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Spatial homogenization and internal layers in a reaction-diffusion system

Dedicated to Professor Jack Hale for his 70th birthday

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ABSTRACT. For a system of reaction-diffusion equations of activator-inhibitor type, we show that solutions undergo at least three stages of dynamical behaviour when the activator diffuses slowly and reacts fast, and the inhibitor diffuses fast. In the first stage, the inhibitor quickly decays to its spatial average (spatial homogenization of the inhibitor). In the second stage, the activator develops internal layers (formation of internal layers). In the third stage, the layers move according to a certain motion law (motion of interfaces) which is described by a system of ordinary differential equations on finite time intervals. Asymptotic behaviour of the solutions of the interface equation is also analyzed. To describe the behaviour of the solutions of the reaction-diffusion equations after the last interface equation becomes powerless, another type of interface equation is proposed.

1. Introduction

The reaction-diffusion system

$$u_t = d_1 \Delta u + f(u, v), \qquad v_t = d_2 \Delta v + rg(u, v)$$

has been employed to model propagation phenomena of chemical waves in excitable media [6], and to describe pattern formation in an activator-inhibitor model [10]. In this system, $d_1 > 0$, $d_2 > 0$ are diffusion rates of u and v, and $r > 0$ measures the ratio of the reaction rates of u and v. Depending upon the relative magnitude among d_1, d_2 and r, it has been found by many authors that the system above, despite its simplicity, is capable of producing various spatio-temporal patterns such as propagating fronts and localized spatial structures [14]. These studies indicate that various patterns observed in reacting and diffusing systems are produced by the interaction between local reaction kinetics and global diffusion effects. It is therefore important to mathematically study

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the characteristics of the reaction-diffusion system according to the magnitudes of (d_1, d_2) and r.

When u diffuses *very slowly* and v both diffuses and reacts *slowly*, namely, if

$$d_1 = \varepsilon^2, \qquad d_2 = \varepsilon, \qquad r = \varepsilon,$$

with $\varepsilon > 0$ being small, then, rescaling the time by $\varepsilon t \to t$, the system is transformed to

$$(\mathrm{P}): \quad u_t = \varepsilon \Delta u + \frac{1}{\varepsilon} f(u, v), \qquad v_t = \Delta v + g(u, v).$$

For sufficiently small $\varepsilon > 0$, *interfacial phenomena* in this system with appropriate nonlinearity (f, g) are well understood by the results in [3]. Roughly speaking, the results in [3] are summarized as follows: Solutions of the system with suitable initial conditions quickly develop internal layers and the location of the layers (*interfaces*) propagates according to a certain motion law.

In this paper, we will deal with the situation where u diffuses *very slowly* and v reacts *slowly*, namely,

$$d_1 = \varepsilon^2, \qquad d_2 = D, \qquad r = \varepsilon.$$

We always understand throughout this paper that the parameter $\varepsilon > 0$ is sufficiently small and $D = O(1)$ as $\varepsilon \to 0$. By rescaling the time as above, the system is recast as follows.

$$(1.1) \quad \begin{cases} u_t = \varepsilon \Delta u + \dfrac{1}{\varepsilon} f(u, v), \qquad v_t = \dfrac{D}{\varepsilon} \Delta v + g(u, v) \quad \text{for } t > 0, x \in \Omega \subset \mathbf{R}^N \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}}, \qquad \text{for } t > 0, x \in \partial\Omega, \\ u(x, 0) = \phi(x), \qquad v(x, 0) = \psi(x) \quad \text{for } x \in \bar{\Omega}. \end{cases}$$

In this system, Ω is a *bounded* domain with smooth boundary, and $\mathbf{n}$ stands for the unit outward normal vector field on $\partial\Omega$. It is interesting to note that the system (1.1) was derived from the system (P) above by rescaling (x, t) appropriately in [9, 13], in order to capture stable mesoscopic structures.

The aim of this paper is to show that results similar to those in [3] are also valid for (1.1) as described below. One difference in our results from those in [3] is the spatial homogenization of the inhibitor (v) at the initial stage.

We describe heuristically the behavior of solutions of (1.1) by using a typical example of the reaction term (f, g), the Bonfoeffer-van der Pol kinetics:

$$f(u, v) = u - u^3 - v, \qquad g(u, v) = u - v.$$

Note that $f(u,v)=0$ has three solutions

$$u = h^-(v), \qquad h^0(v), \qquad h^+(v) \qquad \text{for } v \in (-2/3\sqrt{3}, 2/3\sqrt{3})$$

where $h^-(v) < h^0(v) < h^+(v)$. When $\varepsilon > 0$ is sufficiently small, the largeness of the diffusion rate of v together with the homonegeous Neumann boundary conditions suggests that $v(x,t)$ rapidly decays to its spatial average $\bar{v}(t) = |\Omega|^{-1}\int_\Omega v(x,t)dx$. On the other hand, due to the bistable nature of the ordinaly differential equation $u_t = \varepsilon^{-1}f(u,v)$ with $u = h^{\pm}(v)$ being stable equilibria for $|v| < 2/3\sqrt{3}$, as long as the diffusion effects $\varepsilon\Delta u$ is negligible, $u(x,t)$ will quickly develop transition layers, i.e., $u(x,t)$ tends either to $h^+(\bar{v}(t))$ or to $h^-(\bar{v}(t))$ according to the sign of $u(x,0) - h^0(v(x,0))$. Subsequently the transition layers get sharper and sharper, and eventually location of the layers is so thin that it is considered as a hypersurface, called an interface. Once the transition layers become sharp enough, the diffusion effect $\varepsilon\Delta u$ is no longer negligible and the interface starts to propagate to keep the two competing forces, the local reaction kinetics and the global diffusion effect, in balance. The propagation law of the interface is derived by using asymptotic expansion methods in [11]. To the lowest order, it is given as in (2.5) below.

In this paper we make mathematically rigorous the intuitive statement above. For this purpose, we now state the conditions that the nonlinearity (f,g) has to satisfy:

(A1): The vector field (f,g) is C^∞ on $\mathbf{R}^2$.

(A2): The system of ordinary differential equations

$$u_t = f(u,v), \qquad v_t = g(u,v)$$

has an invariant rectangle

$$\mathscr{R} := \{(u,v) | a_- \le u \le a_+, b_- \le v \le b_+\}.$$

Here $\mathscr{R}$ is said to be an invariant rectangle if the vector field (f,g) points to the interior of $\mathscr{R}$ on the boundary $\partial\mathscr{R}$.

(A3): The nullcline of f, $\{(u,v) | f(u,v) = 0\}$, has exactly three branches of solutions (numbers $\underline{b}$ and $\bar{b}$ with $b_- < \underline{b} < \bar{b} < b_+$ below are suitable constants):

$$C_0 = \{(u,v) | u = h^-(v), v \ge \underline{b}\},$$

$$C_- = \{(u,v) | u = h^0(v), \bar{b} \ge v \ge \underline{b}\},$$

$$C_+ = \{(u,v) | u = h^+(v), v \le \bar{b}\}$$

and

$$\mathscr{R} \supset [h^-(\bar{b}), h^+(\underline{b})] \times [\underline{b}, \bar{b}].$$

(A4): The following inequalities hold true:

$$f_u(h^-(v), v) < 0 \qquad \text{for } v > \underline{b},$$

$$f_u(h^+(v), v) < 0 \qquad \text{for } v < \bar{b},$$

$$f_v(u, v) \le -\delta_0 < 0, \quad g_u(u, v) \ge 0 \qquad \text{for } (u, v) \in \mathscr{R},$$

$$g(h^-(v^*), v^*) < 0 < g(h^+(v^*), v^*),$$

$$g_v(h^\pm(v^*), v^*) < 0,$$

where v^* is a zero of the function $J(v)$ defined by

$$J(v) := \int_{h^-(v)}^{h^+(v)} f(s, v)ds$$

for $v \in [\underline{b}, \bar{b}]$.

(A5): The value $v^* \in (\underline{b}, \bar{b})$ is a simple zero of $J(v)$, namely, $J'(v^*) < 0$.

REMARK 1.1. Note that **(A4)** implies

$$h_v^\pm(v) < 0 \qquad \text{for } v \in (\underline{b}, \bar{b}) \qquad \text{and} \qquad \frac{d}{dv} g(h^\pm(v), v)\bigg|_{v=v^*} < 0.$$

We state our main results in precise terms in §2. Then §3 is devoted to the proof of the main results. We present in §4 a perspective on our results and future projects.

2. Main results

Our first result says that for $\varepsilon > 0$ sufficiently small, the v-component of the solution of (1.1) quickly decays to its spatial average.

THEOREM 2.1. *Suppose that* **(A1)** *and* **(A2)** *are satisfied. Let the initial condition satisfy*

$$(\phi(x), \psi(x)) \in \mathscr{R} \qquad \text{for } x \in \bar{\Omega},$$

and $\phi, \psi \in C^2(\bar{\Omega})$. *We denote by* $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ *the solution of* (1.1). *There exist an* $\varepsilon_0 > 0$ *and a constant* $c^* = c^*(\varepsilon_0, \phi, \psi, \Omega) > 0$ *so that the following estimates are valid for* $\varepsilon \in (0, \varepsilon_0]$.

$$\max_{x \in \bar{\Omega}} |v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)| \le c^* \left[\|\nabla \psi\|_{L^2(\Omega)}^2 \exp\left[-\frac{D\lambda_1}{\varepsilon} t \right] + \frac{M_0^2 |\Omega|}{D^2 \lambda_1} \varepsilon^2 \right]^{1/(N+1)}$$

$$\max_{x \in \bar{\Omega}} |\nabla v^\varepsilon(x,t)| \le c^* \left[\|\nabla \psi\|_{L^2(\Omega)}^2 \exp\left[-\frac{D\lambda_1}{\varepsilon} t \right] + \frac{M_0^2 |\Omega|}{D^2 \lambda_1} \varepsilon^2 \right]^{3/(2N+3)}$$

for $t \geq 2\varepsilon|\log \varepsilon|/D\lambda_1$, *where*

$$M_0 := \sup\{|g(u,v)|\,|(u,v) \in \mathscr{R}\}, \qquad \bar{v}^\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega v^\varepsilon(x,t)dx$$

and λ_1 *is the least positive eigevalue of*

$$\Delta\phi + \lambda\phi = 0 \quad in\ \Omega, \qquad \frac{\partial\phi}{\partial\mathbf{n}} = 0 \quad on\ \partial\Omega.$$

One can easily see that $\bar{v}^\varepsilon(t)$ satisfies an *ordinary differential equation*

$$\text{(2.1)} \qquad \frac{d}{dt}\bar{v}^\varepsilon(t) = \frac{1}{|\Omega|}\int_\Omega g(u^\varepsilon(x,t), v^\varepsilon(x,t))dx, \qquad \bar{v}^\varepsilon(0) = \frac{1}{|\Omega|}\int_\Omega \psi(x)dx,$$

where $|\Omega|$ stands for the N dimensional volume of Ω. It is immediately verified from (2.1) that $\bar{v}^\varepsilon(t)$ satisfies

$$\sup_{x\in\bar{\Omega}}\left|\bar{v}^\varepsilon(t) - \frac{1}{|\Omega|}\int_\Omega \psi(x)dx\right| \leq M_0 t \qquad (t \geq 0).$$

This estimate, combined with the first estimate in the theorem above, implies that at $t = 2\varepsilon|\log\varepsilon|/D\lambda_1$, we have

$$\sup_{x\in\bar{\Omega}}\left|v^\varepsilon(x,t) - \frac{1}{|\Omega|}\int_\Omega \psi(x)dx\right| = O(\varepsilon^{2/(N+1)})$$

i.e., $v^\varepsilon(x,t)$ decays to the spatial average of the initial function in a short time.

We now deal with the generation of internal layers in u-component of the solutions of (1.1). Although the formation of the internal layers is taking place in the same time scale as the spatial homogenization of $v^\varepsilon(x,t)$, it is not technically so easy to analyze the two phenomena simultaneously. As a first step to carry out such an analysis, we need to know in detail the asymptotic $(t \to \infty)$ form of the solution $(u(x,t), v(x,t))$ of

$$\begin{cases} u_t = f(u,v), \qquad v_t = D\Delta v \qquad \text{for } t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 \qquad \text{for } t > 0,\ x \in \partial\Omega, \\ u(x,0) = \phi(x), \qquad v(x,0) = \psi(x) \qquad \text{for } x \in \bar{\Omega}. \end{cases}$$

It is not so easy a task to determine the asymptotic form of the solution to this equation in terms of the initial distribution (ϕ, ψ).

In the next theorem, we suppress the dynamics of the spatial homogenization in v^ε by choosing a special kind of initial functions for v.

Theorem 2.2. *Suppose that* **(A1)** *through* **(A4)** *are satisfied. Let us fix a small* $\sigma > 0$ *and consider the initial function* $\psi(x) \equiv v_0 \in [\underline{b} + \sigma, \overline{b} - \sigma]$. *Let the* *initial condition satisfy*

$$(\phi(x), v_0) \in \mathscr{R} \qquad \textit{for} \ \ x \in \bar{\Omega}$$

and $\phi \in C^2(\bar{\Omega})$.

There exist constants $\varepsilon_0 > 0$, $\tau_1 > 0$ *and* $M_1 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *the following estimates hold true.*

$$h^-(v_0) - M_1\varepsilon|\log \varepsilon| \leq u^\varepsilon(x, \tau_1\varepsilon|\log \varepsilon|) \leq h^+(v_0) + M_1\varepsilon|\log \varepsilon| \quad \textit{for} \ \ x \in \Omega, \tag{2.2}$$

$$u^\varepsilon(x, \tau_1\varepsilon|\log \varepsilon|) \geq h^+(v_0) - M_1\varepsilon|\log \varepsilon| \quad \textit{for} \ \ x \in \{\phi(x) \geq h^0(v_0) + M_1\varepsilon|\log \varepsilon|\}, \tag{2.3}$$

$$u^\varepsilon(x, \tau_1\varepsilon|\log \varepsilon|) \leq h^-(v_0) + M_1\varepsilon|\log \varepsilon| \quad \textit{for} \ \ x \in \{\phi(x) \leq h^0(v_0) - M_1\varepsilon|\log \varepsilon|\}. \tag{2.4}$$

This theorem says that a sharp interface develops in u-component near the set

$$\Gamma_0 = \{x \in \Omega \mid \phi(x) = h^0(v_0)\}$$

in a short time $t = \tau_1\varepsilon|\log \varepsilon|$. This phenomenon is due to the strong bistability of the ordinary differential equation $u_t = \varepsilon^{-1} f(u, v)$. Note that the fast dynamics of v due to the large diffusivity D/ε in (1.1) is suppressed by the choice of the initial funciton for v.

The next stage in the dynamics of solutions to (1.1) is the propagation of the interfaces. By using the method of matched asymptotic expansions, the interface equation for (1.1) is derived in [11]. To the lowest order it reads as follows:

$$\begin{cases} \dfrac{\partial \gamma(y,t)}{\partial t} \cdot v(y,t) = c(v(t)) \\ \dfrac{d}{dt} v(t) = G^-(v(t)) \dfrac{|\Omega^-(t)|}{|\Omega|} + G^+(v(t)) \dfrac{|\Omega^+(t)|}{|\Omega|} \\ v(0) = v_0, \qquad \gamma(y,0) = \gamma_0(y) \equiv y \in \Gamma_0. \end{cases} \tag{2.5}$$

In (2.5) above, it is understood that the domain Ω is devided into two parts $\Omega^-(t)$ and $\Omega^+(t)$ by an inferface $\Gamma(t) \subset \Omega$. Here for each $t \geq 0$, $\Gamma(t)$ is an $N-1$ dimensional hypersurface parameterized by $\gamma(\cdot, t) : \Gamma_0 \ni u \mapsto \gamma(y,t) \in \Gamma(t)$, and $v(y,t)$ is the unit normal vector field on $\Gamma(t)$ at $x = \gamma(y,t)$ pointing to the interior of $\Omega^+(t)$. The symbol $|\Omega|$ (resp. $|\Omega^\pm(t)|$) stads for the N-dimensional volume of Ω (resp. $\Omega^\pm(t)$). The functions $G^\pm(v)$ are respectively defined by

$$G^{\pm}(v) = g(h^{\pm}(v), v) \qquad \text{for } v \in [\underline{b}, \bar{b}].$$

Finally, $c(v)$ is the wave speed of the parabolic equation $u_t = u_{zz} + f(u, v)$, namely, $c(v)$ is the unique value of c so that the following boundary value problem has a solution:

$$\begin{cases} u_{zz} + cu_z + f(u, v) = 0 & \text{for } z \in \mathbf{R}, \\ u(\pm\infty) = h^{\pm}(v), & u(0) = h^0(v). \end{cases} \tag{2.6}$$

We recast (2.5) as an initial value problem for ordinaly differential equations. Let the initial interface Γ_0 be of C^2 class. In a neighborhood of Γ_0, we introduce a coordinate system (r, y) via

$$\Omega \ni x = y + r\nu(y) \qquad (-r_0 < r < r_0, y \in \Gamma_0),$$

where $\nu(y)$ is the unit normal vector field on Γ_0 at y pointing to the interior of $\Omega^+(0)$. We set $\gamma(y, t) = y + r(y, t)\nu(y)$, i.e., $\Gamma(t)$ is expressed as the graph of the function $r(y, t)$ over Γ_0. Since an elementary computation yields

$$\frac{\partial}{\partial t}\nu(y, t) = \nabla_{\Gamma(t)} c(v(t)),$$

where ∇_Γ stands for the gradient operator on a manifold Γ, we have that $\nu(y, t) = \nu(y, 0) = \nu(y)$. Now the first equation in (2.5) is expressed as

$$\frac{\partial r(y, t)}{\partial t} = c(v(t)), \qquad \text{with } r(y, 0) \equiv 0 \text{ for } y \in \Gamma_0.$$

Therefore $b(y, t) := \nabla_{\Gamma(t)} r(y, t)$ satisfies the initial value problem:

$$\frac{\partial}{\partial t} b = 0, \qquad b(y, 0) = 0,$$

which forces $b(y, t) \equiv 0$. Therefore r is independent of $y \in \Gamma_0$.

On the other hand, $|\Omega^-(t)|$ is written in terms of $r(t)$ as

$$|\Omega^-(t)| = |\Omega^-(0)| + \int_0^{r(t)} \int_{\Gamma_0} \sqrt{g(y, s)} dS_y ds$$

where $\sqrt{g(y, s)} dS_y ds = \Pi_{j=1}^{N-1}(1 + s\kappa_j(y)) dS_y ds$ is the volume element in the neighborhood of Γ_0. Here $\kappa_j(y)$ $(j = 1, \dots N-1)$ are the principal curvatures of Γ_0 at y. Therefore, by expanding the volume element as $\Pi_{j=1}^{N-1}(1 + s\kappa_j(y)) = 1 + \sum_{j=1}^{N-1} H_j(y) s^j$, we obtain

$$|\Omega^-(t)| = |\Omega^-(0)| + r(t)|\Gamma_0| + \sum_{j=1}^{N-1}\left(\int_{\Gamma_0} H_j(y) dy\right) \frac{r(t)^{j+1}}{j+1} =: |\Omega^-(0)| + \mathscr{H}(r(t)),$$

where $|\Gamma_0|$ stands for the $N-1$ dimensional volume of Γ_0. Thus the interface equation (2.5) is equivalent to the system of ordinary differential equations:

$$(\mathbf{ODE}):\quad \begin{cases} r_t = c(v) \\ v_t = G^-(v)\dfrac{|\Omega^-(0)| + \mathscr{H}(r)}{|\Omega|} + G^+(v)\dfrac{|\Omega^+(0)| - \mathscr{H}(r)}{|\Omega|} \\ r(0) = 0, \qquad v(0) = v_0. \end{cases}$$

From this reformulation we obtain the following theorem.

THEOREM 2.3. *Suppose that* **(A1)** *through* **(A4)** *are satisfied. Assume that* Γ_0 *is of a* C^2*-hypersurface which is the boundary of* $\Omega^-(0) \subset\subset \Omega$, *and that* $v_0 \in [\underline{b} + \sigma, \bar{b} - \sigma]$.

Then there exists a $T > 0$ *such that the interface equation* (2.5) *has a unique solution* $(v(t), \gamma(y, t))$ *for* $t \in [0, T]$.

As for the relation between the solutions of (2.5) and those of (1.1), we have the following theorem. We emphasize that the accuracy in the approximation of (1.1) by (2.5) crucially depends on the decay estimate in Theorem 2.1 (cf. (2.8) and (2.9) below).

THEOREM 2.4. *In addition to the conditions of Theorems* 2.2 *and* 2.3, *assume that there exists a constant* $l > 0$ *such that*

$$(2.7)\qquad \begin{cases} \phi(x) - h^0(v_0) \geq l\,\mathrm{dist}(x, \Gamma_0) & \text{if } x \in \Omega^+(0) \\ \phi(x) - h^0(v_0) \leq -l\,\mathrm{dist}(x, \Gamma_0) & \text{if } x \in \Omega^-(0). \end{cases}$$

Then there exist constants $\varepsilon_0 > 0$ *and* $M_2 = M_2(T)$ *such that the following estimates hold for* $\varepsilon \in (0, \varepsilon_0]$.

$$(2.8)\qquad |v^\varepsilon(x,t) - v(t)| \leq M_2 \varepsilon^{2/(N+1)} \qquad \text{for } x \in \bar{\Omega}, \qquad t \in [\tau_1 \varepsilon |\log \varepsilon|, T],$$

$$(2.9)\qquad |u^\varepsilon(x,t) - u(x,t)| \leq M_2 \varepsilon^{2/(N+1)}$$

$$\text{for } x \in \{\bar{\Omega} \,|\, \mathrm{dist}(x, \Gamma(t)) > M_2 \varepsilon^{2/(N+1)}\}, \qquad t \in [\tau_1 \varepsilon |\log \varepsilon|, T],$$

where $(v(t), \Gamma(t))$ *is the solution of* (2.5) *and*

$$u(x,t) := \begin{cases} h^+(v(t)) & x \in \Omega^+(t) \cup \Gamma(t) \\ h^-(v(t)) & x \in \Omega^-(t). \end{cases}$$

At this point we should remark that Barles, Bronsard and Souganidis [1] treated the scalar reaction-diffusion equation

$$u_t = \varepsilon \Delta u + \frac{2}{\varepsilon}(u - \mu)(1 - u^2) \qquad t > 0, \qquad x \in \mathbf{R}^N$$

of bistable type and obtained results corresponding to Theorems 2.3 and 2.4. Although the results in [1] are formulated in the framework of viscosity solutions, in terms of our terminology, the interface equation is given by $r_t = 2\mu$ (where $\mu \in (0,1)$ is a constant and the wave speed $c = 2\mu$). Results along the same line were also obtained by Chen [2]. These are results for scalar equations and the wave speed is a constant.

For a system of reaction-diffusion equations, Hilhorst, Logak and Nishiura [8] obtained a result close to ours. They treated the system

$$(\mathrm{RD}): \quad \begin{cases} u_t = \varepsilon \Delta u + \varepsilon^{-1}(1-u^2)(2u - v) \\ \tau v_t = \sigma^{-1}\Delta v + u - \gamma^{-1} v \end{cases}$$

with suitable boundary conditions. As $\sigma \to \infty$, it was shown that (RD) converges to the shadow system:

$$(\mathrm{SS}): \quad \begin{cases} u_t = \varepsilon \Delta u + \varepsilon^{-1}(1-u^2)(2u - \zeta(t)) \\ \tau \dot{\zeta}(t) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x,t)dx - \dfrac{1}{\gamma}\zeta(t). \end{cases}$$

Then, passing to another limit $\tau \to 0$, it was shown that the shadow system (SS) converges to the following non-local Allen-Cahn equation

$$(\mathrm{NLAC}): \quad u_t = \varepsilon \Delta u + \frac{1}{\varepsilon}(1-u^2)\left(2u - \gamma \frac{1}{|\Omega|}\int_\Omega u(x,t)dx\right)$$

for which the interface equation is given by

$$r_t = \gamma \frac{1}{|\Omega|}\int_\Omega u(x,t)dx.$$

Here the wave speed is regulated nonlocally by the distribution of the activator u. In our interface equation (2.5), however, the wave speed is regulated by the value of inhibitor v, and in turn, the value of the inhibitor v is controlled nonlocally by the distribution of the activator u. We note that the interface equation for the shadow system (SS) with $\tau = 1$ is given precisely by (2.5) with $c(\zeta) = \zeta$, although this was not stated in [8]. One has to be careful, however, not to conclude that (SS) captures the essential dynamics of (1.1). There are some aspects in the dynamics of (1.1) that are lost in the process of taking the limit $\sigma \to 0$ to obtain the shadow system. We believe that the interface equation (2.5) describes, generically speaking, only the transient dynamics of solutions to (1.1). For example, we have recently shown in [12] that (2.5) is too crude to give rise to equilibrium solutions of (1.1). Instead, we have shown that the interface equation (4.1)–(4.2) proposed in §4 does capture the equilibrium solutions of (1.1) and their stability property.

As Theorem 2.4 states, the interface equation (2.5) *approximates* the equation (1.1) only on finite time interval $[0, T]$. It is, however, of independent interest to analyze the asymptotic behavior of solutions of (2.5). This is summarized in the following.

Theorem 2.5. *Suppose that* **(A1)** *through* **(A5)** *are satisfied.*

(i) *A pair* (v_0, Γ_0), *where* $\Gamma_0 \subset\subset \Omega$ *is a* C^2*-hypersurface, is an equilibrium solution of* (2.5) *if and only if* $v_0 = v^*$ *(cf.* **(A5)***) and* Γ_0 *subdivides* Ω *into two components* $\Omega^{\pm}$ *such that* $\Omega = \Omega^- \cup \Gamma_0 \cup \Omega^+$,

$$|\Omega^+| = -\frac{G^-(v^*)}{[G]^*}|\Omega|, \qquad |\Omega^-| = \frac{G^+(v^*)}{[G]^*}|\Omega| \tag{2.10}$$

with $[G]^* = G^+(v^*) - G^-(v^*)$.

(ii) *The equilibrium solution* (v^*, Γ_0) *is asymptotically stable relative to* **(ODE)**.

The proof of Theorem 2.5 (i) is trivial. The proof of (ii) is as easy as follows. Linearize **(ODE)** around $(r, v) = (0, v^*)$ to obtain the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 0 & c'(v^*) \\ -[G]^* \dfrac{|\Gamma_0|}{|\Omega|} & \dfrac{G_v^-(v^*)G^+(v^*) - G_v^+(v^*)G^-(v^*)}{[G]^*} \end{bmatrix}.$$

It is easily shown that $c'(v^*) = -\left(\int_{-\infty}^{\infty} u_z^*(z)^2 dz\right)^{-1} J'(v^*) > 0$ (cf. **(A5)**), where $u^*(z)$ is the unique solution of (2.6) with $v = v^*$. The inequalities in **(A4)** imply

$$G_v^-(v^*)G^+(v^*) - G_v^+(v^*)G^-(v^*) < 0.$$

Therefore we find that $\operatorname{trace} \mathbf{A} < 0$ and $\det \mathbf{A} > 0$, which establishes the statement (ii).

Theorem 2.5, however, is not claiming that equilibrium solutions of (1.1) can be thus obtained and are stable. If the initial value (v_0, Γ_0) for (2.5) is such that $v_0 = v^*$ and Γ_0 satisfies (2.10), then Theorem 2.4 loses its power substantially. Once the solutions of the interface equation (2.5) settle down (very close) to the equilibrium states as in Theorem 2.5, it is very likely that another dynamics, which evolves in a slower time scale, takes over. Heuristic discussions on the slower dynamics are given in §4.

From a viewpoint of dynamical system, our results together with some speculations may be summarized as follows:

The solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of (1.1) can be considered as a semiflow on a phase space X;

$$X \ni (\phi, \psi) \mapsto \mathscr{F}^\varepsilon(\phi, \psi, t) = (u^\varepsilon(x, t), v^\varepsilon(x, t)) \in X.$$

Theorems 2.1 and 2.2 may be interpreted as saying that there is a positively invariant subset $\mathscr{A}_0$ in X which quickly attracts its neighborhood under the semiflow $\mathscr{F}^\varepsilon$. The set $\mathscr{A}_0$ consists of pairs of functions $(u(x), v(x))$ such that $u(x)$ has inernal layers and $v(x)$ is nearly constant. Theorem 2.4 may be considered as saying that the interface equation (2.5) describes the dynamics of the semiflow on $\mathscr{A}_0$ on a finite time interval. Then Theorem 2.5 and our speculations above indicate that there exists yet another positively invariant set $\mathscr{A}_1 \subset \mathscr{A}_0$ which attracts its neighborhood. The dynamics of the semiflow on $\mathscr{A}_1$ may be described by another interface equation which we hope to be the one given in Section 4.

3. Proof of theorems

In this section, we prove the theorems stated in Section 2.

The first part in the proof of Theorem 2.1 is a slight modification (which is absolutely necessary for Theorem 2.1) of that in [5]. The second part in the proof contains a new idea which overcomes difficulties one faces when one tries to apply the method in [5] to Theorem 2.1.

The proof of Theorem 2.2 is due to [3]. It is included here for the sake of completeness and reference.

The proof of Theorem 2.4 is also inspired by the method in [3], although we introduced a new step to make the idea in [3] fit to our situation, namely, we approximate the interface equation (2.5) by a genuine interface equation (GIE) at the beginning of Section 3.3.

3.1. Proof of Theorem 2.1. Since the values of the initial condition are contained in the invariant rectangle $\mathscr{R}$, the solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ of (1.1) stays in $\mathscr{R}$ for $t \geq 0$ (see [4]). Therefore we have:

$$|g(u^\varepsilon(x,t), v^\varepsilon(x,t))| \leq M_0 \qquad (x \in \bar{\Omega}, t \geq 0).$$

Let us set $a(t) = (1/2)\|\nabla v^\varepsilon(\cdot\,,t)\|^2_{L^2(\Omega)}$. We obtain the differential inequality

$$\begin{aligned}
\dot{a} &= \int_\Omega \nabla v^\varepsilon \cdot \nabla v^\varepsilon_t dx = -\int_\Omega (\Delta v^\varepsilon) v^\varepsilon_t dx \\
&= -\frac{D}{\varepsilon}\int_\Omega |\Delta v^\varepsilon|^2 dx - \int_\Omega (\Delta v^\varepsilon) g(u^\varepsilon, v^\varepsilon) dx \\
&\leq -\frac{D}{\varepsilon}\int_\Omega |\Delta v^\varepsilon|^2 dx + \frac{k}{2}\int_\Omega |\Delta v^\varepsilon|^2 dx + \frac{M_0^2}{2k}|\Omega|
\end{aligned}$$

for each $k > 0$. Choosing $k = D/\varepsilon$, and using the inequality (cf. [5])

$$\|\Delta v\|^2_{L^2} \geq \lambda_1 \|\nabla v\|^2_{L^2},$$

we have

$$\dot{a} \le -\frac{D\lambda_1}{\varepsilon} a + \varepsilon \frac{M_0^2 |\Omega|}{2D}, \qquad a(0) = \frac{1}{2} \|\nabla\psi\|_{L^2}^2.$$

This differential inequality, together with a Poincaré inequality (see [5])

$$\lambda_1 \|v - \bar{v}\|_{L^2}^2 \le \|\nabla v\|_{L^2}^2,$$

gives

$$(*1) \qquad \|\nabla v^\varepsilon(\cdot, t)\|_{L^2}^2 \le \|\nabla\psi\|_{L^2}^2 \exp\left[-\frac{D\lambda_1}{\varepsilon} t\right] + \frac{M_0^2 |\Omega|}{D^2 \lambda_1} \varepsilon^2,$$

$$(*2) \qquad \|v^\varepsilon(\cdot, t) - \bar{v}^\varepsilon(t)\|_{L^2}^2 \le \lambda_1^{-1} \left[\|\nabla\psi\|_{L^2}^2 \exp\left[-\frac{D\lambda_1}{\varepsilon} t\right] + \frac{M_0^2 |\Omega|}{D^2 \lambda_1} \varepsilon^2\right].$$

We now improve these L^2-estimates to a uniform one by using the following two results.

LEMMA 3.1. *There exists a constant $K_0 > 0$, which is independent of $\varepsilon \in (0, \varepsilon_0]$, such that*

$$\text{(i)} \qquad |v^\varepsilon(x, t) - v^\varepsilon(x', t)| \le K_0 |x - x'| \qquad t \ge 0, \ \ x, x' \in \bar{\Omega},$$

$$\text{(ii)} \qquad |\nabla v^\varepsilon(x, t) - \nabla v^\varepsilon(x', t)| \le K_0 |x - x'|^{3/4} \qquad t \ge 0, \ \ x, x' \in \bar{\Omega}.$$

In order to state the other result, let us define the cone of height $\rho > 0$ by

$$\mathscr{C}(\rho) := \{x \in \mathbf{R}^N \mid x = (x_1, \dots, x_N), x_j \ge 0 \ \ (j = 1, \dots, N), |x| < \rho\}.$$

By $\mathscr{C}(x, \rho)$ we denote the cone $\mathscr{C}(\rho)$, whose vertex is placed at $x \in \bar{\Omega}$, rotated around the vertex appropriately so that it is contained in $\bar{\Omega}$. Since Ω is a smooth bounded domain, there exists $\rho_0 > 0$ such that for each $x \in \bar{\Omega}$ we have $\mathscr{C}(x, \rho) \subset \bar{\Omega}$ for $0 \le \rho \le \rho_0$.

LEMMA 3.2. *Let $w(x)$ be uniformly Hölder (or Lipschitz) continuous with exponent $\nu \in (0, 1]$ and constant $K_0 > 0$, satisfying*

$$\|w\|_{L^2}^2 \le \frac{K_0^2 \omega_N}{2^N (N + 1 + \nu)} \rho_0^{N+2\nu},$$

where ρ_0 is the constant above and ω_N is the surface area of the unit sphere in $\mathbf{R}^N$.

Then w satisfies the estimate

$$\max_{x \in \bar{\Omega}} |w(x)| \le C_{K_0, \nu} \|w\|_{L^2}^{2\nu/(N+2\nu)},$$

where

$$C_{K_0,\nu} = \max\left\{\frac{2N}{N+\nu}\left(\frac{2^N(N+1+\nu)}{\omega_N}\right)^{\nu/(N+2\nu)}, \left(\frac{2^N N(N+\nu)(N+2\nu)}{2\nu^2\omega_N}\right)^{\nu/(N+2\nu)}\right\}$$
$$\times K_0^{N/(N+2\nu)}.$$

We now continue the proof of Theorem 2.1.

Thanks to (∗1) and (∗2), $w(x) := v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)$ satisfies (with $\nu = 3/4$)

$$\|w\|_{L^2}^2 \le \frac{K_0^2\omega_N}{2^N(N+2)}\rho_0^{N+2}, \qquad \|\nabla w\|_{L^2}^2 \le \frac{K_0^2\omega_N}{2^N(N+1+\nu)}\rho_0^{N+2\nu}$$

for $t \ge 2\varepsilon|\log\varepsilon|/D\lambda_1$, since $\|w\|_{L^2}^2 = O(\varepsilon^2)$ and $\|\nabla w\|_{L^2}^2 = O(\varepsilon^2)$ for such t. Therefore applying Lemma 3.2, we obtain the estimates

$$\max_{x\in\bar\Omega}|v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)| \le C_{K_0,1}\|v^\varepsilon(\cdot,t) - \bar{v}^\varepsilon(t)\|_{L^2}^{2/(N+2)}$$

$$\max_{x\in\bar\Omega}|\nabla v^\varepsilon(x,t)| \le C_{K_0,\nu}\|\nabla v^\varepsilon(\cdot,t)\|_{L^2}^{2\nu/(N+2\nu)} \qquad \text{with } \nu = 3/4.$$

On the other hand, for each $q > N$, the Sobolev inequality implies

$$\begin{aligned}
&\max_{x\in\bar\Omega}|v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)| \\
&\quad \le C\left[\int_\Omega |v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)|^q dx + \int_\Omega |\nabla v^\varepsilon(x,t)|^q dx\right]^{1/q} \\
&\quad \le C\left[\max_{x\in\bar\Omega}|v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)|^{q-2}\int_\Omega |v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)|^2 dx\right. \\
&\qquad \left. + \max_{x\in\bar\Omega}|\nabla v^\varepsilon(x,t)|^{q-2}\int_\Omega |\nabla v^\varepsilon(x,t)|^2 dx\right]^{1/q} \\
&\quad \le c^*\left[\|\nabla\psi\|_{L^2}^2 \exp\left[-\frac{D\lambda_1}{\varepsilon}t\right] + \frac{M_0^2|\Omega|}{D^2\lambda_1}\varepsilon^2\right]^\mu \qquad \text{with } \mu = \frac{\nu}{N+2\nu}\frac{q-2}{q} + \frac{1}{q}.
\end{aligned}$$

Taking $q = N+1$ we have

$$\mu = \frac{\nu}{N+2\nu}\frac{N-1}{N+1} + \frac{1}{N+1} \ge \frac{1}{N+1},$$

which completes the proof of Theorem 2.1.

Proof of Lemma 3.1. By rescaling the time as $t/\varepsilon \to t$, we recast the equation for v in (1.1) as

$$v_t + \mathscr{A}v = v + \varepsilon g(u,v) =: G(v,x,t),$$

where $-\mathscr{A}v = D\Delta v - v$. It is well known [7] that $-\mathscr{A}$ generates an analytic semigroup on $L^p(\Omega)$ for $p > 1$; and that there exist constants $C_0 > 0$ and $C_\alpha > 0$ for $\alpha \in (0,1)$ such that

$$\|e^{-t\mathscr{A}}v\|_{L^p} \le C_0 e^{-t}\|v\|_{L^p}, \qquad \|\mathscr{A}^\alpha e^{-t\mathscr{A}}v\|_{L^p} \le C_\alpha t^{-\alpha}e^{-t}\|v\|_{L^p}.$$

Notice that $|v^\varepsilon(x,t)|$ is bounded on $\bar{\Omega} \times [0,\infty)$ and hence that there exists $C > 0$ such that

$$\|G(v,\cdot,t)\|_{L^p} \le C|\Omega|^{1/p}$$

for any $t \ge 0$ and $p > 1$. Therefore we have

$$\begin{aligned}\|\mathscr{A}^\alpha v(t)\|_{L^p} &\le C_0 e^{-t}\|\mathscr{A}^\alpha v(0)\|_{L^p} + C|\Omega|^{1/p}\int_0^t C_\alpha (t-s)^{-\alpha}e^{-(t-s)}ds \\ &\le C_0\|\mathscr{A}^\alpha v(0)\|_{L^p} + C|\Omega|^{1/p}C_\alpha \Gamma(1-\alpha).\end{aligned}$$

It is also known [7] that

$$D(\mathscr{A}^\alpha) \hookrightarrow C^{1+\nu}(\bar{\Omega}) \qquad \text{if } 2\alpha - N/p > 1+\nu \qquad \text{with } \nu \in (0,1).$$

Therefore by choosing $\alpha = 15/16$ and $p > 8N$, we obtain

$$v^\varepsilon(\cdot,t) \in C^{1+\nu}(\bar{\Omega}) \qquad \text{and} \qquad |v^\varepsilon(\cdot,t)|_{C^{1+\nu}(\bar{\Omega})} \le K_0 \qquad \text{for } \nu = \frac{3}{4}$$

with $K_0 > 0$ being independent of $\varepsilon \in (0,\varepsilon_0]$. This completes the proof of Lemma 3.1.

PROOF OF LEMMA 3.2. We give the proof for $\nu = 1$, since other cases are treated in almost the same manner. Let $\bar{x}$ in $\bar{\Omega}$ be such that

$$a := |w(\bar{x})| = \max_{x \in \bar{\Omega}} |w(x)|.$$

By the Lipschitz continuity of w we have:

$$|w(x)| \ge a - K_0|x-\bar{x}| \qquad \text{for } x \in \bar{\Omega} \cap \left\{|x-\bar{x}| \le \frac{a}{K_0}\right\}.$$

There are two case to consider: (1) $\rho_0 < a/K_0$; (2) $a/K_0 \le \rho_0$.

In case (1), since $\mathscr{C}(\bar{x},\rho) \subset \bar{\Omega}$ for $\rho \in [0,\rho_0]$, integrating the squared of the last relation, we have the inequality

$$\begin{aligned}\|w\|_{L^2}^2 &\ge \int_{\mathscr{C}(\bar{x},\rho)} (a - K_0|x-\bar{x}|)^2 dx \\ &= \frac{\omega_N}{2^N}\left(\frac{\rho^N}{N}a^2 - 2K_0\frac{\rho^{N+1}}{N+1}a + K_0^2\frac{\rho^{N+2}}{N+2}\right).\end{aligned}$$

Choosing $\rho > 0$ so that

$$\rho^{N+2} = \frac{(N+2)2^N}{K_0^2 \omega_N} \|w\|_{L^2}^2,$$

we deduce

$$0 \geq \frac{\rho^N}{N} a^2 - 2K_0 \frac{\rho^{N+1}}{N+1} a, \qquad \text{or} \qquad a \leq \frac{2K_0 N}{N+1} \rho.$$

This gives

$$a \leq \frac{2N}{N+1} \left(\frac{2^N (N+2)}{\omega_N} \right)^{1/(N+2)} K_0^{N/(N+2)} \|w\|_{L^2}^2.$$

In case (2), arguing as above, we have

$$\begin{aligned} \|w\|_{L^2}^2 &\geq \int_{\{|x-\bar{x}| \leq a/K_0\} \cap \Omega} (a - K_0 |x - \bar{x}|)^2 dx \\ &\geq \frac{\omega_N}{2^N} \frac{1}{N(N+1)(N+2)} \frac{1}{K_0^N} a^{N+2}, \end{aligned}$$

which gives

$$a \leq \left(\frac{2^N N(N+1)(N+2)}{2\omega_N} \right)^{1/(N+2)} K_0^{N/(N+2)} \|w\|_{L^2}^2.$$

This completes the proof of Lemma 3.2.

3.2. Proof of Theorem 2.2. We modify the function $f(u,v)$ to $\tilde{f}(u,v)$ for $v \in [\underline{b} + \sigma, \overline{b} - \sigma]$ as in [**1**, (3.6) p. 884] so that

$$|f(u,v) - \tilde{f}(u,v)| \leq C_0 \varepsilon |\log \varepsilon|.$$

Let $\tilde{w}(\xi, \tau; v)$ be the solution of

$$\frac{d\tilde{w}}{d\tau} = \tilde{f}(\tilde{w}, v), \qquad \tilde{w}(0) = \xi \in [a_-, a_+].$$

We then have the following (cf. [**1**, Lemma 3.2]):

(i) For $t \geq 0$, $\tilde{w}_\xi(\xi, \tau; v) > 0$.

(ii) There exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\tau \geq (2/k)|\log \varepsilon|$, the following estimates hold:

$$\tilde{w}(\xi, \tau; v) \geq h^+(v) - 2\varepsilon|\log \varepsilon|, \qquad \xi \in [h^0(v) + 2\varepsilon|\log \varepsilon|, a_+] \tag{3.1}$$

$$\tilde{w}(\xi, \tau; v) \leq h^-(v) + 2\varepsilon|\log \varepsilon|, \qquad \xi \in [a_-, h^0(v) - 2\varepsilon|\log \varepsilon|] \tag{3.2}$$

$$h^-(v) - 2\varepsilon|\log \varepsilon| \leq \tilde{w}(\xi, \tau; v) \leq h^+(v) + 2\varepsilon|\log \varepsilon|, \qquad \xi \in [a_-, a_+], \tag{3.3}$$

where $k > 0$ is a constant for which the following estimates are valid:

$$f(u,v) \geq k\min\{u - h^0(v), h^+(v) - u\} \qquad \text{for } u \in [h^0(v), h^+(v)]$$

$$f(u,v) \leq k\max\{u - h^0(v), h^-(v) - u\} \qquad \text{for } u \in [h^-(v), h^0(v)]$$

$$f(u,v) \leq k(h^+(v) - u) \qquad \text{for } u \in [h^+(v), a_+]$$

$$f(u,v) \geq k(h^-(v) - u) \qquad \text{for } u \in [a_-, h^+(v)].$$

(iii) There exists $C_1 > 0$ which depends only on ε_0 and k such that if $\varepsilon \in (0, \varepsilon_0]$ and $0 \leq \tau \leq (2/k)|\log\varepsilon|$, then $|\tilde{w}_{\xi\xi}| \leq C_1\tilde{w}_\xi/\varepsilon$.

Now let us define $u^\pm(x,t)$ by

$$u^\pm(x,t) = \tilde{w}(\phi(x) \pm Mt, t/\varepsilon; v_0 \mp M\varepsilon|\log\varepsilon|).$$

By choosing $M > 0$ large, we will show that $u^-(x,t)$ and $u^+(x,t)$ are respectively a sub-solution and a super-solution of (1.1) on $[0, (2/k)|\log\varepsilon|]$ and satisfy

$$u^-(x,0) \leq u^\varepsilon(x,0) \leq u^+(x,0).$$

Let us first estimate

$$|\varepsilon\Delta u^-| = |\varepsilon\{\tilde{w}_\xi\Delta\phi + \tilde{w}_{\xi\xi}|\nabla\phi|^2\}| \tag{3.4}$$

$$\leq (C_1 + 1)\sup_{x\in\bar{\Omega}}\{\varepsilon|\Delta\phi| + |\nabla\phi|^2\}\tilde{w}_\xi =: C_2\tilde{w}_\xi.$$

On the other hand, we have

$$\begin{aligned} u_t^- - \frac{1}{\varepsilon}f(u^-, v^\varepsilon) &= -M\tilde{w}_\xi + \frac{1}{\varepsilon}[\tilde{f}(u^-, v_0 + M\varepsilon|\log\varepsilon|) - f(u^-, v^\varepsilon)] \\ &= -M\tilde{w}_\xi + \frac{1}{\varepsilon}[\tilde{f}(u^-, v_0 + M\varepsilon|\log\varepsilon|) - f(u^-, v_0 + M\varepsilon|\log\varepsilon|)] \\ &\quad + \frac{1}{\varepsilon}[f(u^-, v_0 + M\varepsilon|\log\varepsilon|) - f(u^-, v^\varepsilon)] \\ &\leq -M\tilde{w}_\xi + C_0|\log\varepsilon| + \frac{1}{\varepsilon}f_v(u^-, \sharp)[v_0 + M\varepsilon|\log\varepsilon| - v^\varepsilon], \end{aligned}$$

where $\sharp$ is a value between $v^\varepsilon(x,t)$ and $v_0 + M\varepsilon|\log\varepsilon|$. By using an easy estimate

$$v^\varepsilon(x,t) \leq v_0 + M_0 t,$$

we have

$$v^\varepsilon \leq v_0 + (2/k)M_0\varepsilon|\log\varepsilon|, \qquad 0 \leq t \leq (2/k)\varepsilon|\log\varepsilon|.$$

Therefore

$$v_0 + M\varepsilon|\log\varepsilon| - v^\varepsilon(x,t) \geq [M - (2M_0/k)]\varepsilon|\log\varepsilon|, \qquad 0 \leq t \leq (2/k)\varepsilon|\log\varepsilon|.$$

This, together with $f_v \leq -\delta_0$, implies

$$u_t^- - \frac{1}{\varepsilon} f(u^-, v^\varepsilon) \leq -M\tilde{w}_\xi + C_0|\log\varepsilon| - \delta_0[M - (2M_0/k)]|\log\varepsilon|. \tag{3.5}$$

We thus conclude from (3.4) and (3.5) that for $0 \leq t \leq (2/k)\varepsilon|\log\varepsilon|$,

$$u_t^- - \varepsilon\Delta u^- - \frac{1}{\varepsilon} f(u^-, v^\varepsilon) \leq -(M - C_2)\tilde{w}_\xi - \delta_0[M - (2M_0/k) - (C_0/\delta_0)]|\log\varepsilon| \leq 0$$

by choosing $M \geq \max\{C_2, (2M_0/k) + (C_0/\delta_0)\}$.

Arguing similarly, we also obtain

$$u_t^+ - \varepsilon\Delta u^+ - \frac{1}{\varepsilon} f(u^+, v^\varepsilon) \geq 0.$$

Applying now the parabolic comparison theorem, we conclude

$$u^-(x,t) \leq u^\varepsilon(x,t) \leq u^+(x,t), \qquad 0 \leq t \leq (2/k)\varepsilon|\log\varepsilon|, \qquad x \in \bar{\Omega}. \tag{3.6}$$

We will now establish the estimates (2.2)–(2.4) with $\tau_1 = 2/k$. We denote by $k_1 > 0$ the Lipschitz constant of $h^0(v), h^-(v)$ and $h^+(v)$ for $v \in [\underline{b} + \sigma, \bar{b} - \sigma]$. By using the second inequality in (3.3) and (3.6),

$$\begin{aligned} u^\varepsilon(x, \tau_1\varepsilon|\log\varepsilon|) &\leq \tilde{w}(\phi(x) + \tau_1\varepsilon|\log\varepsilon|, \tau_1|\log\varepsilon|; v_0 - M\varepsilon|\log\varepsilon|) \\ &\leq h^+(v_0 - M\varepsilon|\log\varepsilon|) + 2\varepsilon|\log\varepsilon| \leq h^+(v_0) + (Mk_1 + 2)\varepsilon|\log\varepsilon|. \end{aligned}$$

Similarly, (3.6) and the first inequality in (3.3) gives

$$u^\varepsilon(x, \tau_1\varepsilon|\log\varepsilon|) \geq h^-(v_0) - (Mk_1 + 2)\varepsilon|\log\varepsilon|.$$

Therefore we have established (2.2) with any $M_1 \geq Mk_1 + 2$.

On the other hand, if $\phi(x) + \tau_1 M\varepsilon|\log\varepsilon| \leq h^0(v_0 - M\varepsilon|\log\varepsilon|) - 2\varepsilon|\log\varepsilon|$, namely, if $\phi(x) \leq h^0(v_0) - (Mk_1 + 2\tau_1 M)\varepsilon|\log\varepsilon|$, the estimates (3.2) and (3.6) allow us to get:

$$\begin{aligned} u^\varepsilon(x, \tau_1\varepsilon|\log\varepsilon|) &\leq \tilde{w}(\phi(x) + \tau_1 M\varepsilon|\log\varepsilon|, \tau_1|\log\varepsilon|; v_0 - M\varepsilon|\log\varepsilon|) \\ &\leq h^-(v_0 - M\varepsilon|\log\varepsilon|) + 2\varepsilon|\log\varepsilon| \leq h^-(v_0) + (Mk_1 + 2)\varepsilon|\log\varepsilon|. \end{aligned}$$

Similarly, (3.1) and (3.6) imply that

$$u^\varepsilon(x, \tau_1\varepsilon|\log\varepsilon|) \geq h^-(v_0) + (Mk_1 + 2)\varepsilon|\log\varepsilon|.$$

provided $\phi(x) \geq h^0(v_0) + (Mk_1 + 2\tau_1 M)\varepsilon|\log\varepsilon|$. Therefore we have established

(2.3) and (2.4) with $M_1 = Mk_1 + 2 + \tau_1 M$, completing the proof of Theorem 2.2.

3.3. Proof of Theorem 2.4. Let $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ be the solution of (1.1) with the initial condition as in Theorem 2.2. We let U^ε and V^ε be defined by

$$U^\varepsilon(x,t) = u^\varepsilon(x, t + \tau_1\varepsilon|\log\varepsilon|), \qquad V^\varepsilon(x,t) = v^\varepsilon(x, t + \tau_1\varepsilon|\log\varepsilon|)$$

and let $\bar{V}^\varepsilon(t)$ stand for the spatial average of $V^\varepsilon(x,t)$. Consider now the initial value problem of moving hypersurfaces (which we call a genuine interface equation):

$$\text{(GIE)} \quad \frac{\partial\gamma^\varepsilon}{\partial t}\cdot \nu^\varepsilon = c(\bar{V}^\varepsilon(t)), \qquad \gamma^\varepsilon(y,0) = y \in \Gamma_0 = \{x \in \Omega \mid \phi(x) = h^0(v_0)\}.$$

This problem has a unique solution $\gamma^\varepsilon(y,t)$ on a time interval $[0,T]$ for some $T > 0$. Let us define $\Gamma^\varepsilon(t) = \{\gamma^\varepsilon(y,t) \mid y \in \Gamma_0\}$ for $t \in [0,T]$. $\Gamma^\varepsilon(t)$ divides Ω into two subdomains $\Omega^{\varepsilon,\pm}(t)$. We define the signed distance function $d^\varepsilon(x,t)$ by:

$$d^\varepsilon(x,t) = \begin{cases} r_0 & \text{if } x \in \Omega^{\varepsilon,+}(t) \text{ and } \operatorname{dist}(x,\Gamma^\varepsilon(t)) \ge r_0 \\ \operatorname{dist}(x,\Gamma^\varepsilon(t)) & \text{if } x \in \Omega^{\varepsilon,+}(t) \text{ and } \operatorname{dist}(x,\Gamma^\varepsilon(t)) \le r_0/2 \\ -\operatorname{dist}(x,\Gamma^\varepsilon(t)) & \text{if } x \in \Omega^{\varepsilon,-}(t) \text{ and } \operatorname{dist}(x,\Gamma^\varepsilon(t)) \le r_0/2 \\ -r_0 & \text{if } x \in \Omega^{\varepsilon,-}(t) \text{ and } \operatorname{dist}(x,\Gamma^\varepsilon(t)) \ge r_0, \end{cases}$$

which is extended smoothly for $x \in \{r_0/2 < \operatorname{dist}(x,\Gamma^\varepsilon(t)) < r_0\}$.

Let us denote by $U(z;v)$ the unique solution of (2.6) for $v \in [\underline{b} + \sigma, \bar{b} - \sigma]$. Notice that $U_z(z;v) > 0$ for $z \in \mathbf{R}$ and that there exist constants $B > 0$ and $\beta > 0$ such that

$$\text{(3.7)} \quad \begin{cases} |U_z(z;v)| + |U_{zz}(z;v)| + |U(z;v) - h^+(v)| \le Be^{-\beta z} & \text{for } z \ge 0 \\ |U_z(z;v)| + |U_{zz}(z;v)| + |U(z;v) - h^-(v)| \le Be^{\beta z} & \text{for } z \le 0. \end{cases}$$

With these preliminaries at our disposal, we now define two functions $U^\pm(x,t)$, which play a crucial role in our proof below, by:

$$(**) \quad U^\pm(x,t) = U\left(\frac{d^\varepsilon(x,t) \pm Le^{mt}\varepsilon^{2/(N+1)}}{\varepsilon}; \bar{V}^\varepsilon(t) \mp L\varepsilon^{2/(N+1)}\right)$$

where $L > 0$ and $m > 0$ are constants to be determined later. The proof of Theorem 2.4 will amount to establishing the following steps:

Step-1: :

$$U_t^+ - \varepsilon\Delta U^+ - \frac{1}{\varepsilon}f(U^+, V^\varepsilon) \ge 0, \qquad U_t^- - \varepsilon\Delta U^- - \frac{1}{\varepsilon}f(U^-, V^\varepsilon) \le 0$$

$$\text{for } x \in \bar{\Omega}, \qquad t \in [0,T],$$

and $\partial U^\pm/\partial\mathbf{n} = 0$ for $x \in \partial\Omega$, $t \in [0,T]$.

Step-2: :

$$U^-(x,0) \le U^\varepsilon(x,0) \le U^+(x,0), \qquad x \in \bar{\Omega}.$$

Step-3: : There exists $L_1 > 0$ such that

$$|U^\varepsilon(x,t) - h^+(\bar{V}(t))| \le L_1 \varepsilon^{2/(N+1)} \qquad \text{if } d^\varepsilon(x,t) \ge L_1 \varepsilon^{2/(N+1)} \qquad t \in [0,T],$$

$$|U^\varepsilon(x,t) - h^-(\bar{V}(t))| \le L_1 \varepsilon^{2/(N+1)} \qquad \text{if } d^\varepsilon(x,t) \le -L_1 \varepsilon^{2/(N+1)} \qquad t \in [0,T].$$

Step-4: : There exists $L_2 > 0$ such that

$$\text{(3.8)} \quad \begin{cases} |\gamma^\varepsilon(y,t) - \gamma(y, t + \tau_1 \varepsilon |\log \varepsilon|)| \le L_2 \varepsilon^{2/(N+1)} \\ |\bar{V}^\varepsilon(t) - v(t + \tau_1 \varepsilon |\log \varepsilon|)| \le L_2 \varepsilon^{2/(N+1)} \end{cases} \qquad \text{for } t \in [0,T], \quad y \in \Gamma_0,$$

where $(v(t), \gamma(y,t))$ is the unique solution of (2.5).

Once Step-3 and Step-4 are established, Theorem 2.4 is immediately obtained as follows. By using Theorem 2.1 and the second inequality in (3.8),

$$\begin{aligned} |v^\varepsilon(x,t) - v(t)| &\le |v^\varepsilon(x,t) - \bar{v}^\varepsilon(t)| + |\bar{v}^\varepsilon(t) - v(t)| \\ &\le \left(c^* \frac{M_0^2 |\Omega|}{D^2 \lambda_1^2} + L_2 \right) \varepsilon^{2/(N+1)}. \end{aligned}$$

If we choose $M_2 \ge 2L_2$ then for $x \in \bar{\Omega}$ such that $\operatorname{dist}(x, \Gamma(t)) \ge M_2 \varepsilon^{2/(N+1)}$, we have either

$$d^\varepsilon(x, t - \tau_1 \varepsilon |\log \varepsilon|) \ge \operatorname{dist}(x, \Gamma(t)) - L_2 \varepsilon^{2/(N+1)} \ge L_2 \varepsilon^{2/(N+1)},$$

or

$$d^\varepsilon(x, t - \tau_1 \varepsilon |\log \varepsilon|) \le -\operatorname{dist}(x, \Gamma(t)) + L_2 \varepsilon^{2/(N+1)} \le -L_2 \varepsilon^{2/(N+1)},$$

and hence, by Step-3, it follows

$$\begin{aligned} |u^\varepsilon(x,t) - h^\pm(v(t))| &\le |u^\varepsilon(x,t) - h^\pm(\bar{v}^\varepsilon(t))| + |h^\pm(\bar{v}^\varepsilon(t)) - h^\pm(v(t))| \\ &\le (L_1 + k_1 L_2) \varepsilon^{2/(N+1)}. \end{aligned}$$

Therefore, by choosing M_2 as

$$M_2 = \max\left\{ 2L_2, L_1 + k_1 L_2, c^* \frac{M_0^2 |\Omega|}{D^2 \lambda_1^2} + L_2 \right\},$$

we have established Theorem 2.4.

We now prove Step-1 through Step-4.

Step-1: In the sequel, the functions U, U_z, U_{zz} and U_v are all evaluated at

$$(z,v) = \left(\frac{d^\varepsilon(x,t) + Le^{mt}\varepsilon^{2/(N+1)}}{\varepsilon}, \overline{V}^\varepsilon(t) - L\varepsilon^{2/(N+1)}\right).$$

According to the definition of U^+, we easily find

$$U_t^+ - \varepsilon\Delta U^+ - \frac{1}{\varepsilon}f(U^+, V^\varepsilon) = \mathrm{I}_1 + \frac{1}{\varepsilon}\mathrm{I}_2 U_z - \frac{1}{\varepsilon}\mathrm{I}_3,$$

where

$$\mathrm{I}_1 = U_v\overline{V}_t^\varepsilon - U_z\Delta d^\varepsilon - \frac{1}{\varepsilon}U_{zz}[|\nabla d^\varepsilon|^2 - 1],$$

$$\mathrm{I}_2 = d_t^\varepsilon + c(\overline{V}^\varepsilon - L\varepsilon^{2/(N+1)}) + mLe^{mt}\varepsilon^{2/(N+1)},$$

$$\mathrm{I}_3 = f(U^+, v^\varepsilon) - f(U^+, \overline{V}^\varepsilon - L\varepsilon^{2/(N+1)}).$$

Since $|\nabla d^\varepsilon| = 1$ for $|d^\varepsilon| \le r_0/2$ and U_{zz} has the decay property as in (3.7), one easily find that $|\mathrm{I}_1| \le C_1$ for $t \in [0,T]$ with some constant $C_1 > 0$ which is independent of ε. To estimate I_2, let $k_2 > 0$ denote the Lipschitz constant of $c(v)$ for $v \in [\underline{b} + \sigma, \overline{b} - \sigma]$. It then follows that

$$\begin{aligned}\mathrm{I}_2 &= d_t^\varepsilon + c(\overline{V}^\varepsilon) + mLe^{mt}\varepsilon^{2/(N+1)} + c(\overline{V}^\varepsilon - L\varepsilon^{2/(N+1)}) - c(\overline{V}^\varepsilon)\\ &\ge d_t^\varepsilon + c(\overline{V}^\varepsilon) + mLe^{mt}\varepsilon^{2/(N+1)} - k_2L\varepsilon^{2/(N+1)}.\end{aligned}$$

Note that $d_t^\varepsilon + c(\overline{V}^\varepsilon) = 0$ when $d^\varepsilon = 0$. This is because, by definition, we have

$$d^\varepsilon(\gamma^\varepsilon(y,t),t) = 0$$

which, upon differentiation with respect to t, gives:

$$0 = \nabla d^\varepsilon(\gamma^\varepsilon, t)\cdot\frac{\partial\gamma^\varepsilon}{\partial t} + d_t^\varepsilon(\gamma^\varepsilon, t) = \frac{\partial\gamma^\varepsilon}{\partial t}\cdot v^\varepsilon + d_t^\varepsilon = c(\overline{V}^\varepsilon) + d_t^\varepsilon.$$

Therefore, by the smoothness of d^ε and the mean value theorem, there exists a constant $k_3 > 0$, which is independent of ε, such that $|d_t^\varepsilon + c(\overline{V}^\varepsilon)| \le k_3|d^\varepsilon|$. By using these observations, we now continue to estimate I_2:

$$\begin{aligned}\mathrm{I}_2 &\ge d_t^\varepsilon + c(\overline{V}^\varepsilon) + mLe^{mt}\varepsilon^{2/(N+1)} - k_2L\varepsilon^{2/(N+1)}\\ &\ge -k_3|d^\varepsilon| + mLe^{mt}\varepsilon^{2/(N+1)} - k_2L\varepsilon^{2/(N+1)}\\ &\ge -k_3|d^\varepsilon + Le^{mt}\varepsilon^{2/(N+1)}| + (m - k_3)Le^{mt}\varepsilon^{2/(N+1)} - k_2L\varepsilon^{2/(N+1)}\\ &\ge -k_3|d^\varepsilon + Le^{mt}\varepsilon^{2/(N+1)}| + (m - k_2 - k_3)L\varepsilon^{2/(N+1)}.\end{aligned}$$

By choosing m large, $m > k_2 + k_3$, we get $\mathrm{I}_2 \ge -k_3|d^\varepsilon + Le^{mt}\varepsilon^{2/(N+1)}|$ which, together with (3.7), gives rise to

$$\frac{1}{\varepsilon}\mathrm{I}_2 U_z \geq -k_3\left|\frac{d^\varepsilon + Le^{mt}\varepsilon^{2/(N+1)}}{\varepsilon}\right| U_z\left(\frac{d^\varepsilon + Le^{mt}\varepsilon^{2/(N+1)}}{\varepsilon}; \overline{V}^\varepsilon - L\varepsilon^{2/(N+1)}\right)$$
$$\geq -k_3 C_2,$$

for some constant $C_2 > 0$ which is independent of ε.

I_3 is estimated as follows.

$$\mathrm{I}_3 = f_v(U^+, \sharp)[V^\varepsilon - \overline{V}^\varepsilon + L\varepsilon^{2/(N+1)}],$$

where $\sharp$ is a value between V^ε and $\overline{V}^\varepsilon - L\varepsilon^{2/(N+1)}$. From Theorem 2.1, there exists a constant $L_0 > 0$ such that

$$|V^\varepsilon(x,t) - \overline{V}^\varepsilon(t)| \leq L_0\varepsilon^{2/(N+1)}, \qquad x \in \bar{\Omega} \qquad \text{for } t \geq 0.$$

Therefore, by choosing $L > L_0$ and using $f_v \leq -\delta_0$ (see **(A4)**), we have

$$U_t^+ - \varepsilon\Delta U^+ - \frac{1}{\varepsilon}f(U^+, V^\varepsilon) \geq -C_1 - C_2 + \delta_0(L - L_0)\varepsilon^{(1-N)/(N+1)} \geq 0.$$

Similarly, we can show

$$U_t^- - \varepsilon\Delta U^- - \frac{1}{\varepsilon}f(U^-, V^\varepsilon) \leq 0.$$

That $\partial U^\pm/\partial \mathbf{n} = 0$ follows from the definition of $U^\pm$.

Step-2: In order to show that $U^+(x,0) \geq U^\varepsilon(x,0)$, we note that there exists a $k_4 > 0$ such that $h^\pm(v-a) - h^\pm(v) \geq k_4 a$ for $a > 0$ and $\underline{b} + \sigma \leq v - a$, $v \leq \overline{b} - \sigma$ (cf. Remark 1.1). From the second condition in (2.7), it follows that if $d^\varepsilon(x,0) \leq -(M_1/l)\varepsilon|\log\varepsilon|$ then

$$\phi(x) - h^0(v_0) \leq -l\,\mathrm{dist}(x,\Gamma_0) = ld^\varepsilon(x,0) \leq -M_1\varepsilon|\log\varepsilon|.$$

Therefore, by Theorem 2.2, $U^\varepsilon(x,0) \leq h^-(v_0) + M_1\varepsilon|\log\varepsilon|$. On the other hand,

$$U^+(x,0) = U\left(\frac{d^\varepsilon(x,0) + L\varepsilon^{2/(N+1)}}{\varepsilon}; \overline{V}^\varepsilon(0) - L\varepsilon^{2/(N+1)}\right)$$
$$\geq h^-(\overline{V}^\varepsilon(0) - L\varepsilon^{2/(N+1)}) \geq h^-(v_0) + k_4[L\varepsilon^{2/(N+1)} - |\overline{V}^\varepsilon(0) - v_0|]$$
$$\geq h^-(v_0) + k_4(L - L_0)\varepsilon^{2/(N+1)}.$$

Therefore for $\varepsilon_0 > 0$ small and $L > L_0$, it holds that

$$U^+(x,0) \geq h^-(v_0) + L_0\varepsilon^{2/(N+1)} \geq h^-(v_0) + M_1\varepsilon|\log\varepsilon| \geq U^\varepsilon(x,0)$$

for $\varepsilon \in (0,\varepsilon_0]$.

If $d^\varepsilon(x,0) \geq -(M_1/l)\varepsilon|\log\varepsilon|$, then it follows that

$$U^+(x,0) \geq U\left(-\frac{M_1}{l}|\log\varepsilon| + L\varepsilon^{(1-N)/(N+1)}; \bar{V}^\varepsilon(0) - L\varepsilon^{2/(N+1)}\right).$$

By choosing ε_0 small, we have

$$L\frac{\varepsilon^{(1-N)/(N+1)}}{|\log\varepsilon|} - \frac{M_1}{l} \geq \frac{2}{\beta} \qquad \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{3.9}$$

Therefore (3.7) gives

$$\begin{aligned} U^+(x,0) &\geq h^+(\bar{V}^\varepsilon(0) - L\varepsilon^{2/(N+1)}) - B\varepsilon^2 \geq h^+(v_0) + k_4(L - L_0)\varepsilon^{2/(N+1)} \\ &\geq h^+(v_0) + M_1\varepsilon|\log\varepsilon| \geq U^\varepsilon(x,0). \end{aligned}$$

We have thus established $U^\varepsilon(x,0) \leq U^+(x,0)$ for $x \in \bar{\Omega}$.

Similar arguments yield $U^\varepsilon(x,0) \geq U^-(x,0)$ for $x \in \bar{\Omega}$.

Step-3: By applying the parabolic comparison theorem, Step-1 and Step-2 imply that

$$U^-(x,t) \leq U^\varepsilon(x,t) \leq U^+(x,t) \qquad \text{for } (x,t) \in \bar{\Omega} \times [0,T].$$

If we choose

$$L_1 \geq \max\{2Le^{mT}, k_1 L + B\},$$

then for $d^\varepsilon(x,t) \geq L_1\varepsilon^{2/(N+1)}$ we have:

$$\begin{aligned} U^\varepsilon(x,t) \geq U^-(x,t) &\geq U(Le^{mT}\varepsilon^{2/(N+1)}; \bar{V}^\varepsilon(t) + L\varepsilon^{2/(N+1)}) \\ &\geq h^+(\bar{V}^\varepsilon(t) + L\varepsilon^{2/(N+1)}) - B\exp[-\beta Le^{mT}\varepsilon^{(1-N)/(N+1)}] \\ &\geq h^+(\bar{V}^\varepsilon(t)) - k_1 L\varepsilon^{2/(N+1)} - B\varepsilon^2 \geq h^+(\bar{V}^\varepsilon(t)) - L_1\varepsilon^{2/(N+1)}. \end{aligned}$$

In the fourth inequality above, we used the fact that $\beta Le^{mT}(\varepsilon^{(1-N)/(N+1)}/|\log\varepsilon| \geq 2$ for $\varepsilon \in (0, \varepsilon_0]$ from (3.9).

On the other hand, we have

$$U^\varepsilon(x,t) \leq U^+(x,t) \leq h^+(\bar{V}^\varepsilon(t)) + k_1 L\varepsilon^{2/(N+1)} \leq h^+(\bar{V}^\varepsilon(t)) + L_1\varepsilon^{2/(N+1)}.$$

Therefore the first inequality in Step-3 is established. The second inequality in Step-3 follows from the same line of arguments.

Step-4: From Step-3, we can rewrite the equation for $(\bar{V}^\varepsilon(t), \gamma^\varepsilon(t))$ as:

$$\begin{aligned} \bar{V}_t^\varepsilon(t) &= \frac{1}{|\Omega|}\int_\Omega g(U^\varepsilon(x,t), V^\varepsilon(x,t))dx \\ &= G^-(\bar{V}^\varepsilon(t))\frac{|\Omega^{\varepsilon,-}(t)|}{|\Omega|} + G^+(\bar{V}^\varepsilon(t))\frac{|\Omega^{\varepsilon,+}(t)|}{|\Omega|} + O(\varepsilon^{2/(N+1)}), \\ \frac{\partial\gamma^\varepsilon}{\partial t}\cdot\nu^\varepsilon &= c(\bar{V}^\varepsilon(t)) \end{aligned}$$

$$\bar{V}^{\varepsilon}(0) = \frac{1}{|\Omega|}\int_{\Omega} v^{\varepsilon}(x, \tau_1 \varepsilon |\log \varepsilon|))dx = v_0 + O(\varepsilon^{2/(N+1)})$$

$$\gamma^{\varepsilon}(y, 0) = y \in \Gamma_0.$$

Comparing this initial value problem with (2.5), one can find $L_2 > 0$ that makes (3.8) true, by virture of the continuous dependence of solutions on the initial conditions and on the vector fields. This completes the proof of Step-4.

4. Discussion

We have established in this paper that the solutions of (1.1) exhibit at least three different types of dynamic behaviors for a class of initial conditions. Although the spatial homogenization of $v^{\varepsilon}(x, t)$ and the development of internal layers in $u^{\varepsilon}(x, t)$ take place in the same time scale, the particular choice of initial conditions as in Theorem 2.2 enables us to observe these two phenomena separately. The motion of interfaces, the third type of dynamic behavior, is described by the system of ordinary differential equations (2.5). The asymptotic behavior of the last equations is shown to be rather simple, namely, solutions converge to an equilibrium of (2.5). As indicated at the end of Section 2, it is natural to ask the question:

What would happen to the motion of the internal layer solutions of (1.1), after the solutions of (2.5) have reached equilibrium states?

It is likely that the location of the internal layer evolves according to a slower dynamics which is described by another interface equation. In fact, such an interface equation can be read off from the computation in [7]. Rescaling the time by $\varepsilon t \to t$, the equation is given by:

$$\frac{\partial \gamma(y, t)}{\partial t} \cdot v(y, t) = -H(y, t) + c'(v^*) b(y, t) - A(t) \tag{4.1}$$

with

$$A(t) = -\frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} H(y, t) dS_y + c'(v^*) \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} b(y, t) dS_y,$$

where $H(y, t)$ is the sum of principal curvatures of $\Gamma(t)$ at $\gamma(y, t)$, dS_y the surface measure on $\Gamma(t)$, and $b(y, t)$ is the value of the function $V(x, t)$ which is a unique solution of

$$\text{(4.2)} \quad \begin{aligned} &\text{(a)} \quad D\Delta V = -G^{\pm}(v^*), \qquad x \in \Omega^{\pm}(t) \qquad \frac{\partial V}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega, \\ &\text{(b)} \quad V(\cdot, t) \in C^1(\bar{\Omega}), \qquad \int_{\Omega} V(x, t) dx = 0. \end{aligned}$$

The initial conditions for (4.1) and (4.2) are:

$$\gamma(y,0) = \gamma_0(y), \qquad V(y,0) = V_0(y)$$

with V_0 safisfying (4.2) for $t = 0$. The interface equation (4.1)–(4.2) was also derived in [9] by a reasoning different from ours.

It is not our intention here to present the detail of how to derive (4.1)–(4.2). Instead, let us comment on the compatiblity of (4.1) and (4.2).

For any given interface $\Gamma(t) \subset\subset \Omega$ with $\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t)$, the problem (4.2) has a unique solution if and only if

$$|\Omega^-(t)| = \frac{G^+(v^*)}{[G]^*}|\Omega|, \qquad |\Omega^+(t)| = -\frac{G^-(v^*)}{[G]^*}|\Omega|, \tag{4.3}$$

namely, the volume of $\Omega^\pm(t)$ is independent of t. On the other hand, due to the nonlocal nature, the solution $\Gamma(t)$ of (4.1) evolves in such way that the volume of $\Omega^\pm(t)$ is preserved:

$$\frac{d}{dt}|\Omega^\pm(t)| = \mp\int_{\Gamma(t)} \frac{\partial\gamma(y,t)}{\partial t}\cdot \nu(y,t)dS_y = 0.$$

Therefore, (4.1) and (4.2) are compatible.

When $\Gamma(t)$ has several connected components $\Gamma_j(t)$ $(j = 1,\dots,k)$, the equation (4.1) should be replaced by k equations

$$\frac{\partial\gamma_j(y,t)}{\partial t}\cdot \nu_j(y,t) = -H^j(y,t) + c'(v^*)b^j(y,t) - A(t) \qquad (j = 1,\dots,k)$$

where $A(t)$, which is independent of j, is given by

$$A(t) = \frac{1}{|\Gamma(t)|}\sum_{j=1}^{k}\left(-\int_{\Gamma_j(t)} H^j(y,t)dS_y + c'(v^*)\int_{\Gamma_j(t)} b^j(y,t)dS_y\right).$$

In the above, γ_j is a parametrization of Γ_j, ν_j the unit normal vector on Γ_j pointing to $\Omega^+(t)$, H^j the sum of principal curvatures on Γ_j, and b^j is the value of V on Γ_j. In this context, a one dimensional version ($N = 1$ and $\Omega = (0,1)$) of (4.1)–(4.2) is given by:

$$(-1)^{j+1}\dot{\gamma}_j(t) = c'(v^*)\left(b^j(t) - \frac{1}{k}\sum_{i=1}^{k} b^i(t)\right) \qquad (j = 1,\dots,k),$$

$$DV_{xx}(x,t) = -G^\pm(v^*) \qquad x \in \Omega^\pm(t), \qquad V_x(0,t) = 0 = V_x(1,t), \tag{4.4}$$

$$V(\cdot,t) \in C^1[0,1], \qquad \int_0^1 V(x,t)dx = 0, \qquad b^j(t) = V(\gamma_j(t),t) \;\; (j = 1,\dots,k).$$

In (4.4),

$$\Gamma(t) = \{\gamma_1(t), \dots, \gamma_k(t)\} \qquad \text{with } \gamma_0(t) \equiv 0 < \gamma_1(t) < \cdots < \gamma_k(t) < \gamma_{k+1}(t) \equiv 1,$$

and $\Omega^{\pm}(t)$ is given by

$$\Omega^-(t) = \bigcup\nolimits_{0 \le 2j \le k} (\gamma_{2j}(t), \gamma_{2j+1}(t)), \qquad \Omega^+(t) = \bigcup\nolimits_{0 \le 2j-1 \le k} (\gamma_{2j-1}(t), \gamma_{2j}(t)).$$

It is easy to see that (4.4) with suitable initial conditions is well posed and that the solutions can be explicitly written down by an elementary computation. Note also that there are only $k-1$ independent equations in the first line of (4.4).

When the domain Ω is the unit disk in $\mathbf{R}^N$ $(N \ge 2)$, assuming that the interfaces $\Gamma_j(t)$ are concentric spheres with radius $\rho_j(t)$ and that $V(x,t)$ is radially symmetric, our interface equation reduces to (4.2) coupled with the following

$$(4.5) \qquad (-1)^{j+1}\dot{\rho}_j(t) = (-1)^j (N-1)\frac{1}{\rho_j(t)} - (N-1)\frac{\sum_{i=1}^k (-1)^i \rho_i(t)^{N-2}}{\sum_{i=1}^k \rho_i(t)^{N-1}} + c'(v^*)\left(b^j(t) - \frac{\sum_{i=1}^k \rho_i(t)^{N-1} b^i(t)}{\sum_{i=1}^k \rho_i(t)^{N-1}}\right).$$

We have recently established in [12] that an equilibrium solution of (4.5) gives rise to an internal layer solution of (1.1). It is our future projects to show the well posedness of (4.1)–(4.2) and to clarify the relationship between the solutions of this problem and those of (1.1).

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