A formula for the geometric genus of surface singularities

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Contents

- $(X, o) \subset (\mathbb{C}^n, o)$: a normal complex surface singularity.
- Σ: the link of (X, o). $\Sigma = X \cap S_\varepsilon^{2n-1}$ $(0 < \varepsilon \ll 1)$
  We may assume X is a cone over Σ.
- $\pi : \widetilde{X} \to X$: a resolution with exceptional set $E := \pi^{-1}(o)$.
  a) π is proper and $\widetilde{X}$ is smooth,
  b) $\widetilde{X} \setminus E \cong X \setminus \{o\}$,

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Definition (The geometric genus)

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By duality,

$$p_g(X, o) = \dim_{\mathbb{C}} \frac{H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}))}$$

Hypersurfaces (Counting "lower monomials")

Suppose $X = \{f = 0\}$, $f \in \mathbb{C}[x_1, x_2, x_3]$.
$\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Kimio Watanabe, 1980

f is weighted homogeneous of degree d with weights (q_1, q_2, q_3)
$\Rightarrow p_g(X, o) = \#\left\{(a_1, a_2, a_3) \in (\mathbb{Z}_+)^3 \;\middle|\; \sum_{i=1}^{3}(a_i + 1)q_i \leq d\right\}$.

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Merle–Teissier, 1980

f is non-degenerate w.r.t. its Newton boundary
$\Rightarrow p_g(X, o) = \#\left\{a \in (\mathbb{Z}_+)^3 \mid a + (1, 1, 1) \notin \mathrm{Int}(\Gamma_+)\right\}$.

Hypersurfaces (Invariants from resolution process)

Any double point is defined by a function of type $x_1^2 + g(x_2, x_3)$.

Horikawa, 1975

$f = x_1^2 + g(x_2, x_3)$
$\Rightarrow p_g(X, o) = \frac{1}{2} \sum_{i=1}^{n} \gamma_i(\gamma_i + 1)$, where $\gamma_i = \left\lfloor \frac{m_i}{2} \right\rfloor - 1$, m_i is the multiplicity of curve singularity which is the center of i-th blowing up in the embedded resolution process for $\{g = 0\} \subset \mathbb{C}^2$.

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Ashikaga (1999) obtained a formula for $x_1^n + g(x_2, x_3)$.
Tomari (1985) obtained a formula for any hypersurface singularity.

$\{E_v\}_{v\in\mathcal{V}}$: the irreducible components of E.
Then $H_2(\widetilde{X},\mathbb{Z}) = \sum_{v\in\mathcal{V}} \mathbb{Z}E_v$.
Let $Z = \min\{D \in \sum \mathbb{Z}_+E_v \mid D\cdot E_v \le 0,\ \ \forall v \in \mathcal{V}\}$: the Artin cycle.

Definition

For $D \in \sum_{v\in\mathcal{V}} \mathbb{Z}_+E_v$

- $p_a(D) = 1 - h^0(\mathcal{O}_D) + h^1(\mathcal{O}_D) = 1 + \frac{1}{2}D\cdot(D + K_{\widetilde{X}})$

$p_a(Z)$ is independent of the choice of resolution.

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- (Artin, 1966) $p_a(Z) = 0 \Rightarrow p_g = 0$.
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- (Némethi, 1999) (X, o) is Gorenstein, $p_a(Z) = 1$, $\mathbb{Q}$HS link $\Rightarrow p_g =$ the length of the “elliptic sequence”
(S.S.-T Yau introduced the elliptic sequence and gave bounds of these invariants)

Assume that $\pi : \widetilde{X} \to X$ is a good resolution,
i.e., E_v are nonsingular, $E_v \cdot E_w = 1$ or 0 for $v \neq w$.

- $I(E) = (E_v \cdot E_w)$: the intersection matrix of E; it is negative-definite.
- Γ: the weighted dual graph of E
  - $E_v \mapsto$ vertex v;
  - $E_v \cap E_w \mapsto$ edge connectin v and w;
  - vertex v has weight $(E_v \cdot E_v,\ \ g(E_v))$.

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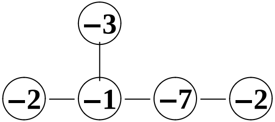
The graph Γ and the link Σ determine each other (Neumann).

- An invariant of a singularity is topological if it is determined by the resolution graph.

In general, Gorenstein condition and p_g are not topological.

The following graph ($\forall$ vertex $\leftrightarrow \mathbb{P}^1$)



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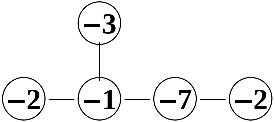

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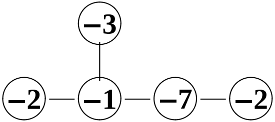

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- X_1: $x^2 + y^3 + z^{13} = 0$, $p_g = 2$
- X_2 (non Gorenstein):
  $$\operatorname{rank}\begin{pmatrix} x & y & z \\ y - 3w^2 & z + w^3 & x^2 + 6wy - 2w^3 \end{pmatrix} < 2, \quad p_g = 1$$

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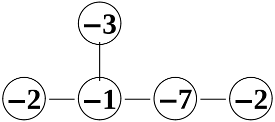

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Main question

When (How) can we compute p_g from the graph?

Let E_v be an arbitrary component of E.

- Let $F_1, \dots, F_m$ be the connected components of $E - E_v$.
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Notation

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Problem

Find a class $\mathcal{R}$ of surface singularities, which satisfies the following: For any (X, o) in $\mathcal{R}$, there exists a good resolution $\widetilde{X} \to X$ such that

- $c(\widetilde{X}, v)$ can be computed from the graph for any $v \in \mathcal{V}$,
- every (X_i, x_i) is in $\mathcal{R}$.

There exisits $E_v^* \in \sum_{i \in \mathcal{V}} \mathbb{Q} E_i$ satisfying $E_v^* \cdot E_i = -\delta_{vi}$ $(\forall i)$.

- Let k be a positive integer such that $kE_v^* \in L$.
- $I_n = H^0(\mathcal{O}_{\widetilde{X}}(-nkE_v^*))$.
- $G := \bigoplus_{n \geq 0} I_n / I_{n+1}$.
- $H_G(t) := \sum_{n \geq 0} (\dim G_n) t^n$: the Hilbert series of G.

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Definition

Let $F(t) = \sum_{i \geq 0} a_i t^i$ be a formal power series. If $\psi(n) := \sum_{i=0}^{nk-1} a_i$ is a polynomial function of n for some $k \in \mathbb{N}$, then the constant term of $\psi(n)$ is independent of $k \in \mathbb{N}$. This constant is called the periodic constant of $F(t)$ and denote it by $\mathrm{pc}\, F$.

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Lemma

Assume $F(t) = p(t) + r(t)/q(t)$, where p, q, and r are polynomials with $\deg r < \deg q$. Then $\mathrm{pc}\, F = p(1)$.

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Let $i \in \mathcal{E}$.

- We say that $\widetilde{X}$ satisfies the weak end curve condition (weak ECC) at E_i if there exist $n \in \mathbb{N}$ and $f \in H^0(\mathcal{O}_{\widetilde{X}})$ such that $H = \mathrm{div}(f) - nE_i^*$ has no component of E.

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- We say that $\widetilde{X}$ satisfies the ECC at E_i if it satisfies the weak ECC at E_i and the divisor H can be chosen as a divisor with irreducible support.
- We say simply that $\widetilde{X}$ satisfies the (weak) ECC if it satisfies the (weak) ECC at E_i for every $i \in \mathcal{E}$.

Proposition

Assume that $\widetilde{X}$ satisfies the weak ECC. Then

1. $p_g(X, o) = \mathrm{pc}\, H_G + \sum_{i=1}^{m} p_g(X_i, x_i)$. *(i.e.,* $c(\widetilde{X}, v) = \mathrm{pc}\, H_G$*)*
2. $\widetilde{X}_i$ *satisfies the weak ECC.*

Proposition

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1. $p_g(X, o) = \mathrm{pc}\, H_G + \sum_{i=1}^m p_g(X_i, x_i)$. *(i.e., $c(\widetilde{X}, v) = \mathrm{pc}\, H_G$)*
2. $\widetilde{X}_i$ *satisfies the weak ECC.*

The point of (1) is that $\mathcal{L}_n := \mathcal{O}_{\widetilde{X}}(-nkE_v^*)$ ($n \gg 0$) has no base points in E.

If $\pi' : \widetilde{X} \to X'$ is the contraction of $E - E_v$, then

$$0 = H^1(\pi'_* \mathcal{L}_n) \to H^1(\mathcal{L}_n) \to H^0(R^1\pi' \mathcal{L}_n) \to 0.$$

From the exact sequence

$$0 \to \mathcal{L}_n \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{nkE_v^*} \to 0$$

we have

$$p_g(X, o) = \dim I_0/I_n - \chi(\mathcal{O}_{nkE_v^*}) + h^1(\mathcal{L}_n).$$

Corollary (Tomari–Watanabe, 1989)

Suppose X is a hypersurface with multiplicity d. Then $p_g(X, o) = \frac{1}{6}d(d-1)(d-2)$ if the blowing up of X at o has only rational singularities. (E.g. superisolated singularities)

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The Hilbert series of the tangent cone is $F(t) = \dfrac{1-t^d}{(1-t)^3}$.

$$\mathrm{pc}\, F = \binom{d}{3} = \frac{1}{6}d(d-1)(d-2)$$

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Theorem (End Curve Theorem (Neumann-Wahl))

(X, o) *is a splice quotient*
$\Leftrightarrow$ *The minimal good resolution of* (X, o) *satisfies the ECC.*

Definition

Let $i \in \mathcal{E}$.

- $\widetilde{X}$ satisfies the weak ECC at E_i if there exist $n \in \mathbb{N}$ and $f \in H^0(\mathcal{O}_{\widetilde{X}})$ such that $H = \mathrm{div}(f) - nE_i^*$ has no component of E.
- $\widetilde{X}$ satisfies the ECC at E_i if it satisfies the weak end curve condition at E_i and the divisor H can be chosen as a divisor with irreducible support.
- $\widetilde{X}$ satisfies the ECC if it satisfies the ECC at E_i for every $i \in \mathcal{E}$.

Splice quotients includes

- weighted homogeneous singularities with $\mathbb{Q}$HS links.
- Rational singularities are splice quotients.
- Minimally elliptic singularities with $\mathbb{Q}$HS links.

Definition

- $L := \sum_{v \in \mathcal{V}} \mathbb{Z} E_v$.
- $L^* := \sum_{v \in \mathcal{V}} \mathbb{Z} E_v^*$ $\quad (E_w \cdot E_v^* = -\delta_{wv})$
- $H := L^*/L$.

  Then $\quad H \cong H_1(\Sigma, \mathbb{Z})$, $\quad |H| = |\det I(E)|$.

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- $H := L^*/L$.
  Then $H \cong H_1(\Sigma, \mathbb{Z})$, $|H| = |\det I(E)|$.
- For $h \in H$,
  $$\theta(h, i) := \exp(2\pi \sqrt{-1}\; h \cdot E_i^*)$$
  where $\cdot$ is $(L^*/L) \times L^* \to \mathbb{Q}/\mathbb{Z}$.
- $(m_{ij}) = |H|(-I(E))^{-1}$
- $$Z^{SW}_{\Gamma,w}(t) = \frac{1}{|H|} \sum_{h \in H} \prod_{i \in \mathcal{V}} (1 - \theta(h, i)\, t^{m_{wi}})^{\delta_i - 2}.$$
  $\delta_i = (E - E_i) \cdot E_i$.

Assume (X, o) is splice quotient.

Theorem (O, 2008)

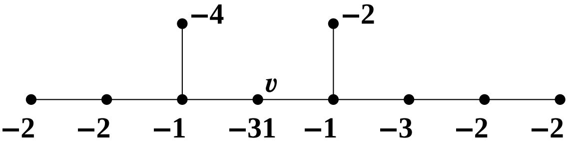
1. $p_g(X, o) = \mathrm{pc}\, Z^{SW}_{\Gamma,w} + \sum_{i=1}^{m} p_g(X_i, x_i)$.
2. *Each* (X_i, x_i) *is a splice quotient*

Corollary

If Γ *is a star-shaped graph with center* v*, then* $p_g(X, o) = \mathrm{pc}\, Z^{SW}_{\Gamma,w}$.

Example

The following graph is realized by a Gorenstein splice quotient.



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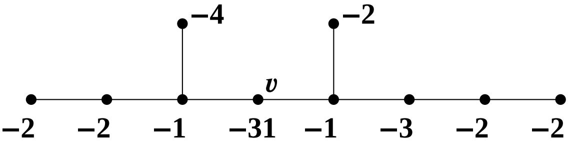

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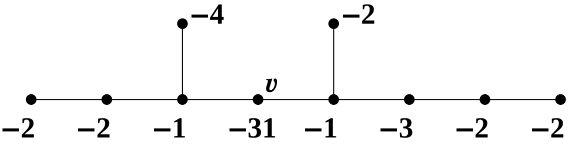

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$$p_g = 6 + 1 + 1 = 8$$

Casson Invariant Conjecture

- Assume (X, o) is complete intersection.
- F: the Milnor fiber of (X, o). (Then $\Sigma \approx \partial F$)
- $\sigma(F)$: the signature.
- $\lambda(\Sigma)$: the Casson invariant.

Casson Invariant Conjecture (Neumann-Wahl)

Assume that $H_1(\Sigma, \mathbb{Z}) = 0$. Then $\lambda(\Sigma) = \sigma(F)/8$.

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By Laufer–Durfee, CIC $\Leftrightarrow$ $p_g(X, o) + \lambda(\Sigma) + \dfrac{K^2 + s}{8} = 0$.

$K = K_{\widetilde{X}}, \quad s = \#\mathcal{V}.$

A generalization of CIC

In the following, we do not assume that $H = \{0\}$, the existence of smoothing.

Seiberg–Witten Invariant Conjecture (Némethi–Nicolaescu)

$$p_g(X, o) + \mathrm{sw}(\Sigma) + \frac{K^2 + s}{8} \leq 0.$$

If (X, o) is $\mathbb{Q}$-Gorenstein, then "=" holds.

$H_1(\Sigma, \mathbb{Z}) = 0 \Rightarrow \mathrm{sw}(\Sigma) = \lambda(\Sigma)$ (Casson invariant).

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Theorem (Némethi–Nicolaescu)

SWIC is true for quotient singularities, weighted homogeneous singularities, and $z^n = g(x, y)$.

A recursion formula

For (X_i, x_i), define Σ_i, s_i, K_i^2 as Σ, s, K^2.

Theorem (Braun–Némethi, 2010)

$$\mathrm{sw}(\Sigma) + \frac{K^2 + s}{8} = -\mathrm{pc}\, Z_{\Gamma,w}^{SW} + \sum_{i=1}^{m} \left(\mathrm{sw}(\Sigma_i) + \frac{K_i^2 + s_i}{8} \right).$$

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$p_g(X, o) = \mathrm{pc}\, Z^{SW}_{\Gamma,w} + \sum_{i=1}^{m} p_g(X_i, x_i)$.

Corollary

SWIC and CIC are true for splice quotients.

A recursion formula

For (X_i, x_i), define Σ_i, s_i, K_i^2 as Σ, s, K^2.

Theorem (Braun–Némethi, 2010)

$$\mathrm{sw}(\Sigma) + \frac{K^2 + s}{8} = -\mathrm{pc}\, Z^{SW}_{\Gamma,w} + \sum_{i=1}^{m} \left(\mathrm{sw}(\Sigma_i) + \frac{K_i^2 + s_i}{8} \right).$$

$$p_g(X, o) = \mathrm{pc}\, Z^{SW}_{\Gamma,w} + \sum_{i=1}^{m} p_g(X_i, x_i).$$

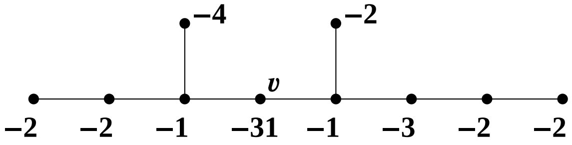
Corollary

SWIC and CIC are true for splice quotients.

There are counterexamples for SWIC; no for CIC.

A counterexample for SWIC (Luengo-Velasco, Melle-Hernández, and Némethi)

The following graph is realized by a superisolated singulaity



Then $-Z^2 = 5$,

$$p_g = \frac{5(5-1)(5-2)}{6} = 10.$$

($p_g = 8$ if it is splice quotient)

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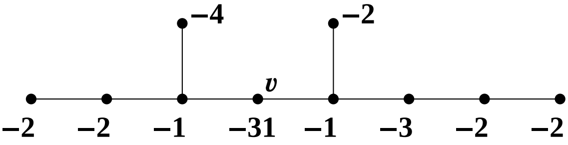

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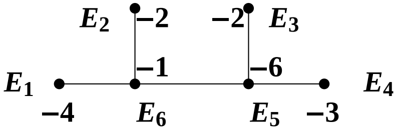
Problem

Characterize singularities satisfying SWIC (or $\mathrm{pc}\, Z^{SW} = \mathrm{pc}\, H_G$).
CIC is true?

Non splice quotient

The following graph is not realized by a splice quotient, but realized by

- a hypersurface singularity with $p_a(Z) = 1$,
- its minimal good resolution satisfies the weak ECC.



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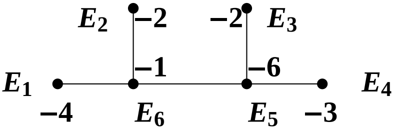

$$p_g = \mathrm{pc}\, Z^{SW}_{\Gamma,5} = \mathrm{pc}\, Z^{SW}_{\Gamma,6} = 2$$

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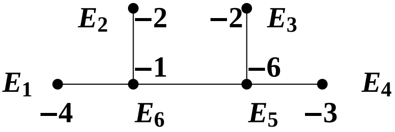

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The class of singularities satisfying SWIC is bigger than the class of splice quotients.