Time-interacting fields and actions in positive topological field theories

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§Introduction

Topological Field Theory & Gluing

- [1] M.F. Atiyah (1988): *Topological quantum field theory*
- axioms of TFTs for smooth oriented manifolds
- **$(n+1)$-dim. TFT Z** (over comm. ground ring R with 1)
- M^n closed manifold $\longmapsto$ **state module** $Z(M)$ (f.g. over R)
- W^{n+1} compact manifold $\longmapsto$ **state sum** $Z_W \in Z(\partial W)$
- **gluing axiom:** $(M^n, N^n, P^n) \rightsquigarrow$ contraction product

$$\langle \cdot, \cdot \rangle : \quad Z(M \sqcup N) \otimes_R Z(N \sqcup P) \quad \longrightarrow \quad Z(M \sqcup P)$$

  s.t. $Z_{\boldsymbol{W}} = \langle Z_{\boldsymbol{W'}}, Z_{\boldsymbol{W''}} \rangle$ whenever $\boldsymbol{W} : M \xrightarrow{\boldsymbol{W'}} N \xrightarrow{\boldsymbol{W''}} P$

- **further axioms:** $Z(-)$ is functorial w.r.t. diffeomorphisms, Z_W is diffeomorphism invariant; $Z(-M) = Z(M)^*$ dual module; disjoint union $Z(M \sqcup N) \cong Z(M) \otimes_R Z(N)$, $Z_{W \sqcup V} \cong Z_W \otimes_R Z_V$; normalizations $Z(\emptyset) = R$, $Z_\emptyset = 1 \in R$, $Z_{M \times [0,1]} = \mathrm{id}_{Z(M)}$

Examples (Gluing)

- $W^{n+1}\colon M^n \xrightarrow{W'} N^n \xrightarrow{W''} P^n$
- Euler characteristic (n odd):

$$\chi(W) = \chi(W') + \chi(W'')$$

- Novikov additivity (compatibly oriented bordisms):

$$\sigma(W) = \sigma(W') + \sigma(W'')$$

- Pontrjagin numbers ($n = 7$, compatibly oriented bordisms, $M = P = \emptyset$, $\boldsymbol{H^3(N^7) = H^4(N^7) = 0}$):

$$p_1^2[W] = p_1^2[W'] + p_1^2[W'']$$

$\rightsquigarrow$ [10] **Milnor's invariant $\boldsymbol{\lambda(N^7)}$**

A Convenient Setting for Topological Invariants

GOAL:
Exploit concept of TFT as a source of inspiration for constructing powerful (differential) topological invariants of manifolds!

IDEA (M. Banagl [3], 2015):
Find formulation of Atiyah's axioms for TFTs over **semirings**!

- accept certain deviations from Atiyah's axioms
- obtain **positive TFTs** as framework for topological invariants
- avoid measure theoretic difficulties in Feynman's path integral

TODAY:
Present features of positive TFT, and a high dimensional example.

- Banagl's work [2, 3, 4]: theory of semirings and semimodules, axioms for positive TFTs, framework of quantization
- our work [12, 16, 17, 18]: time-interacting fields and actions, improving Banagl's example of a positive TFT based on fold maps, computations of aggregate invariant for exotic spheres

§Semi-Algebra

Semirings and Semimodules

Definition

1. A **semiring** is a tuple $S = (S, +, \cdot, 0, 1)$, where
   - $(S, +, 0)$ comm. monoid
   - $(S, \cdot, 1)$ monoid

   satisfying $a(b+c) = ab + ac$, $(a+b)c = ac + bc$, and $0 \cdot a = a \cdot 0 = 0$.
2. A (left) **semimodule** over the semiring S is a comm. monoid $M = (M, +, 0_M)$ with scalar multiplication $S \times M \to M$, $(s, m) \mapsto sm$, such that $(rs)m = r(sm)$, $r(m+n) = rm + rn$, $(r+s)m = rm + sm$, $1m = m$, $r0_M = 0_M = 0m$.

Example

- natural numbers $\mathbb{N} = \{0, 1, \dots\}$ form semiring $(\mathbb{N}, +, \cdot)$
- Boolean semiring $\mathbb{B} = \{0, 1\}$, require $1 + 1 = 1$
- semiring of formal power series $\mathbb{B}[\![q]\!]$ is an $\mathbb{N}[\tau]$-semimodule

Eilenberg-Completeness

[6] S. Eilenberg (1974): *Automata, Languages, and Machines*

Definition

1. A comm. monoid $(M, +, 0)$ is **complete** if "+" is extended to
$$\sum : \quad \{m_i\}_{i \in I} \longmapsto \sum_{i \in I} m_i \in M$$
satisfying *Fubini's law*:
$I = \dot{\bigcup}_{j \in J} I_j \quad \Rightarrow \quad \sum_{i \in I} m_i = \sum_{j \in J} \sum_{i \in I_j} m_i.$
2. A semiring S is called **complete** if $(S, +, 0)$ is complete, and $\sum$ satisfies infinite distributivity.
3. An S-semimodule M is called **complete** if the monoid M is complete, and $\sum$ satisfies infinite distributivity.

Eilenberg swindle: If S is an Eilenberg-complete **ring**, then

$$s := 1 + 1 + \cdots = 1 + (1 + \dots) = 1 + s \Rightarrow 0 = 1 \Rightarrow S = 0.$$

Continuous Monoids

- $(M, +, 0)$ comm. monoid
- suppose M is **idempotent**, i.e., $m + m = m$ for all $m \in M$
- then, M has natural partial order "$\leq$" given by

$$m \leq m' \qquad \Leftrightarrow \qquad m + m' = m$$

Definition

An idempotent complete monoid $(M, +, 0, \Sigma)$ is called **continuous** if for all families $(m_i)_{i \in I}$, $m_i \in M$, and for all $c \in M$, $\sum_{i \in F} m_i \leq c$ for all finite $F \subset I$ implies $\sum_{i \in I} m_i \leq c$.

Lemma

Let $(M, +, 0, \Sigma)$ be a continuous, idempotent, complete monoid. Then, for any families $(m_i)_{i \in I}$ and $(n_j)_{j \in J}$ of elements in M for which $\{m_i;\ i \in I\} = \{n_j;\ j \in J\}$ as subsets of M, we have

$$\sum_{i \in I} m_i = \sum_{j \in J} n_j.$$

A Completed Tensor Product

- M, N continuous idempotent complete comm. monoids
- suppose M, N are complete **bi**semimodules over a semiring S

Theorem (Banagl [4], 2016)

There exists a continuous idempotent complete monoid $M \widehat{\otimes}_S N$ which has the structure of a complete S-bisemimodule, and a $S_S S$-linear bicontinuous map $\widehat{\alpha}\colon M \times N \to M \widehat{\otimes}_S N$ such that the following **universal property** *holds. For every continuous idempotent complete monoid P which has the structure of a complete S-bisemimodule, and for every $S_S S$-linear bicontinuous map $\varphi\colon M \times N \to P$, there exists a unique S-bisemimodule homomorphism $\widehat{\varphi}\colon M \widehat{\otimes}_S N \to P$ such that the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[rr, "\varphi"] \arrow[dr, "\widehat{\alpha}"'] & & P \\
& M \widehat{\otimes}_S N \arrow[ur, "\widehat{\varphi}"'] &
\end{tikzcd}$$

§Positive TFTs

Banagl's Axioms for Positive Topological Field Theory

- Q continuous idempotent complete comm. monoid
- Q^c, Q^m complete semirings having additive monoid Q
- $\widehat{\otimes}_c$, $\widehat{\otimes}_m$ tensor products for bisemimodules over Q^c, Q^m
- all manifolds are smooth and unoriented
- define **$(n+1)$-dim. positive TFT Z** (over Q^c, Q^m)
- M^n closed manifold $\longmapsto$ **state module** $Z(M)$, a continuous idempotent complete two-sided semialgebra over Q^c, Q^m
- W^{n+1} compact manifold $\longmapsto$ **state sum** $Z_W \in Z(\partial W)$
- **gluing axiom:** $(M^n, N^n, P^n) \rightsquigarrow$ contraction product

$$\langle \cdot, \cdot \rangle \colon Z(M \sqcup N) \widehat{\otimes}_c Z(N \sqcup P) \quad \longrightarrow \quad Z(M \sqcup P),$$

  s.t. $Z_{\boldsymbol{W}} = \langle Z_{\boldsymbol{W'}}, Z_{\boldsymbol{W''}} \rangle$ whenever $\boldsymbol{W} \colon M \xrightarrow{\boldsymbol{W'}} N \xrightarrow{\boldsymbol{W''}} P$
- **further axioms:** $Z(-)$ functorial w.r.t. diffeomorphisms; pseudo-isotopy invariance; $Z(M \sqcup N) \cong Z(M) \widehat{\otimes}_{m/c} Z(N)$; $Z_{W \sqcup V} \cong Z_W \widehat{\otimes}_m Z_V$; diffeomorphism invariance of Z_W

Constructing a Positive TFT from Fields and Actions

- **system of fields** $\mathcal{F}$: have sets of fields $\mathcal{F}(W^{n+1})$, $\mathcal{F}(M^n)$, axioms (restriction, disjoint union, diffeomorphisms, gluing)
- $\mathbf{C}$ small strict monoidal category
- **C-valued action functional** $\mathbb{T}$ **on fields**: have maps $\mathbb{T}_W\colon \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{C})$ for all W, require certain axioms
- S complete semiring, $Q = \{\mathrm{Mor}(\mathbf{C}) \to S\}$ complete monoid
- Q^c composition ("$\circ$") semiring; Q^m monoidal ("$\otimes$") semiring
- **state modules:** $Z(M^n) = \{\mathcal{F}(M) \to Q \mid \text{constraint equation}\}$
- **state sum (partition function)** $Z_W \in Z(\partial W)$: $f \in \mathcal{F}(\partial W)$,

$$Z_W(f) = \sum_{F \in \mathcal{F}(W),\, F|_{\partial W} = f} \chi_{\mathbb{T}_W(F)} \quad \in Q$$

$$\boldsymbol{Z_W(f) = \int_{\mathcal{F}(W;f)} e^{iS_W(F)}\, \mathrm{d}\,\mu_W}$$

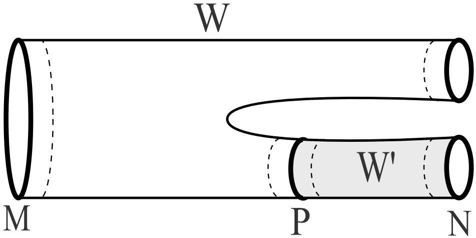
Theorem (Banagl [3], 2015)

The above process of **quantization** *yields a positive TFT Z.*

§Time-Interaction

Bordisms and Submanifolds

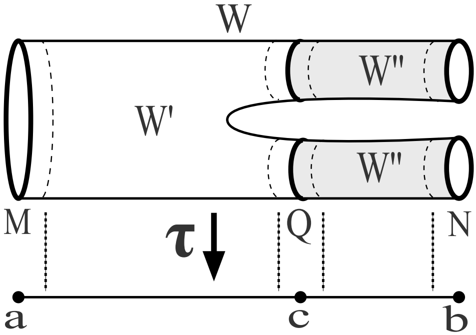
- M, N, P closed n-mflds.; W,W', W'' compact $(n+1)$-mflds.
- equip submanifolds $M, N, P \subset W$ with germs of **framings**



- W is a **collared bordism** from M to N
- M, N, P are (codim. 1) **framed submanifolds** of W
- W' is **collared subbordism** of W; W' collared from P to N
- given collared bordisms W' from M to N and W' from N to P, have **canonical gluing** $W' \cup_N W''$ with N as framed codim. 1 submanifold, and W', W'' as collared subbordisms
- a **diffeomorphism** of collared bordisms respects ingoing and outgoing boundaries, and is identity map in collar direction

Time Functions on Collared Bordisms

- **time function** $\boldsymbol{\tau}$ on collared bordism W from M to N



- M, N, Q are $\boldsymbol{\tau}$**-consistent** framed submanifolds of W
- W', W'' are $\boldsymbol{\tau}$**-consistent** collared subbordism of W
- τ restricts to time functions $\tau|_{W'}$, $\tau|_{W''}$ on W', W''
- a diffeomorphism of collared bordisms is **time-consistent** if it covers an orientation preserving diffeomorphism of intervals

System of Time-Interacting Fields

- have sets of fields $\mathcal{F}(M^n)$, and $\mathcal{F}(W^{n+1})$ for W collared
- for each time function τ on W have subset $\mathcal{F}_\tau(W) \subset \mathcal{F}(W)$
- **restriction maps:**
  $\mathcal{F}(W) \to \mathcal{F}(W_0)$, when W_0 is union of components of W
  $\mathcal{F}(M) \to \mathcal{F}(M_0)$, when M_0 is union of components of M
  $\mathcal{F}(W) \to \mathcal{F}(P)$, when P is union of components of ∂W
  $\mathcal{F}_\tau(W) \to \mathcal{F}_{\tau|_{W_0}}(W_0)$, when $W_0 \subset W$ is collared, τ-consistent
  $\mathcal{F}_\tau(W) \to \mathcal{F}(Q)$, when $Q \subset W$ is framed, τ-consistent
- **disjoint union:** $\mathcal{F}(W' \sqcup W'') \xrightarrow{\cong} \mathcal{F}(W') \times \mathcal{F}(W'')$; $M \sqcup M'$
- **$\boldsymbol{\tau}$-gluing:** $\mathcal{F}_\tau(W' \cup_N W'') \xrightarrow{\cong} \mathcal{F}_{\tau|_{W'}}(W') \times_{\mathcal{F}(N)} \mathcal{F}_{\tau|_{W''}}(W'')$
- **contravariant action of time-consistent diffeomorphisms**
- **time interaction:** $\exists\ \mathcal{F}(W) \to \mathcal{F}_\tau(W)$ such that whenever $P \subset \partial W$ is a union of components of ∂W, we have

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\exists"] \arrow[d, "\mathrm{res}"'] \arrow[dr, phantom, "\square"] & & \mathcal{F}_\tau(W) \arrow[dll, "\mathrm{res}"] \\ \mathcal{F}(P) & & \end{tikzcd}$$

System of Time-Interacting Action Functionals

- **C** small strict monoidal category
- $\mathcal{F}$ system of time-interacting fields on M^n, W^{n+1} (collared)
- have functions $\mathbb{T}_W \colon \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{C})$ for W^{n+1} collared
- **disjoint union:** $\mathbb{T}_{W' \sqcup W''}(F) = \mathbb{T}_{W'}(F|_{W'}) \otimes \mathbb{T}_{W''}(F|_{W''})$
- **τ-gluing:** $\mathbb{T}_{W' \cup_N W''}(F) = \mathbb{T}_{W'}(F|_{W'}) \circ \mathbb{T}_{W''}(F|_{W''})$
- **action of time-consistent diffeomorphisms:** require that $\mathbb{T}_W(\phi^* F) = \mathbb{T}_{W'}(F)$ holds under the bijection $\phi^* \colon \mathcal{F}_{\tau'}(W') \to \mathcal{F}_\tau(W)$ induced by time-consistent diffeomorphism $\phi \colon W \to W'$
- **time interaction:**

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[r] \arrow[dr, "\mathbb{T}_W"'] & \mathcal{F}_\tau(W) \arrow[r, "\mathrm{incl}"] \arrow[d, phantom, "\square"] & \mathcal{F}(W) \arrow[dl, "\mathbb{T}_W"] \\
& \mathrm{Mor}(\mathbf{C}) &
\end{tikzcd}$$

Quantization of Time-Interacting Fields and Actions

- $\mathcal{F}$ system of time-interacting fields on M^n, W^{n+1} (collared)
- $\mathbf{C} = (\mathbf{C}, \otimes, I)$ small strict monoidal category
- $\mathbb{T}$ system of $\mathbf{C}$-valued time-interacting action functionals on $\mathcal{F}$
- S continuous idempotent Eilenberg-complete semiring
- $Q = \{\mathrm{Mor}(\mathbf{C}) \to S\}$ continuous idempotent complete monoid
- Q^c, Q^m complete semirings, Q underlying additive monoid
- define **state module** $Z(M)$ of closed manifold M^n by

$$Z(M) = \{z\colon \mathcal{F}(M) \to Q \mid z(\phi^* f) = z(f) \text{ for all } \phi \in \mathrm{Diff}_0(M)\}$$

- define **state sum** $Z_W \in Z(\partial W)$ of collared bordism W^{n+1} by

$$Z_W(f) = \sum_{\substack{F \in \mathcal{F}(W), \\ F|_{\partial W} = f}} \chi_{\mathbb{T}_W(F)} \in Q, \qquad f \in \mathcal{F}(\partial W)$$

Theorem (W. [19], 2018)

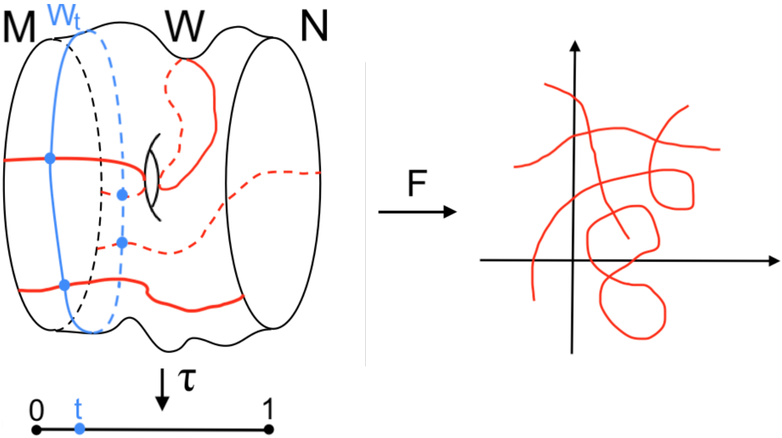
The above process of **quantization** *yields a positive TFT Z.*

§Application

Step 1: System of Time-Interacting Fields

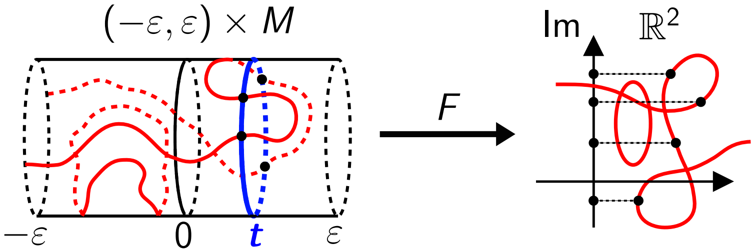
$F\colon W^{n+1} \to \mathbb{R}^2$ is called **fold map** if F looks at every singular point $c \in \boldsymbol{S(F)}$ in suitable coordinates centered at c and $F(c)$ like

$$(t, x_1, \ldots, x_n) \mapsto (t, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$



Step 1: System of Time-Interacting Fields

- define set $\mathcal{F}(M)$ of fields on (connected) closed manifold M^n
- a **field on M** is a $(\{0\} \times M)$-germ represented by a fold map



such that for all $t \neq 0$ we have $S(F) \pitchfork \{t\} \times M$, and $\mathrm{Im} \circ F$ is **injective** on $S(F) \cap \{t\} \times M$

- the projection $(0, \varepsilon) \times M \to (0, \varepsilon)$ restricts to **finite covering**

$$S(F) \cap (0, \varepsilon) \times M \longrightarrow (0, \varepsilon)$$

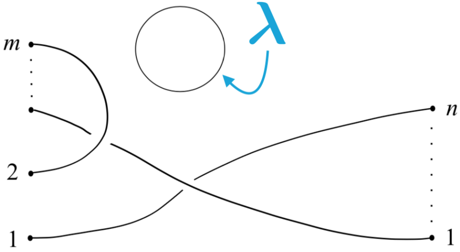
- components of $S(F) \cap (0, \varepsilon) \times M$ have **canonical ordering**

Step 1: System of Time-Interacting Fields

- W^{n+1} collared bordism from M to N with time-function τ
- define sets $\mathcal{F}(W)$, $\mathcal{F}_\tau(W)$ of (τ-interacting) fields on W
- a **field on W** is a triple (F, f_M, f_N), where $f_M \in \mathcal{F}(M)$ and $f_N \in \mathcal{F}(N)$, and $F\colon W \setminus \partial W \to \mathbb{R}^2$ is a fold map that extends for suitable $\varepsilon > 0$ the fold maps $f_M|_{(0,\varepsilon)\times M}$ and $f_N|_{(-\varepsilon,0)\times N}$
- a field (F, f_M, f_N) on W is **τ-interacting** if F restricts for every τ-consistent framed submanifold $P \subset W$ to a field on P
- in general, a field on W^{n+1} might not be τ-interacting for all τ
- check field axioms except for time interaction: restriction, disjoint union, τ-gluing, action of time-consistent diffeomorphisms

Step 2: Action Functional

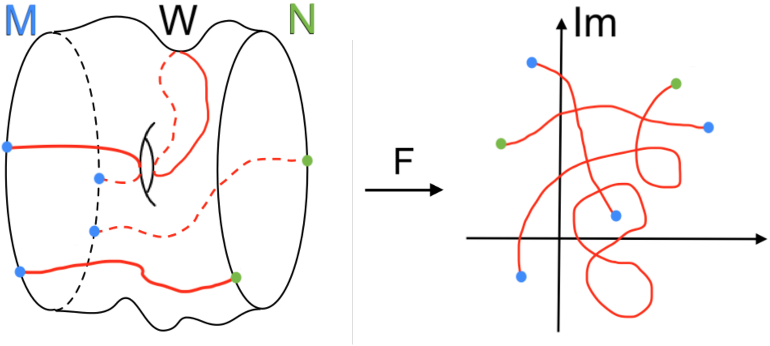
- categorify Brauer algebras [5] (representation theory of $O(n)$)
- symmetric strict monoidal **Brauer category** $(\mathbf{Br}, \otimes, [0], b)$
- $\mathrm{Ob}\,\mathbf{Br}$: $[0] = \emptyset,\ [1] = \{1\},\ [2] = \{1, 2\},\ \ldots$
- $\mathrm{Hom}_{\mathbf{Br}}([m], [n])$: morphisms look like



- no over- underpass information (compare Turaev [15])
- $[m] \otimes [n] = [m + n]$; $\otimes$ of morphisms by vertical stacking
- braiding $b =$  $\in \mathrm{Hom}_{\boldsymbol{Br}}([2], [2])$
- enrichment [12]: **chromatic Brauer category** $(\mathbf{cBr}, \otimes, [0], b)$
- use countable number of colors to label components

Step 2: Action Functional

- want $\mathbb{T}_W \colon \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{Br})$ for W^{n+1} collared from M to N
- associate to field (F, f_M, f_N) on W morphism $[m] \to [m']$ in **Br**
- identify canonically ordered set $S(F) \cap M$ with $[m]$
- similarly, $S(F) \cap N \cong [m']$
- morphism $[m] \to [m']$ is naturally induced by **fold pattern**



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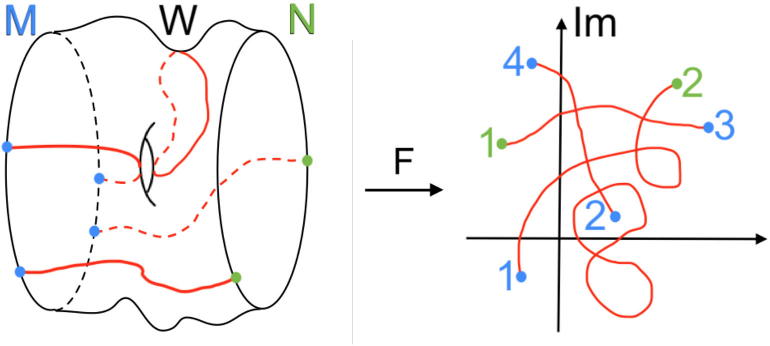

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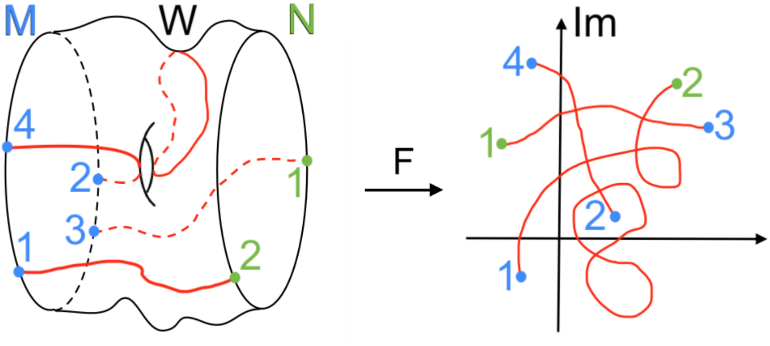

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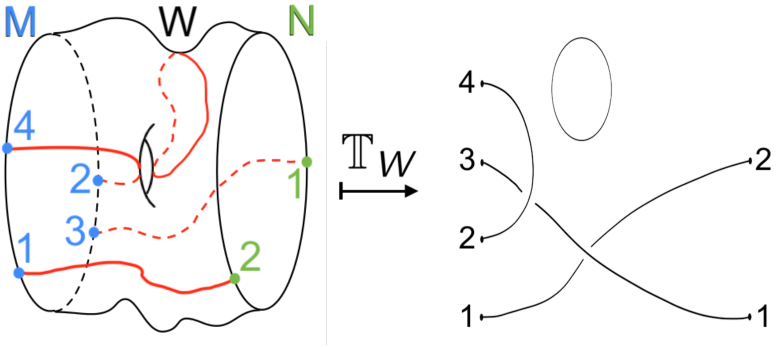
- check action axioms except for time interaction: disjoint union, τ-gluing, action of time-consistent diffeomorphisms



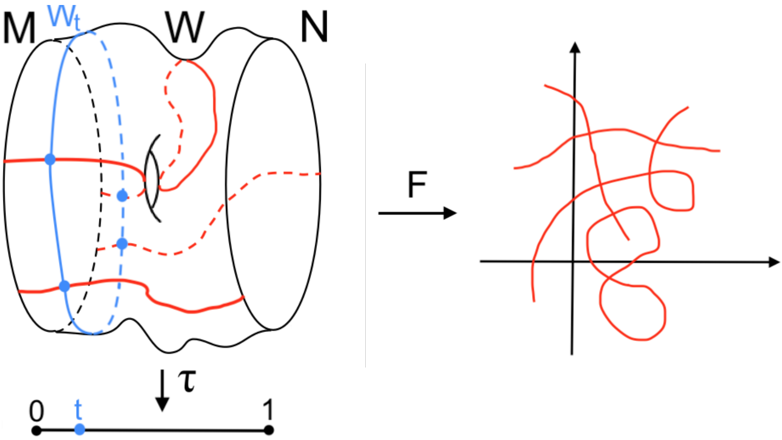
Step 2: Action Functional

- W^{n+1} collared bordism from M to N with time-function τ
- **Question (M. Banagl): Are all fold patterns of fields on W also realized by τ-interacting fields?**
- in other words: do fields and action satisfy the time-interaction axioms?

Theorem (W. [16, 19], 2018)

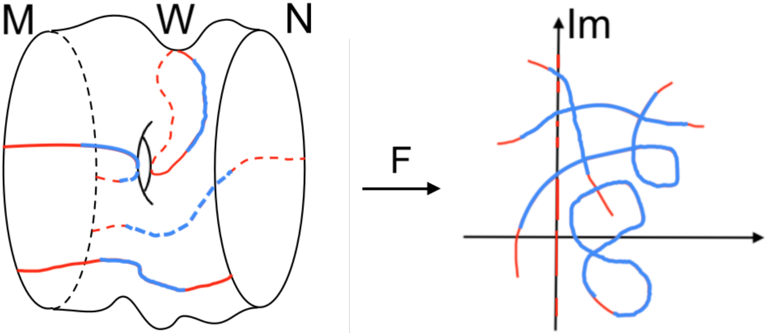
For every field (F, f_M, f_N) on W, there is a τ-interacting field (G, f_M, f_N) on W such that $\mathbb{T}_W(F, f_M, f_N) = \mathbb{T}_W(G, f_M, f_N)$.

Sketch of Proof



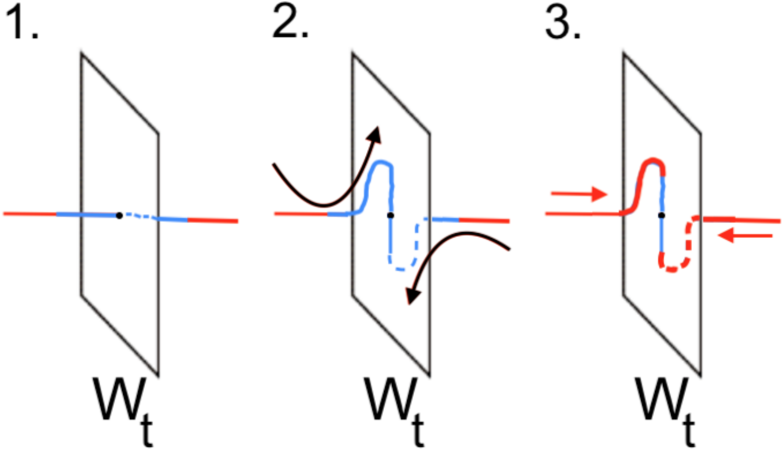
- modify $\boldsymbol{S(F)}$ by precomposing F with automorphism of W
- achieve that $\boldsymbol{\tau^{-1}(t)} \pitchfork \boldsymbol{S(F)}$ for all $t \in \mathrm{Reg}(\tau) \setminus$ **finite set**

Sketch of Proof (continued)



- modify ***F*(*S*(*F*)) slightly** by perturbing F, not changing ***S*(*F*)**
- achieve that $\mathrm{Im} \circ F$ is injective on ***S*(*F*)** $\setminus$ **open intervals**

Sketch of Proof (continued)



- modify $\boldsymbol{S(F)}$ by precomposing F with automorphism of W
- for all $t \in \mathrm{Reg}(\tau) \setminus$ **finite set**, still have $\boldsymbol{\tau^{-1}(t)} \pitchfork \boldsymbol{S(F)}$, and in addition, $\mathrm{Im} \circ F$ is **injective** on $\boldsymbol{\tau^{-1}(t)} \cap \boldsymbol{S(F)}$

Step 3: Quantization

- complete additive monoid $Q = \{\mathrm{Mor}(\mathbf{Br}) \to \mathbb{B}\}$
- write $Q = \bigoplus_{m,n \geq 0} Q_{m,n}$, where $Q_{m,n} = \{\mathrm{Hom}([m],[n]) \to \mathbb{B}\}$
- product "$\cdot$" of **composition semiring** Q^c induced by

$$\cdot\colon Q_{m,p} \times Q_{p,n} \to Q_{m,n}, \quad (f' \cdot f'')(\gamma) = \sum_{\boldsymbol{\gamma = \alpha \circ \beta}} f'(\alpha) f''(\beta)$$

- product "$\times$" of **monoidal semiring** Q^m induced by

$$\times\colon Q_{m,n} \times Q_{r,s} \to Q_{m+r,n+s}, \quad (f' \times f'')(\gamma) = \sum_{\boldsymbol{\gamma = \alpha \otimes \beta}} f'(\alpha) f''(\beta)$$

- comm. monoid $\{t^k;\ k \in \mathbb{N}\} \cong (\mathbb{N}, +, 0)$ acts on $Q_{m,n}$ via

$$(t^k, f) \mapsto (\varphi \mapsto f(\lambda^{\otimes k} \otimes \varphi))$$

- **monoid semiring** $\mathbb{N}[\{t^k;\ k \in \mathbb{N}\}] = \mathbb{N}[t]$
- isomorphism of $\mathbb{N}[t]$-semimodules

$$Q_{m,n} \xrightarrow{\cong} \bigoplus_{\substack{\varphi\colon [m] \to [n] \\ \text{loop-free}}} \mathbb{B}[\![\boldsymbol{q}]\!], \qquad f \mapsto \left(\sum_{k=0}^{\infty} f(\lambda^{\otimes k} \otimes \varphi) \boldsymbol{q}^k \right)_{\varphi}$$

Step 3: Quantization

- **partition function** $Z_W \in Z(\partial W)$ of bordism W^{n+1}:

$$Z_W(f) = \sum_{F \in \mathcal{F}(W;f)} \chi_{\mathbb{T}_W(F)} \quad \in Q_{m(f),m'(f)}, \qquad f \in \mathcal{F}(\partial W)$$

Theorem (Banagl [3], 2015)

If $n+1 \geq 3$, then $Z_W(f)$ is for all $f \in \mathcal{F}(\partial W)$ a rational function

$$Z_W(f) = \frac{P_f(q)}{1-q^2}, \qquad P_f(q) \in \mathbb{B}[\![q]\!] \oplus \cdots \oplus \mathbb{B}[\![q]\!].$$

Theorem (W. [16], 2017)

If $n+1 \geq 2$, then $Z_W(f)$ is for all $f \in \mathcal{F}(\partial W)$ a rational function

$$Z_W(f) = \frac{Q_f(q)}{1-q}, \qquad Q_f(q) \in \mathbb{B}[\![q]\!] \oplus \cdots \oplus \mathbb{B}[\![q]\!].$$

For $n+1 = 2$, $Q_f(q)$ is known [17]. For $n+1 > 2$, $\deg Q_f(q) \leq n$.

Step 4: Linearization

- **Vect** category of real vector spaces and linear maps
- "$\otimes$" Schauenburg tensor product [14]
- $(\mathbf{Vect}, \otimes, \mathbb{R}, b)$ symmetric strict monoidal category
- $\mathbf{C} = (\mathbf{C}, \otimes, I)$ small strict monoidal category
- **linear representation**: strict monoidal functor $Y \colon \mathbf{C} \to \mathbf{Vect}$
- define ***Y*-linearization** $\mathbb{L}$ of **C**-valued action functional $\mathbb{T}$ by

$$\mathbb{L}_W \colon \mathcal{F}(W) \xrightarrow{\mathbb{T}_W} \mathrm{Mor}(\mathbf{C}) \xrightarrow{Y} \mathrm{Mor}(\mathbf{Vect})$$

- $\mathbb{L}$ is **Vect**-valued system of action functionals

Theorem (Müller [11], W. [16], 2015)

All (non-trivial) symmetric linear representations of **Br** *are faithful.*

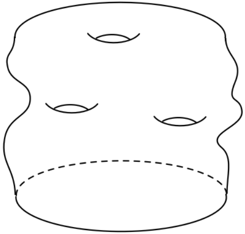
Theorem (Müller-W. [12], 2019)

There exist symmetric linear representations $Y \colon \mathbf{cBr} \to \mathbf{Vect}$.

Aggregate Invariant

- M^n oriented closed n-manifold
- $\mathrm{Cob}(M^n)$ set of all oriented nullbordisms W^{n+1} of M^n

$\exists\ W^{n+1}$ oriented



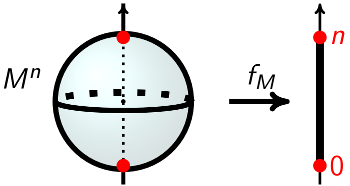
$M^n = \partial W^{n+1}$

- define **aggregate invariant**:

$$\mathfrak{A}(M^n) := \sum_{W^{n+1} \in \mathrm{Cob}(M^n)} Z_W \quad \in Z(M)$$

Application: Detecting Exotic Smooth Spheres

- $n \geq 5$
- [9, 10] **exotic sphere** Σ^n: closed smooth manifold which is homeomorphic, but not diffeomorphic to S^n
- **FACT.** $M^n = S^n$ and $M^n = \Sigma^n$ have Morse number 2:



Theorem (Banagl [3, 4], 2015)

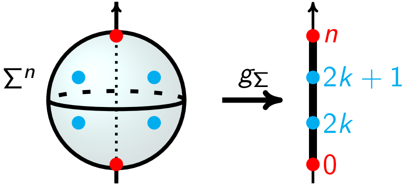
$$M^n \cong S^n \quad \Longleftrightarrow \quad \mathfrak{A}(M^n)(\bar{f}_M) \notin q \cdot Q_{2,2}$$

Proof.

use methods of Saeki [13] based on Stein factorization $\square$

Application: Detecting Exotic Kervaire Spheres

- $n = 4k + 1, \ k \geq 1$
- Σ_K^n: unique **Kervaire sphere** of dimension n (see [8])
- Σ_K^n is exotic whenever $n \notin \{5, 13, 29, 61, 125\}$ (see [7])
- on exotic sphere Σ^n, choose a Morse function

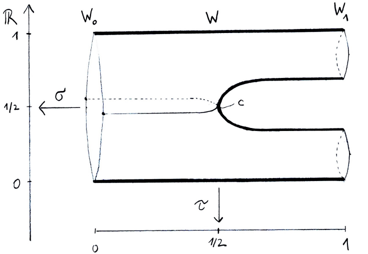


Theorem (W. [16, 18], 2017)

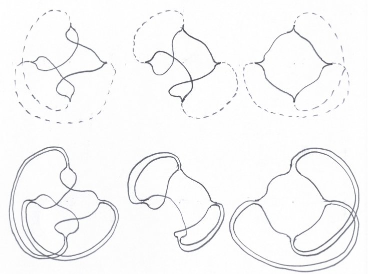
Let $n \geq 237$ and $n \equiv 13 \pmod{16}$. Then, for an exotic sphere Σ^n,

$$\Sigma^n \cong \Sigma_K^n \quad \Longleftrightarrow \quad \mathfrak{A}(\Sigma^n)(\overline{g}_\Sigma) \ \notin \ q \cdot Q_{2,2}$$

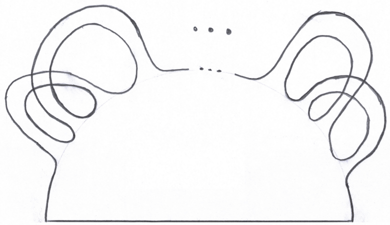
Proof – Main Ingredients



Handle-Extension Thm. (Gay-Kirby)



Cusp Elimination (Levine) & Cretaion



Stein Factorization

Two-Index Thm. (Cerf-Hatcher-Wagoner)

Thank you for your attention!

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