The harmonic product of $\delta(x_1,\ldots,x_n)$ and $\delta(x_1)$ and two combinatorial identities

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Abstract. In the framework of nonstandard analysis, Bang-He Li and the author defined the product of any two distributions on R^n via their harmonic representations. The product of $\delta(x_1,\ldots,x_n)$ and $\delta(x_1)$ was calculated by Kuribayashi and the author in [LK]. In this paper, the result of [LK] is improved to

$$\delta(x_1,\ldots,x_n) \circ \delta(x_1) = \frac{1}{2\pi\rho}\delta(x_1,\ldots,x_n) \mod \text{ infinitesimals}$$

where ρ is a positive infinitesimal. Moreover two combinatorial identities are obtained as byproducts.

Thirty years ago, Bremermann and Durand [BD] defined the products of distributions with one variable by using analytic representations. It was shown by Itano [I1][I2] and further by Bang-He Li and the author [LL1] that this multiplication is very broad, i.e. if the product of two distributions exists for several other multiplications, then the same product is obtained for this multiplication. So a problem that interested people was "What is a generalization of this multiplication to distributions with several variables?".

Itano [I1] showed by an example that for distributions with more than one variables, analytic representation can not offer well-defined multiplication. Bang-He Li and the author [LL2] at last found that a suitable generalization of this multiplication is the one via harmonic representations; For one variable, harmonic and analytic representations are essentially the same. Bang-He Li [L] adapted the multiplication of Bremermann and Durand into the framework of nonstandard analysis. Its generalization to multiple variables via harmonic representations [LL2] was also written in this framework. The merit to use nonstandard analysis is that one needs not to worry about the existence of products anymore, and when taking the finite part (if exists), we return to some kind of standard product.

In the case of one variable, systematic results for singular distributions have been obtained by the author. We quote only the survey paper [LL4]. For multiple variables, only few calculations have been made in [LL4] and the paper of Kuribayashi and the author [LK]. In this paper, we improve the result of [LK] to the neatest form.

Denote by $\mathscr{D}(R^n)$ the Schwartz spase consisting of complex-valued C^∞-functions on R^n with compact supports, and $\mathscr{D}'(R^n)$ its dual, i.e. the space of Schwarts distributions.

For $T \in \mathscr{D}'(R^n)$, its harmonic representation $\hat{T}$ is a harmonic function on $R^n \times R_+$ such that

$$\lim_{y \to 0_+} \int_{R^n} \hat{T}(x_1, \ldots, x_n, y)\phi(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$= \int_{R^n} T(x_1, \ldots, x_n)\phi(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

for any $\phi(x_1, \ldots, x_n) \in \mathscr{D}(R^n)$.

Harmonic representation exists for any $T \in \mathscr{D}'(R^n)$, and the difference of two such representations extends to a harmonic function on $R^n \times R$ skew-symmetric for y (cf. [LL3]).

Denote by C the complex field, $^*C$ a nonstandard model of C, R the real field and $\rho \in {}^*R$ a positive infinitesimal. Let

$$^\rho C = \{x \in {}^*C | \text{for some finite integer } n,\ |x| < \rho^{-n}\},$$

$$\theta = \text{the set of all infinitesimals in } {}^*C \text{ and}$$

$$^\rho C' = {}^\rho C / \theta.$$

Then $^\rho C'$ is a complex vector space, and we call a complex linear functional of $\mathscr{D}(R^n) \to {}^\rho C'$ a hyperdistribution on R^n.

Suppose $S, T \in \mathscr{D}'(R^n)$, $\hat{S}$, $\hat{T}$ are harmonic representations of S and T, and $^*\hat{S}$, $^*\hat{T}$ are the nonstandard extensions of $\hat{S}$ and $\hat{T}$ respectively.

Let $\psi : {}^\rho C \to {}^\rho C'$ be the homomorphism modulo θ. Then

$$\varphi \to \psi(\langle \hat{S}(x, \rho)\hat{T}(x, \rho), \varphi(x)\rangle)$$

defines a complex linear functional of $\mathscr{D}(R^n) \to {}^\rho C'$, i.e., a hyperdistribution on R^n, and we call this hyperdistribution the harmonic product of S and T, denoted by $S \circ T$ in [LL1] (see also [O]).

Let $\delta(x_1, \ldots, x_n)$ be the δ-function on R^n. The harmonic product of $\delta(x_1, \ldots, x_n)$ and $\delta(x_1)$ has been calculated in [LK] to get

$$\delta(x_1, \ldots, x_n) \circ \delta(x_1) = A(1, n)\delta(x_1, \ldots, x_n)$$

where

$$A(1,n) = \frac{2\pi}{c_1 c_n \rho} \prod_{j=1}^{n-3} \int_0^{\pi} \sin^j \theta \, d\theta \int_0^{\infty} \int_0^{\pi} \frac{t^n \sin^{n-2}\xi \, dt d\xi}{(1+t^2)^{(n+1)/2}(1+t^2 \cos^2 \xi)}$$

and

$$c_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\dfrac{n+1}{2}\right)}.$$

Furthermore, for odd n,

$$A(1,2k+1) = \frac{2k}{\rho\pi}\left(\frac{1}{4} + \sum_{j=1}^{k-1}\sum_{p=0}^{j-1}\binom{k-1}{j}\binom{j-1}{p}\frac{(-1)^j(2p+1)!!(2j-2p-1)!!}{(2j+2)!!(2p+1)}\right)$$

and for even n,

$$A(1,2k+2)$$

$$= \frac{2k+1}{2\rho\pi}\left(1 + \sum_{j=0}^{k-1}\sum_{r=0}^{j}\sum_{p=0}^{j-r}\sum_{s=0}^{r}\sum_{h=0}^{p+s}\binom{2k}{k+1+j}\binom{j-r}{p}\binom{r}{s}\binom{p+s}{h}\right.$$

$$\left. \times \frac{(-1)^{j+p+r+1}\Gamma(k-j)\Gamma(1-h+(p+s+j+1)/2)}{2^{2k-2+p+s-j}\Gamma(k-h+(p+s-j+3)/2)}\right)$$

where $\Gamma(x)$ is the Gamma function.

It was also calculated in [LK] that

$$A(1,2k+1) = \frac{1}{2\pi\rho}, \qquad \text{for } k = 1,2,3,4,5,6$$

and

$$A(1,2k+2) = \frac{1}{2\pi\rho}, \qquad \text{for } k = 0,1,2,3.$$

It is a result of Bang-He Li [L] that

$$A(1,1) = \frac{1}{2\pi\rho}, \quad \text{i.e., } \delta(x_1) \circ \delta(x_1) = \frac{1}{2\pi\rho}\delta(x_1)$$

(see also [O] [DR]).

So it might be conjectured that $A(1,n) = \dfrac{1}{2\pi\rho}$ for all $n \geq 1$. Here we find a very simple way to prove that it is indeed so, i.e., we have

THEOREM 1.   *For any* $n \in N$,

$$\delta(x_1, \ldots, x_n) \circ \delta(x_1) = \frac{1}{2\pi\rho}\delta(x_1, \ldots, x_n)$$

PROOF.   A harmonic representation of $\delta(x_1, \ldots, x_n)$ is the Poisson kernel

$$\hat{\delta}(x_1, \ldots, x_n; y) = c_n^{-1}y(|x|^2 + y^2)^{-(n+1)/2}, \qquad y > 0$$

and a harmonic representation of $\delta(x_1)$ is

$$\hat{\delta}(x_1, y) = c_1^{-1}y(x_1^2 + y^2)^{-1}.$$

For $\phi \in \mathscr{D}(R^n)$, it has been proved in [LK] that

$$\int_{R^n} \hat{\delta}(x_1, \ldots, x_n; \rho)\hat{\delta}(x_1, \rho)\phi(x)dx$$

$$= \phi(0) \int_{R^n} \hat{\delta}(x_1, \ldots, x_n; \rho)\hat{\delta}(x_1, \rho)dx \text{ mod infinitesimals}$$

So, for $n \geq 3$

$$A(1, n) = \frac{\rho^2}{c_1 c_n} \int_{R^n} \frac{dx}{(x_1^2 + x_2^2 + \cdots + x_n^2 + \rho^2)^{(n+1)/2}(x_1^2 + \rho^2)}$$

$$= \frac{\rho^2}{c_1 c_n} \int_{R^{n-2}} \frac{dx_1 \cdots dx_{n-2}}{(x_1^2 + \rho^2)} \int_{R^2} \frac{dx_{n-1}dx_n}{(x_1^2 + x_2^2 + \cdots + x_n^2 + \rho^2)^{(n+1)/2}}.$$

By using polar coordinate, we have

$$\int_{R^2} \frac{dx_{n-1}dx_n}{(x_1^2 + x_2^2 + \cdots + x_n^2 + \rho^2)^{(n+1)/2}}$$

$$= \int_0^{2\pi} d\theta \int_0^\infty \frac{r\,dr}{(x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + \rho^2 + r^2)^{(n+1)/2}}$$

$$= \frac{2\pi}{(n-1)(x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + \rho^2)^{(n-1)/2}}.$$

Since $(n-1)c_n = 2\pi c_{n-2}$,

$$A(1, n) = \frac{\rho^2}{c_1 c_{n-2}} \int_{R^{n-2}} \frac{dx_1 \cdots dx_{n-2}}{(x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + \rho^2)^{(n-1)/2}(x_1^2 + \rho^2)} = A(1, n-2)$$

and the proof is complete by using the known results

$$A(1, 1) = A(1, 2) = \frac{1}{2\pi\rho}.$$

As corollaries of Theorem 1 and the results of [LK], we obtain two combinatorial identities:

PROPOSITION 2.   *For any* $k \in N,\ k \geq 1,$

$$\sum_{j=1}^{k-1}\sum_{p=0}^{j-1}\binom{k-1}{j}\binom{j-1}{p}\frac{(-1)^{j}(2p+1)!!(2j-2p-1)!!}{(2j+2)!!(2p+1)} = \frac{1}{4k} - \frac{1}{4}$$

PROPOSITION 3.   *For any* $k \in N,\ k \geq 1,$

$$\sum_{j=0}^{k-1}\sum_{r=0}^{j}\sum_{p=0}^{j-r}\sum_{s=0}^{r}\sum_{h=0}^{p+s}\binom{2k}{k+1+j}\binom{j-r}{p}\binom{r}{s}\binom{p+s}{h}$$

$$\times\frac{(-1)^{j+p+r+1}\Gamma(k-j)\Gamma(1-h+(p+s+j+1)/2)}{2^{2k-2+p+s-j}\Gamma(k-h+(p+s-j+3)/2)} = \frac{1}{2k+1} - 1$$

Acknowledgement

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