbit of a fatgraph spine $[\mathcal{G} \hookrightarrow S]$ is just the fatgraph $\mathcal{G}$ (modulo $\mathrm{Aut}(\mathcal{G})$).
- This fatgraph $\mathcal{G}$ is the combinatorial type of the fatgraph spine $\mathcal{G} \hookrightarrow S$.

Pitfall. We want to consider the groupoid associated to the Mod(S)-action on the set $\mathrm{SPINE}(S)$; however, this action is not free.

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Spines with a doe

We can remedy the non-freeness of the Mod(S)-action by considering the enlarged set

$$\mathrm{SPINE}^{doe}(S) := \{\varphi : (\mathcal{G}, \vec{e}) \hookrightarrow S \ : \text{ is a spine with a doe}\}/\text{isotopy}, \quad (2)$$

where $\vec{e}$ is a distinguished oriented edge of $\mathcal{G}$.

The isotopy class of a spine $(\mathcal{G}, \vec{e}) \hookrightarrow S$ is denoted as $[(\mathcal{G}, \vec{e}) \hookrightarrow S]$.

Fact. The $\mathrm{Mod}(S)$ action on $\mathrm{SPINE}^{doe}(S)$ is free. Whence

Definition

The *Mapping Class Groupoid* $\mathbf{MCG}(S)$ is the groupoid associated to the $\mathsf{Mod}(S)$ action on $\mathrm{SPINE}^{doe}(S)$. (Mosher)

In other words,

$$\mathbf{MCG}(S) := [\mathrm{SPINE}^{doe}(S)/\mathsf{Mod}(S)]$$

- The Mod(S)-orbit of a fatgraph spine $[\mathcal{G} \hookrightarrow S]$ is just the combinatorial fatgraph with a doe $[\mathcal{G}, \vec{e}]$ $\Longrightarrow$ the objects of $\mathbf{MCG}(S)$ are **isotopy classes of trivalent fatgraphs** $[\mathcal{G}, \vec{e}]$ with a doe. (just trivalent graphs without the embedding)
- Morphisms are of $\mathbf{MCG}(S)$ the Mod(S)-orbits of the pairs $([(\mathcal{G}, \vec{e}) \hookrightarrow S], [(\mathcal{G}', \vec{e}') \hookrightarrow S])$.
- Isotropy groups of $\mathbf{MCG}(S)$ are isomorphic to Mod(S).

This last proposition is almost tautological, based on the freeness of the Mod(S) **action.**

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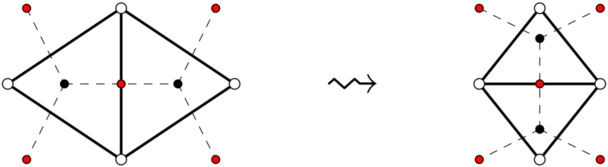
Flips

Given an ideal triangulation of S with an arc of this triangulation, one obtains a new ideal triangulation by applying a *a flip*, as below



A flip can be viewed as an operation on the trivalent fatgraph spines dual to the triangulation, which is also called a flip. (or"H-I move").

Lemma

(Whitehead) Any two trivalent fatgraph spines of S are connected via a finite sequence of flips.

Whitehead's lemma provides the non-trivial content of this story.

Flips are well-defined on isotopy classes of spines with a doe, i.e. if we define

$$X := \{((\mathcal{G}, e), f) \,:\, (\mathcal{G}, e) \in \mathrm{SPINE}(S) \text{ and } f \text{ is an edge of } \mathcal{G}\},$$

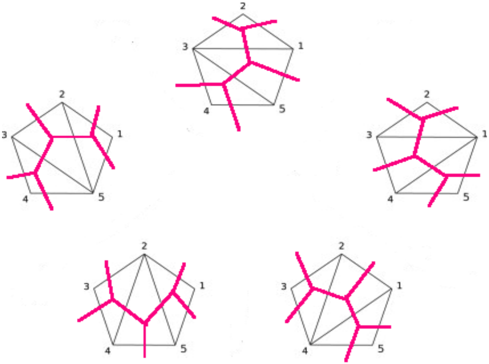
then we may define φ as a map $X \to X$ (of order 4).

Corollary

MCG(S) *is generated by flips and doe moves.*

What is a doe move?

Remark. **MCG**(S) is not freely generated by flips and doe moves. Among others, one has the famous pentagon relation



Remark Another way to obtain a free Mod(S)-action is to consider labeled trivalent fatgraph spines (spines with an enumeration of their edges).

Penner gave a complete presentation of the groupoid associated to this action.

Main observation, again

A fatgraph flip does not define nor is defined by some homeomorphism between the graphs in question. In contrast with this,

Lemma

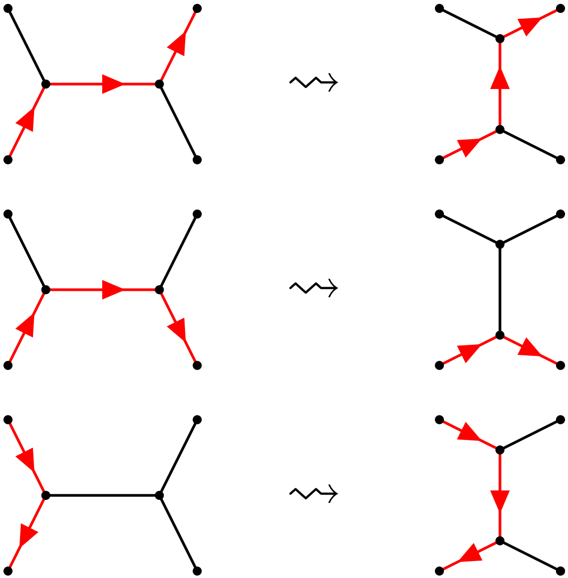
Flips induce isomorphisms of fundamental groups of fatgraphs; i.e. for every pair of edges e, f of $\mathcal{G}$, there are isomorphisms

$$\phi_e : \pi_1(\mathcal{G}, f) \to \pi_1(\mathcal{G}', \phi_e(f)) \quad (3)$$

More naturally, flips induce isomorphisms of fundamental groupoids

$$\phi_e : \Pi_1(\mathcal{G}) \to \Pi_1(\mathcal{G}') \quad (4)$$

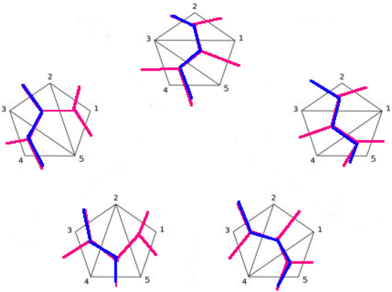
Proof.....



Main theorem

Hence, it is natural to define the groupoid $\mathbf{\Pi MCG}(S)$, whose objects are combinatorial types of fatgraph spines of S and whose morphisms are flip-induced isomorphisms between their fundamental groupoids.

Note that these isomorphisms respects the relations inside $\mathbf{MCG}(S)$:



Theorem

If S is finite, then $\mathbf{MCG}$ *and* $\mathbf{\Pi MCG}(S)$ *are canonically isomorphic.*

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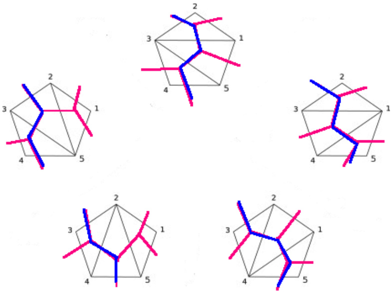


Theorem

If S is finite, then $\mathbf{MCG}$ and $\mathbf{\Pi MCG}(S)$ are canonically isomorphic.

Idea of proof.

A tedious idea would be to find a presentation of **MCG**(S) and show that the natural map between **ΠMCG**(S) and **MCG**(S) is an isomorphism.

A better ide is to apply the fundamental groupoid functor to the construction of **MCG**.

First, every fatgraph spine with a doe $\varphi : (\mathcal{G}, \vec{e}) \hookrightarrow S$ gives rise to an isomorphism (homotopy equivalence) of groupoids

$$\varphi^* : \Pi_1(\mathcal{G}, \vec{e}) \hookrightarrow \Pi_1(S). \tag{5}$$

Isotopies of S acts on the set of such isomorphisms and we have the corresponding set

$$\Pi\mathrm{SPINE}^{doe}(S) := \left\{\varphi^* : \Pi_1(\mathcal{G}, \vec{e}) \hookrightarrow \Pi_1(S) \;:\; \varphi \in \mathrm{SPINE}^{doe}(S)\right\}/\text{isotopy}, \tag{6}$$

$\mathrm{Mod}(S)$ acts freely on $\Pi\mathrm{SPINE}^{doe}(S)$.

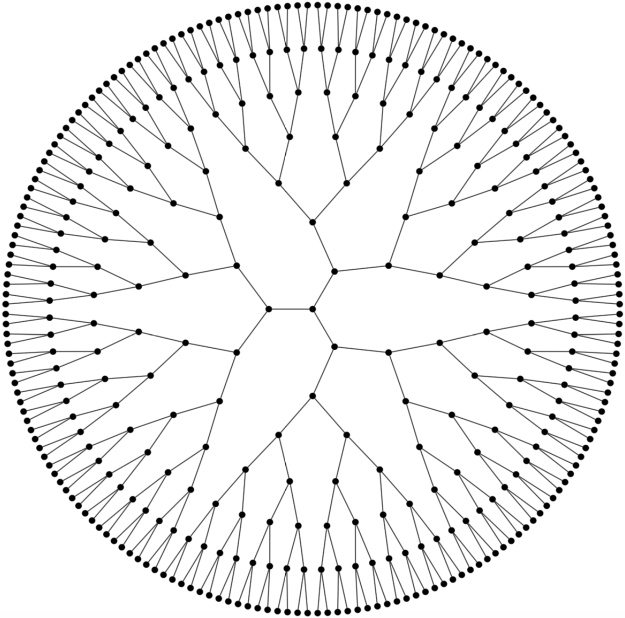
Since $\Pi\mathrm{SPINE}^{doe}$ and SPINE^{doe} are in canonical bijection; and since the $\mathrm{Mod}(S)$ action on them respects this bijection, the result follows.

More generally, define the groupoid $\mathbf{\Pi MCG}(\mathcal{G})$, whose objects are fatgraphs with a doe that can be obtained from $\mathcal{G}$ by a finite sequence of flips doe moves and morphisms are isomorphisms between their fundamental groupoids induced by these operations.

Theorem

If $\mathcal{F}$ is the infinite planar trivalent tree, then $\mathbf{\Pi MCG}(\mathcal{F})$ has just one object and it is isomorphic to Thompson's group.

Proof.(with Ayberk Zeytin) The action on the fundamental groupoid of $\mathcal{F}$ extends to an action on the space of ends ($\sim$ paths to infinity) $\partial\mathcal{F}$. There is an identification $\partial\mathcal{F} \to S^1$ (the circle) via continued fractions, which is compatible with the flip action. The resulting flip action is precisely $\mathrm{PPSL}_2(\mathrm{Z})$, i.e. the Thompson's group T.



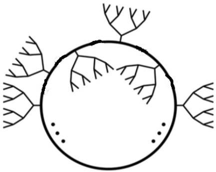
Charks

Another instance to which we may apply the previous theorem is...

Definition

A **a chark** $\mathcal{G}$ is a trivalent fatgraph with just one cycle and with no pending vertices. We also require that this cycle do not encircle a puncture.

In other words, a chark has just one cycle and otherwise it looks like the infinite planar tree:



Charks

At this point we make contact with arithmetic, which was our point of departure in this adventure:

- Every chark (without a doe) represents in a natural way the class of a binary quadratic form, such that;
- Each doe can be identified with a binary quadratic form in this class.

Lemma

The flip action is transitive on the set of charks.

To repeat things, we may identify the objects of $\mathbf{\Pi MCG}(\mathcal{G})$ with the set of binary quadratic forms in a natural way.

(Here, $\mathcal{G}$ is any chark.)

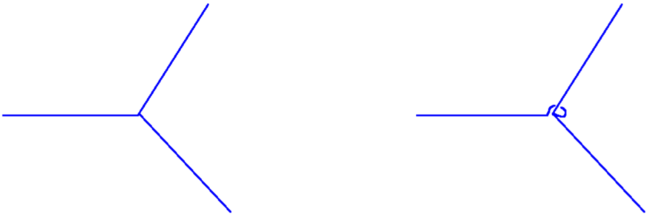
The isotropy groups of the groupoid $\mathbf{\Pi MCG}(\mathcal{G})$ appears to be $\mathrm{T} \times \mathrm{T}$. (on-going work)

We expect to obtain true hybrids of mapping class groups and Thompson groups for more sophisticated graphs (i.e. pair of pants graphs). We haven't studied these cases in depth yet.

It appears to be an important and difficult problem to determine the isotropy groups of the groupoids so obtained.

Shuffles

A **shuffle** is an operation which modifies a trivalent fatgraph at a given vertex as follows:



Shuffles

- Shuffles don't modify the underlying topological graph $\implies$ induce trivial isomorphisms of their fundamental groupoids
- Shuffles change the genus and the number of punctures of $\mathcal{G}$.
- By applying finitely many flips and shuffles to $\mathcal{G}$, we can obtain every trivalent fatgraph whose π_1 is isomorphic to $\pi_1(\mathcal{G})$, if $\pi_1(\mathcal{G})$ is finitely generated.

Remark. Shuffles can be defined on trivalent fatgraph spines as well; however, they change the ambient surface.

The Outer Mapping Groupoid

Definition

The objects of the *outer mapping groupoid* **OMG** are fatgraphs $\mathcal{G}$ with a doe, and

$$\mathrm{Mor}_{\mathbf{OMG}}(\mathcal{G},\mathcal{G}') :=$$

$$\left\{\text{isomorphisms induced by flips, shuffles \& doe moves } \Pi_1^{\mathcal{G}} \to \Pi_1^{\mathcal{G}'}\right\}.$$

If $\mathcal{G}$ is finite with $d = 2g + n - 1$, then the connected component of $\mathcal{G}$ inside **OMG** contains the groupoids $\mathbf{\Pi MCG}(S_g^n)$ with $d = 2g + n - 1$ as a subgroupoid.

Speculative part

Theorem

- *If $\mathcal{G} \hookrightarrow S$ is a finite trivalent fatgraph spine, then*

$$\textit{(isotropy group)}\ \mathrm{Aut}_{\mathbf{OMG}}(\mathcal{G}) \simeq \mathrm{Out}(\pi_1(S))$$

- *If $\mathcal{F}$ is the infinite trivalent planar tree, then*

$$\textit{(isotropy group)}\ \mathrm{Aut}_{\mathbf{OMG}}(\mathcal{F}) \simeq \mathrm{V}$$

- *For other infinite graphs, this construction permits us to define their Outer/Thomson hybrid groups.*

(not completely proven yet)

A trivalent fatgraph is (almost) the same thing as a subgroup of $\mathrm{PSL}_2(Z)$.

The set of subgroups constitute a category (a poset) under the inclusion of subgroups, denoted **Sub**.

Hence the objects sets of **ΠMCG** and **ΠMCG** can be identified with the category **Sub**.

The category **Sub**. is reverse-equivalent to the category **Cov** of trivalent fatgraphs under fatgraph coverings. These fatgraphs are nothing but dessins.

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In turn, this category is equivalent to the category of surfaces under branched coverings at the punctures. This appearance of branched coverings is not forced; these coverings are important from the arithmetic point of view.

The structures of **ΠMCG** and **ΠMCG** groupoids are compatible with this structure. (Connection with branched coverings of moduli spaces along the compactification locus)

Which permits to pass to the limit and define the profinite case.

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Table of contents

References & Acknowledgements

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- Anonymous internet contributor from whom I have borrowed some of the the images

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