A Zariski-van Kampen presentation of elliptic Artin group

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(joint work with Kyoji Saito)

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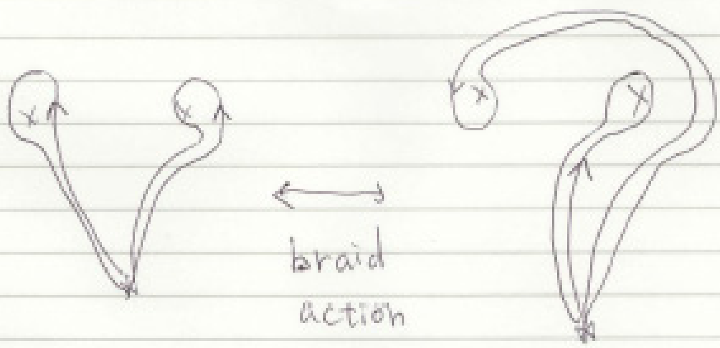
§1

In this talk, we will give a Zariski-van Kampen presentation of fundamental groups of the complement of discriminant divisors of semi-versal deformations for simply elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (introduced by Kyoji Saito (1974)).

When the fundamental groups are presented by using Zariski-van Kampen method, the generator system is called a Zariski-van Kampen generator and the presentation is called a Zariski-van Kampen presentation.

Of course, there exists an ambiguity of a choice of a generator system, where any two Zariski-van Kampen generator systems can be transformed to each other by an action of braid.



After some attempt to change generator system, we would like to give a Zariski-van Kampen presentation correspond-ing to "the elliptic Dynkin diagram (explamed in §3).

However, we have not been able to take a suitable generator system yet.

§2 discriminant divisor

Simply elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ are hypersurface singularities introduced by K. Saito (1974).

$$\widetilde{E}_6 : f_{\widetilde{E}_6} := x^3 + y^3 + z^3 - \lambda xyz \qquad (\lambda^3 \neq 27)$$
$$\widetilde{E}_7 : f_{\widetilde{E}_7} := x^4 + y^4 + z^2 - \lambda xyz \qquad (\lambda^4 \neq 64)$$
$$\widetilde{E}_8 : f_{\widetilde{E}_8} := x^6 + y^3 + z^2 - \lambda xyz \qquad (\lambda^6 \neq 432)$$

Since for each λ the fundamental group π_1 of complement of discriminant divisor is isomorphic, we assume that $\lambda = 0$.

$X \in \{\tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$.

Thm (Kas - Schlessinger)

The semi-versal deformation of X is given by

$$\begin{array}{ccc} \mathcal{X} := \left\{ (x, y, z, \underline{t}) \in \mathbb{C}^3 \times \mathbb{C}^\mu \;\middle|\; f_X + \sum_{i=1}^{\mu} t_i p_i = 0 \right\} & \ni & (x, y, z, \underline{t}) \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{C}^\mu & \ni & \underline{t} \end{array}$$

where the p_i determine a $\mathbb{C}$ basis of the vector space $\mathbb{C}[x, y, z] \big/ \left(\frac{\partial f_X}{\partial x}, \frac{\partial f_X}{\partial y}, \frac{\partial f_X}{\partial z} \right)$

and $\mu = \dim_{\mathbb{C}} \left(\mathbb{C}[x, y, z] \big/ \left(\frac{\partial f_X}{\partial x}, \frac{\partial f_X}{\partial y}, \frac{\partial f_X}{\partial z} \right) \right)$.

Due to this theorem, we can give an explicit form of P_i.

e.g. $\widetilde{E}_6$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
1	x	y	z	xy	yz	zx	xyz

$\dashv$

We explain how to compute the explicit form of defining eq. of discriminant divisors.

$$F_X(x,y,z,\underline{t}) := f_X + \sum_{i=1}^{\mu} t_i P_i$$

$$\begin{cases} F_X(x,y,z,\underline{t}) = 0 \\ \dfrac{\partial F_X}{\partial x}(x,y,z,\underline{t}) = 0 \\ \dfrac{\partial F_X}{\partial y}(x,y,z,\underline{t}) = 0 \\ \dfrac{\partial F_X}{\partial z}(x,y,z,\underline{t}) = 0 \end{cases}$$

By eliminating the valiables x, y, z ,

we obtain the defining eq. $\Delta_X(\underline{t})$

of the discriminant divisor for X.

We give suitable weights to the

variables t_i so that $\Delta_X(\underline{t})$ is a

weighted homogeneous polynomial.

$$D_X := \{ \underline{t} \in \mathbb{C}^\mu \mid \Delta_X(\underline{t}) = 0 \}.$$

We call the isomorphic class of

$\pi_1(\mathbb{C}^\mu \setminus D_X)$ <u>elliptic Artin group</u>.

§3 Y. Saito - M. Shiota presentation

prehistory

In 1983, H van der Lek gave a presentation of $\pi_1\left(W_X \backslash V_{\mathbb{C}}^{reg.}\right)$ using affine Dynkin diagrams. Moreover, in 1991 Hiroshi Yamada gave an another presentation of $\pi_1\left(W_X \backslash V_{\mathbb{C}}^{reg.}\right)$ in terms of the elliptic Dynkin diagrams. After that, Yoshihisa Saito and Midori Shiota arranged the presentations such that they correspond to the elliptic Dynkin diagrams.

K. Saito defined a notion of the marked elliptic root systems that is a genera lization of finite or affine root systems.

Attaching each marked elliptic root system he introduced a diagram, so called the elliptic Dynkin diagram.

Question Why do we attempt to give a Zariski-van Kampen presentation?

→ In the case of Arnold's 14 exceptional singularities, there is no description of regular orbit space.

Therefore, we have to use Zariski-van Kampen method to give a presentation of π_1 for Arnold's 14 exceptional singularities.

Notation

$$: \quad g_\alpha g_\beta = g_\beta g_\alpha$$

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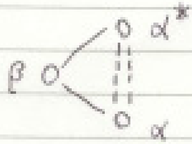
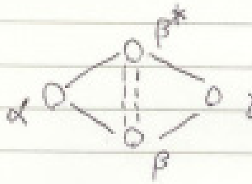


$$: \quad g_\beta g_\alpha g_{\alpha^*} g_\beta g_\alpha g_{\alpha^*} = g_\alpha g_{\alpha^*} g_\beta g_\alpha g_{\alpha^*} g_\beta$$



$$: \quad g_\beta g_{\beta^*} g_\alpha g_\gamma g_\beta g_{\beta^*} g_\gamma = g_\gamma g_\beta g_{\beta^*} g_\gamma g_\alpha g_\beta g_{\beta^*}$$

Thm (Y. Saito - M. Shiota)

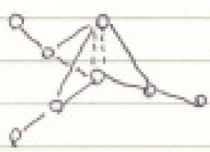
$$\pi_1(\mathbb{C}^\mu \setminus D_X) \cong \left\langle g_1, \cdots, g_\mu \;\middle|\; \begin{array}{l} \text{the associated} \\ \text{relations} \end{array} \right\rangle$$

§4 Main result ($\tilde{E}_6$ case)

compute $\pi_1(\mathbb{C}^8 \setminus D_{\tilde{E}_6})$ using Zariski-van Kampen method.

Due to the Lefschetz hyperplane section theorom, we can take the section as

$$\begin{cases} t_5 = \cdots = t_7 = t_8 = 0 & \left(\begin{array}{l}\text{due to the} \\ \text{weighted homogeneity} \\ \text{of } \Delta_{\tilde{E}_6}(\underline{t})\end{array}\right) \\ t_2 + t_3 + t_4 = 1 & \\ t_2 + \nu t_3 - \dfrac{\nu}{1+\nu} = 0 & \text{where } \nu^2 - \nu + 1 = 0. \end{cases}$$

In order to preserve $\mathbb{Z}/3\mathbb{Z}$ symmetry 

we take the later two sections.

$$\pi_1(\mathbb{C}^8 \setminus D_{\tilde{E}_6}) \cong \pi_1(\mathbb{C}^2 \setminus C_{\tilde{E}_6})$$

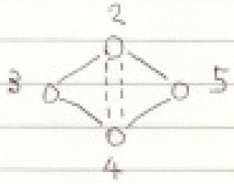
<u>Thm</u> (K. Saito - I)

$$\pi_1(\mathbb{C}^8 \setminus D_{\tilde{E}_6}) \cong \left\langle g_1, \cdots, g_8 \;\middle|\; R_{\tilde{E}_6} \right\rangle$$

$$R_{\tilde{E}_6} = \left\{ \begin{array}{l} 15=51,\ 65=56,\ 85=58, \\ 17=71,\ 37=73,\ 87=78, \\ 264=642=426, \\ 121=212,\ 474=747,\ 282=828, \\ 484=848,\ 252=525,\ 454=545, \\ 141=414,\ 232=323, \\ \underline{4253425=5425342},\ + \text{some relations} \end{array} \right\}$$

<u>Remark</u>

We can detect



two special generators

corresponding to two vertices $\underset{2}{\circ}$, $\underset{4}{\circ}$.

§5 VKCURVE presentation

VKCURVE is name of a software made by David Bessis & Jean Michel.

The software VKCURVE is a computer algebra package that computes a Zariski-van Kampen presentation of the fundamental group of the complement of complex algebraic curves.

The best one for $\widetilde{E}_6$ computed by VKCURVE is the following:

$$\left\langle a, b, c, d, e, f, g, h \;\middle|\; R'_{\widetilde{E_6}} \right\rangle$$

$R'_{\widetilde{E_6}}$

$bd = db$, $ce = ec$, $fg = gf$,

$ada = dad$, $aea = eae$, $afa = faf$,

$beb = ebe$, $bfb = fbf$, $cdc = dcd$,

$cfc = fcf$, $dgd = gdg$, $dhd = hdh$,

$ege = geg$, $hfh = fhf$, $hgh = ghg$,

$cgb = gbc = bcg$, $def = efd = fde$,

$adca = dcad$, $dchd = chdc$,

$eghe = gheg$, $egae = aega$,

$fbaf = afba$, $bhfb = fbhf$,

$h^{-1}chf^{-1}af = f^{-1}afchc^{-1}$, $ada^{-1}ghg^{-1} = ghg^{-1}ada^{-1}$,

$bhb^{-1}e^{-1}ae = e^{-1}aeh^{-1}bh$

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