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"The logarithms of Dehn twists"

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$g \geqq 1$

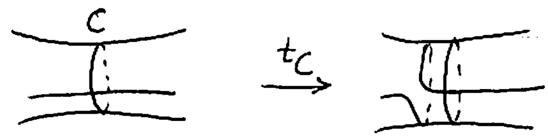
$\Sigma = \Sigma_{g,1} =$ 

$* \in \partial\Sigma$

$\pi = \pi_1(\Sigma, *)$ free group of rank $2g$

$H = H_1(\Sigma ; \mathbb{Q})$

<u>Dehn twist</u> $C \subset \Sigma$ simple closed curve (SCC)



$t_C \in MCG(\Sigma \text{ rel } \partial\Sigma)$ right handed Dehn twist

<u>transvection formula</u> (Picard−Lefschetz formula)

$\forall X \in H$

$$t_{C*}X = X - (X \cdot [C])[C] \in H$$

$$t_{C*} = 1_H - [C]^{\otimes 2} \in \mathrm{Hom}(H, H) = H^* \otimes H \overset{\text{P.d.}}{=} H \otimes H$$

$\Longrightarrow$ <u>Goal</u> : describe the action.

$$t_C \curvearrowright \pi \hookrightarrow \mathbb{Q}\pi \hookrightarrow \widehat{\mathbb{Q}\pi} := \varprojlim_{m\to\infty} \mathbb{Q}\pi / I\pi^m$$

$I\pi = \{ \sum_{x\in\pi} a_x x \in \mathbb{Q}\pi ;\ \sum a_x = 0 \}$ augmentation ideal

<u>Coordinate on</u> $\widehat{\mathbb{Q}\pi}$ ---- symplectic expansion (Massuyeau)

$\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ completed tensor algebra.

$\widehat{T}_p := \prod_{m \geqq p} H^{\otimes m} \subset \widehat{T}$, $p \geqq 1$, two-sided ideal

$\Delta : \widehat{T} \to \widehat{T} \hat{\otimes} \widehat{T}$ coproduct $\left(\Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X \quad (\forall X \in H) \right)$

$$\widehat{T} \hat{\otimes} \widehat{T} = \varprojlim_{m\to\infty} \left(\widehat{T} \otimes \widehat{T} \Big/ \bigoplus_{p+q \geqq m} \widehat{T}_p \otimes \widehat{T}_q \right)$$

$\Longrightarrow$ $\widehat{T}$: complete Hopf algebra

<u>Definition</u> (Massuyeau)

$\theta : \pi \to \hat{T}$ symplectic expansion

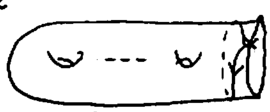
$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\forall x, \forall y \in \pi \quad \theta(xy) = \theta(x)\theta(y)$

2) $\forall x \in \pi \quad \theta(x) = 1 + [x] + \text{higher terms}$

3) $\forall x \in \pi \quad \Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$ group-like

4) $\theta(\xi) = \exp(\omega) \left(= \sum_{k=0}^{\infty} \frac{1}{k!} \omega^k\right)$

where



$\xi \in \pi$

$\omega = \sum_{i=1}^{g} A_iB_i - B_iA_i \in H^{\otimes 2}$ symplectic form

$\{A_i, B_i\} \subset H$ symplectic basis

<u>Examples</u> (1) (K.) harmonic Magnus expansions/$\mathbb{R}$ ($\Leftarrow$ harmonic forms, Green operator)

(2) (Massuyeau) LMO expansions ($\Leftarrow$ Lê-Murakami-Ohtsuki functor)

(3) (Kuno) combinatorial symplectic expansions ($\Leftarrow$ free generators on π)

(4) (Bene-K-Kuno-Penner) symplectic fatgraph expansions

$\theta : \pi \to \hat{T}$ symplectic expression

$\Longrightarrow$ extends to

$$\begin{array}{ccc} \widehat{\mathbb{Q}\langle\xi\rangle} & \hookrightarrow & \widehat{\mathbb{Q}\pi} \\ \downarrow\wr & \circlearrowleft & \downarrow\wr \\ \mathbb{Q}[[\omega]] & \hookrightarrow & \hat{T} \end{array}$$

isomorphism of pairs of complete Hopf algebras

$\forall \varphi \in MCG(\Sigma \operatorname{rel} \partial\Sigma)$

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi} & \xrightarrow[\cong]{\theta} & \hat{T} \\ \varphi \downarrow & & \downarrow\wr \ \exists! \ T^\theta(\varphi) \in Aut(\hat{T}) \\ \widehat{\mathbb{Q}\pi} & \xrightarrow[\cong]{\theta} & \hat{T} \end{array}$$

the total Johnson map (K.)

$T^\theta : MCG(\Sigma, \partial\Sigma) \to Aut(\hat{T})$ <u>injective</u> homomorphism

$T^\theta(\varphi) \circ \varphi_*^{-1}|_H = 1_H + \sum_{k=1}^{\infty} \tau_k^\theta(\varphi), \quad \tau_k^\theta(\varphi) \in H^* \otimes H^{\otimes(k+1)} \overset{\text{P.d.}}{=} H^{\otimes(k+2)}$

$\tau_k^\theta : MCG(\Sigma \operatorname{rel} \partial\Sigma) \to H^{\otimes(k+2)}$

an extension of the k^{th} Johnson homomorphism.

<u>Our result</u>: an "explicit" formula for $T^\theta(t_C)$

<u>Invariant of an unoriented loop on Σ</u> : the logarithm of a Dehn twist.

$\ell = \ell^{\theta} := \log \theta : \pi \to \hat{\mathcal{L}} := \{u \in \hat{T} ; \Delta u = u \hat{\otimes} 1 + 1 \hat{\otimes} u\}$

$$\log \theta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\theta(x) - 1)^n \in \hat{T}$$

$$N : \hat{T} \to \hat{T}_1 = H \otimes \hat{T} \overset{\text{P.d.}}{=} H^* \otimes \hat{T} = \mathrm{Der}(\hat{T})$$

$$\left(\begin{array}{l} N|_{H^{\otimes 0}} = 0 \\ N(X_1 \cdots X_p) := \sum_{i=1}^{p} X_i \cdots X_p X_1 \cdots X_{i-1}, \quad p \geq 1, \ X_j \in H \end{array} \right.$$

cyclic symmetrizer

$$\left(N(\hat{T}) = N(\hat{T}_1) = \mathrm{Der}_{\omega}(\hat{T}) := \{ D \in \mathrm{Der}(\hat{T}) ; D\omega = 0 \} \right)$$

$x \in \pi$

$$L(x) = L^{\theta}(x) \overset{\text{def}}{=} \frac{1}{2} N(\ell(x)\ell(x)) = N\theta\left(\frac{1}{2}(\log x)^2\right) \in \mathrm{Der}_{\omega}(\hat{T})$$

$$\frac{1}{2}(\log x)^2 \in \widehat{\mathbb{Q}\pi}$$

$$L(xyx^{-1}) = L(y)$$

$$\left(\because \ N(uv) = N(vu) \quad (\forall u \ \forall v \in \hat{T}) \right)$$

$$L(x^{-1}) = L(x)$$

$$\left(\because \ \ell(x^{-1}) = -\ell(x) \right)$$

$\Longrightarrow$ L : an invariant of an unoriented loop on Σ

$$L(C) \in \mathrm{Der}_{\omega}(\hat{T}) \quad (C \subset \Sigma \text{ closed curve})$$

<u>Theorem</u> (Kuno-K.) $\theta : \pi \to \hat{T}$ symplectic expansion.

$C \subset \Sigma$ SCC.

$$T^{\theta}(t_C) = e^{-L(C)} \left(= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L(C)^k \right) \text{ on } \hat{T} \ (\overset{\theta}{\cong} \widehat{\mathbb{Q}\pi})$$

in other words

$$-L(C) = \log T^{\theta}(t_C)$$

<u>low-degree cases</u> $L(C) = \sum_{m=2}^{\infty} L_m$, $L_m \in H^{\otimes m}$ of degree $m-2$ as derivations of $\hat{T}$

(non-separating cases)

<u>degree 0</u> $L_2 = [C]^{\otimes 2} \in H^{\otimes 2} \overset{\text{P.d.}}{=} \mathrm{Hom}(H, H)$

$$t_{C*} = 1_H - L_2 \in \mathrm{Hom}(H, H) \quad (\text{transvection formula})$$

<u>degree 1</u> (Kuno's original formula) C = conj. class of $x \in \pi$

$$\tau_1^{\theta}(t_C) = -L_3 = -[x] \wedge \ell_2(x) \in \Lambda^3 H$$

(matches Morita's explicit computations)

$$\left(\ell_2(x) = (\text{the degree 2 part of } \ell(x)) \in \Lambda^2 H \right)$$

<u>degree 2</u>

$$\tau_2^{\theta}(t_C) = -L_4 + \frac{1}{2}[L_2, L_4] + \frac{1}{2} L_3^2$$

(separating cases)

<u>degree 2</u> (Morita)

$$\tau_2(t_C) = -L_4$$

<u>degree 3</u>

$$\tau_3^\theta(t_C) = -L_5.$$

<u>Nilpotent quotient</u>

$\Gamma_k = \Gamma_k(\pi)$; lower central series of π, $k \geq 1$

$\Gamma_1 = \pi$, $\quad \Gamma_{k+1} = [\Gamma_1, \Gamma_k]$

$N_k := \pi / \Gamma_{k+1}$ the k^{th} nilpotent quotient.

$\theta \bmod \hat{T}_{k+1} : N_k \hookrightarrow 1 + \hat{T}_1 / 1 + \hat{T}_{k+1}$ injective homomorphism

$\ell(x) = \sum_{k=1}^{\infty} \ell_k(x)$, $\quad \ell_k(x) \in \hat{\mathcal{L}} \cap H^{\otimes k}$, $\quad \ell_1(x) = [x]$

$L_m = \frac{1}{2} \sum_{k=1}^{m-1} N(\ell_k(x)\ell_{m-k}(x))$ depends only on $\ell_1(x), \cdots, \ell_{m-1}(x)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 〃 on $\theta \bmod \hat{T}_m$

$\Rightarrow \quad L_m : N_{m-1} \to H^{\otimes m}$

<u>Corollary 1</u> $\quad C$: SCC on Σ. $\quad C =$ conj. class of $x \in \pi$

The action of t_C on N_k depends only on the conjugacy class of x in N_k —

<u>Corollary 2</u>

One can define the "Dehn twist" along a <u>non-simple</u> CC on $\widehat{\mathbb{Q}\pi}$ by $e^{-L(C)}$ —

<u>Idea of Proof of Theorem</u> : geometric interpretation of derivations of $\hat{T}$

$$N(\hat{T}_1) = \mathrm{Der}_\omega(\hat{T}) \curvearrowright \hat{T}$$

$\hat{\pi} := \pi/\text{conj} = [S^1, \Sigma] = \{\text{free loops on } \Sigma\}/\simeq$

$| \ | : \pi \to \hat{\pi}$, $\quad x \mapsto |x|$ $\quad$ quotient map

$$\begin{tikzcd} \mathbb{Q}\pi \arrow[r, "N\theta"] \arrow[d, "| \ |"'] \arrow[dr, phantom, "\circlearrowleft"] & N(\hat{T}_1) = \mathrm{Der}_\omega(\hat{T}) \\ \mathbb{Q}\hat{\pi} \arrow[ur, dashed, "\exists ! \, \lambda_\theta"'] & \end{tikzcd} \quad (\because \ N\theta(xyx^{-1}) = N\theta(y))$$

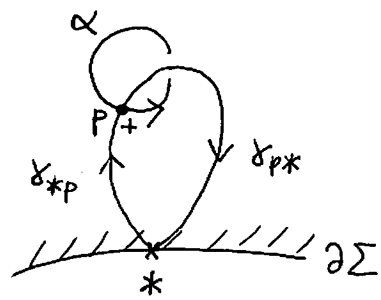
$\mathbb{Q}\hat{\pi}$: Goldman Lie algebra of Σ

Theorem (Kuno–K.) θ : symplectic expansion

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\pi & \xrightarrow{\sigma} & \mathbb{Q}\pi \\ {\scriptstyle -\lambda_\theta \otimes \theta} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \theta} \\ \mathrm{Der}_\omega(\hat{T}) \otimes \hat{T} & \underset{\text{derivation}}{\longrightarrow} & \hat{T} \end{array}$$

where $\alpha \in \hat{\pi}$, $\gamma \in \pi$ in general position

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p : \alpha, \gamma)\, \overrightarrow{\gamma_{*p}\, \alpha_p\, \gamma_{p*}}$$



$\varepsilon(p : \alpha, \gamma) = \pm 1$, local intersection number

$\alpha_p \in \pi_1(\Sigma, p)$ loop along α with basepoint p

γ_{*p} path from $*$ to p $\quad$ γ_{p*} path from p to $*$

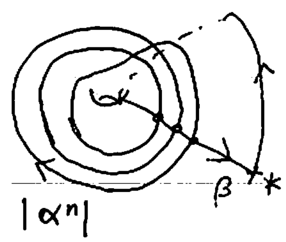
$\sigma : \mathbb{Q}\hat{\pi} \to \mathrm{Der}(\mathbb{Q}\pi)$ Lie algebra homomorphism

example

$\forall n \geq 0$

$$\sigma(|\alpha^n|)(\beta) = n\beta\alpha^n$$

($\because$)



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$f(x)$: formal power series in $x-1$

$$\sigma(|f(\alpha)|)(\beta) = \beta\alpha f'(\alpha) \in \widehat{\mathbb{Q}\pi}$$

On the other hand,

$$\begin{cases} t_\alpha(\alpha) = \alpha \\ t_\alpha(\beta) = \beta\alpha \end{cases} \rightsquigarrow \text{"}(\log t_\alpha)\text{"}(\beta) = \beta \log \alpha$$

$$\log x = x f'(x)$$

$$\Rightarrow \quad f(x) = \int_1^x \frac{1}{t} \log t \, dt = \frac{1}{2}(\log x)^2$$

$$N\theta\left(\tfrac{1}{2}(\log x)^2\right) = \tfrac{1}{2} N(\ell(x)\ell(x)) = L^\theta(x) \quad //$$