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On non-singular stable maps of 3-manifolds with boundary into the plane

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Abstract. Let M be a compact connected orientable 3-manifold with non-empty boundary and $f: M \to \mathbf{R}^2$ a stable map. In this paper we study the existence of an immersion or embedding lift of f to $\mathbf{R}^n$ $(n \geq 3)$ with respect to the standard projection $\mathbf{R}^n \to \mathbf{R}^2$. We also characterize the orientable 3-dimensional handlebody in terms of stable maps which have only a restricted class of singularities. Moreover, by using the concept of an embedding lift of a certain map of a 2-dimensional polyhedron into $\mathbf{R}^2$, we give a characterization of S^3.

1. Introduction

Let M be a smooth manifold, $f: M \to \mathbf{R}^m$ a smooth map and $\pi: \mathbf{R}^n \to \mathbf{R}^m$ $(n > m)$ a standard projection. Then we ask if there exists an immersion or embedding $g: M \to \mathbf{R}^n$ which satisfies $f = \pi \circ g$. Such a map g is called an *immersion* or *embedding lift* of f.

In this paper, M will be a compact connected orientable 3-manifold with non-empty boundary, of class C^∞. Let $f: M \to \mathbf{R}^2$ be a stable map. We ask if there exists an immersion or embedding lift of f to $\mathbf{R}^n$ $(n \geq 3)$ with respect to the standard projection $\pi: \mathbf{R}^n \to \mathbf{R}^2$, $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2)$. A point x in M is called a *singularity* if $\operatorname{rank} df_x < 2$. $S(f)$ denotes the set of singularities of f. Our main result is the following theorem.

Theorem 1. *Let M be a compact connected orientable 3-manifold with non-empty boundary and $f: M \to \mathbf{R}^2$ a stable map. We consider the condition* (I): *For any $r \in \mathbf{R}^2$, $f^{-1}(r)$ is either empty or homeomorphic to a finite disjoint union of closed intervals and points. Then the following two conditions are equivalent.*
(a) *f has an immersion lift to $\mathbf{R}^3$.*
(b) *$S(f) = \emptyset$ and f satisfies the condition* (I).

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By Whitehead [13], there exists an immersion $i : M \to \mathbf{R}^3$ for every compact connected orientable 3-manifold M with non-empty boundary. Thus $f = \pi \circ i$ satisfies $S(f) = \emptyset$ and the condition (I), provided that f is stable. We show that a submersion $f : M \to \mathbf{R}^2$ whose restriction to ∂M is stable, is a stable map in Lemma 2 of §3. Hence, after a slight perturbation of i, we may assume that $f = \pi \circ i$ is a stable map. Moreover, it is not difficult to prove that the space of non-singular stable maps is open and dense in the space of submersions of M to $\mathbf{R}^2$ by using Lemma 2.

Based on the arguments in the proof of Theorem 1, we consider the structure of source manifolds of a certain class of stable maps. For a stable map $f : M \to \mathbf{R}^2$ with $S(f) = \emptyset$, the normal forms around points of ∂M consist exactly of four types: regular, $\mathscr{F}_I$, $\mathscr{F}_{II}$ and $\mathscr{C}$ (for details, see §3 and 4). A point of ∂M is of regular type (or of type $\mathscr{C}$) if it is a regular point (resp. a cusp point) of $f|\partial M$. Fold points of $f|\partial M$ are classified into two types: $\mathscr{F}_I$ and $\mathscr{F}_{II}$. We consider a stable map which has only points of regular type or of type $\mathscr{F}_I$ on ∂M. Such a map is called a *boundary special generic map*.

Theorem 2. *A compact connected orientable* 3*-manifold M with non-empty boundary is an orientable* 3*-dimensional handlebody (i.e., M is diffeomorphic to $\natural^k(S^1 \times D^2)$, $k \geq 0$) if and only if there exists a boundary special generic map $f : M \to \mathbf{R}^2$.*

The tool for the proof of Theorems 1 and 2 is the Stein factorization which consists of 2-dimensional polyhedron W_f, $q_f : M \to W_f$ and $\bar{f} : W_f \to \mathbf{R}^2$ with $f = \bar{f} \circ q_f$. Although W_f is not a manifold, we can define an embedding lift of $\bar{f}$ and get the following theorem.

Theorem 3. *Let $\hat{M}$ be a closed, connected, orientable* 3*-manifold. Suppose that there exists a stable map $f : \hat{M} - \operatorname{Int} D^3 \to \mathbf{R}^2$ with $S(f) = \emptyset$ and the condition* (I). *If there exists an embedding lift $g_e : W_f \to \mathbf{R}^3$ of $\bar{f}$, then $\hat{M}$ is homeomorphic to S^3.*

The paper is organized as follows. In §2 we recall some fundamental concepts: stable maps, Stein factorizations and etc. In §3 we clarify the local normal forms of f on the neighborhoods of singular points of $f|\partial M$. In §4 we investigate the semi-local structures of f around simple or non-simple points of ∂M and the Stein factorization. In §5 we prove Theorem 1 using the Stein factorization. In §6 we consider the existence problem of an embedding lift to $\mathbf{R}^n$ and get Proposition 10 which guarantees the existence of an embedding lift for $n \geq 5$. Moreover we give some examples which have no embedding lifts for $n = 3, 4$. In §7, we prove Theorems 2 and 3.

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2. Preliminaries

Let M be a smooth 3- or 2-dimensional manifold with or without boundary. We denote by $C^\infty(M, \mathbf{R}^2)$ the set of the smooth maps of M into $\mathbf{R}^2$ with the Whitney C^∞ topology. For a smooth map $f : M \to \mathbf{R}^2$, $S(f)$ denotes the singular set of f; i.e., $S(f)$ is the set of the points in M where the rank of the differential df is strictly less than two. A smooth map $f : M \to \mathbf{R}^2$ is *stable* if there exists an open neighborhood $N(f)$ of f in $C^\infty(M, \mathbf{R}^2)$ such that every g in $N(f)$ is *right-left equivalent* to f; i.e., there exist diffeomorphisms $\phi : M \to M$ and $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ satisfying $g = \varphi \circ f \circ \phi^{-1}$.

We quote an explicit description of a stable map from a closed 3-manifold $\hat{M}$ into $\mathbf{R}^2$.

Lemma 1. ([7]) *Let $\hat{M}$ be a closed 3-manifold. Then a smooth map $f : \hat{M} \to \mathbf{R}^2$ is stable if and only if f satisfies the following local and global conditions. For each point $p \in \hat{M}$ there exist local coordinates centered at p and $f(p)$ such that f is expressed by one of the following four types:*

(**I**) $(u, x, y) \mapsto (u, x)$, p: *regular point*,

(**II**) $(u, x, y) \mapsto (u, x^2 + y^2)$, p: *definite fold point*,

(**III**) $(u, x, y) \mapsto (u, x^2 - y^2)$, p: *indefinite fold point*,

(**IV**) $(u, x, y) \mapsto (u, y^2 + ux - x^3)$, p: *cusp point*.

Also f should satisfy the following global conditions:

($\mathbf{G}_1$) *if p is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$, and*
($\mathbf{G}_2$) $f|S(f) - \{\text{cusps}\}$ *is an immersion with normal crossings.*

Let us recall the definition of the Stein factorization. Let M be a compact orientable 3-manifold with or without boundary, and let $f : M \to \mathbf{R}^2$ be a stable map. For p, $p' \in M$, we define $p \sim p'$ if $f(p) = f(p')$ and p, p' are in the same connected component of $f^{-1}(f(p)) = f^{-1}(f(p'))$. Let W_f be the quotient space of M under this equivalence relation and we denote by $q_f : M \to W_f$ the quotient map. By the definition of the equivalence relation, we have a unique map $\bar{f} : W_f \to \mathbf{R}^2$ such that $f = \bar{f} \circ q_f$. The quotient space W_f or more precisely the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbf{R}^2 \\ & {\scriptstyle q_f} \searrow \quad \nearrow {\scriptstyle \bar{f}} & \\ & W_f & \end{array}$$

is called the *Stein factorization* of f. In general, W_f is not a manifold, but is homeomorphic to a 2-dimensional finite CW complex. This fact has been obtained for the case $\partial M = \emptyset$ in [7] and [9] (see also [6]). In the case where $\partial M \neq \emptyset$ with $S(f) = \emptyset$ and the condition (I), this will be shown in §4.

3. Local normal forms of f around singular points of $f|\partial M$

Our purpose of this section is to investigate the local normal forms of a stable map f around singular points of $f|\partial M$.

Throughout this section, M is a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbf{R}^2$ is a stable map with $S(f) = \emptyset$. Since f is stable, $f|\partial M$ is also stable by [10, p. 2564, Lemma].

Recall the theorem of Whitney ([14]): *Let N be a closed 2-manifold, and let $h : N \to \mathbf{R}^2$ be a stable map. Then for each point x in N, there exist local coordinates (x_1, x_2) centered at x and (y_1, y_2) centered at $h(x)$ such that h is given by one of the following local normal forms:*

(i) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1, x_2),$ $\quad$ x: *regular point*,

(ii) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1^2, x_2),$ $\quad$ x: *fold point*,

(iii) $(x_1, x_2) \mapsto (y_1, y_2) = (-x_1^3 + x_1 x_2, x_2),$ $\quad$ x: *cusp point*.

PROPOSITION 1. *Let x be a fold point of $f|\partial M$. Then there exist local coordinates (T, X_1, X_2) of M centered at x and (Y_1, Y_2) of $\mathbf{R}^2$ centered at $f(x)$ such that f is given by one of the local normal forms $(Y_1, Y_2) = (X_1^2 \pm T, X_2)$, where ∂M corresponds to $\{T = 0\}$ and* $\operatorname{Int} M$ *corresponds to $\{T > 0\}$.*

PROOF. By the theorem of Whitney, for $x \in \partial M$, we can choose local coordinates (t, x_1, x_2) centered at x and (y_1, y_2) centered at $f(x)$ such that $f|\partial M$ is expressed by $(0, x_1, x_2) \mapsto (x_1^2, x_2)$, where ∂M corresponds to $\{t = 0\}$ and $\operatorname{Int} M$ corresponds to $\{t > 0\}$. Then we put $f(t, x_1, x_2) = (\varphi(t, x_1, x_2), \psi(t, x_1, x_2))$ so that

$$\varphi(0, x_1, x_2) = x_1^2,$$

$$\varphi(0, x_1, x_2) = x_2.$$

Since the Jacobian matrix of f at $x = (0, 0, 0)$ is

$$Jf(0) = \begin{pmatrix} \dfrac{\partial \varphi}{\partial t}(0) & 0 & 0 \\ \dfrac{\partial \psi}{\partial t}(0) & 0 & 1 \end{pmatrix}$$

and $\operatorname{rank} Jf(0) = 2$ by our assumption that $S(f) = \emptyset$, we obtain $(\partial \varphi / \partial t)(0) \neq 0$.

Then, we define the map $\Phi : (t, x_1, x_2) \mapsto (T, X_1, X_2)$ by

$$\begin{cases} T = \varphi(t, x_1, x_2) - x_1^2, \\ X_1 = x_1, \\ X_2 = \psi(t, x_1, x_2). \end{cases}$$

By the condition $(\partial\varphi/\partial t)(0) \neq 0$, we see that the determinant of the Jacobian matrix of Φ at $(0,0,0)$, $|J\Phi(0)|$, is not equal to 0, since

$$J\Phi(0) = \begin{pmatrix} \dfrac{\partial\varphi}{\partial t}(0) & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{\partial\psi}{\partial t}(0) & 0 & 1 \end{pmatrix}.$$

Hence, (T, X_1, X_2) forms local coordinates. Then we get $f(T, X_1, X_2) = (\varphi(t, x_1, x_2), \psi(t, x_1, x_2)) = (X_1^2 + T, X_2)$. Moreover, $\{t = 0\}$ corresponds to $\{T = 0\}$ by this coordinate change, since $\Phi(0, x_1, x_2) = (\varphi(0, x_1, x_2) - x_1^2, x_1, \psi(0, x_1, x_2)) = (0, x_1, x_2)$.

Then on a neighborhood of x, $\{t \geq 0\}$ corresponds to $\{T \geq 0\}$ or to $\{T \leq 0\}$ by Φ. By replacing T with $-T$ if necessary, we may always assume that $\{T > 0\}$ corresponds to $\operatorname{Int} M$ and $\{T = 0\}$ corresponds to ∂M. According to this change of coordinates, f is expressed either by $(T, X_1, X_2) \mapsto (X_1^2 + T, X_2)$ or by $(T, X_1, X_2) \mapsto (X_1^2 - T, X_2)$. This completes the proof. □

Proposition 2. *Let x be a cusp point of $f|\partial M$. Then there exist local coordinates (T, X_1, X_2) of M centered at x and (Y_1, Y_2) of $\mathbf{R}^2$ centered at $f(x)$ such that f is given by the local normal form $(Y_1, Y_2) = (-X_1^3 + X_1X_2 + T, X_2)$, where ∂M corresponds to $\{T = 0\}$ and* Int M *corresponds to $\{T > 0\}$.*

Proof. By the theorem of Whitney, for $x \in \partial M$, we can choose local coordinates (t, x_1, x_2) centered at x and (y_1, y_2) centered at $f(x)$ such that $f|\partial M$ is expressed by $(0, x_1, x_2) \mapsto (-x_1^3 + x_1x_2, x_2)$, where ∂M corresponds to $\{t = 0\}$ and $\operatorname{Int} M$ corresponds to $\{t > 0\}$. Then we put $f(t, x_1, x_2) = (\varphi(t, x_1, x_2), \psi(t, x_1, x_2))$ so that

$$\varphi(0, x_1, x_2) = -x_1^3 + x_1x_2,$$

$$\psi(0, x_1, x_2) = x_2.$$

In this case, we consider the map $\Phi : (t, x_1, x_2) \mapsto (T, X_1, X_2)$ defined by

$$\begin{cases} T = \varphi(t, x_1, x_2) + x_1^3 - x_1\psi(t, x_1, x_2), \\ X_1 = x_1, \\ X_2 = (t, x_1, x_2). \end{cases}$$

Then, by an argument similar to that in the proof of Proposition 1, we see that (T, X_1, X_2) forms local coordinates. So, by the same reason, we get the local normal form $f(T, X_1, X_2) = (-X_1^3 + X_1X_2 \pm T, X_2)$. However, these two types of normal forms coincide with each other through the changes of coordinates $(T, X_1, X_2) \mapsto (T, -X_1, X_2)$ and $(Y_1, Y_2) \mapsto (-Y_1, Y_2)$. This completes the proof. □

We can also obtain the following proposition.

Proposition 3. *Let x be a regular point of $f|\partial M$. Then there exist local coordinates (T, X_1, X_2) of M centered at x and (Y_1, Y_2) of $\mathbf{R}^2$ centered at $f(x)$ such that f is given by the local normal form $(Y_1, Y_2) = (X_1, X_2)$, where ∂M corresponds to $\{T = 0\}$ and $\operatorname{Int} M$ corresponds to $\{T > 0\}$.*

Now, we show the following Lemma 2. This lemma guarantees the existence of a stable map which satisfies the condition (b) of Theorem 1 as explained in §1.

Lemma 2. *Let M be a compact 3-manifold with non-empty boundary and $f : M \to \mathbf{R}^2$ a submersion such that $f|\partial M$ is a stable map. Then f is also stable.*

Proof. Let us prepare a notion of the infinitesimal stability of Mather ([4, p. 73] and [11]) modified for the case $\partial M \neq \emptyset$ as follows. Let $\alpha : M \to \mathbf{R}^2$ be a smooth map and $\pi_{\mathbf{R}^2} : T\mathbf{R}^2 \to \mathbf{R}^2$ the canonical projection. A smooth map $w : M \to T\mathbf{R}^2$ is called a *vector field along* α if w satisfies $\alpha = \pi_{\mathbf{R}^2} \circ w$. Then we say that α is *strongly infinitesimally stable* if for every w, a vector field along α, there always exist a vector field s on M whose restriction to ∂M is a vector field on ∂M (i.e., each vector of s on ∂M is tangent to ∂M) and a vector field t on $\mathbf{R}^2$ such that

$$w = (d\alpha) \circ s + t \circ \alpha,$$

where $d\alpha : TM \to T\mathbf{R}^2$ is the differential of α.

By using an argument similar to that of Mather [11], we can show that a strongly infinitesimally stable map is stable. Thus, it is sufficient to prove that f is strongly infinitesimally stable.

Since $f|\partial M$ is stable and hence infinitesimally stable, for any w, $w|\partial M$ is expressed by $w|\partial M = d(f|\partial M) \circ s_\partial + t_\partial \circ (f|\partial M)$, where s_∂ is a vector field on ∂M and t_∂ is a vector field on $\mathbf{R}^2$. It is easy to see that there exists a vector field $\overline{s_\partial}$ on M such that $\overline{s_\partial}|\partial M = s_\partial$. If we define the new vector field w' along f by $w' = w - (df) \circ \overline{s_\partial} - t_\partial \circ f$, then w' satisfies $w'|\partial M = 0$. By the argument in the proof of [4, p. 78, Proposition 2.1], we see that there exists a smooth subbundle H complementary to $\operatorname{Ker}(df)$ in TM and that the isomorphism $df_x : H_x \to T_{f(x)}\mathbf{R}^2$ $(x \in M)$ induces an isomorphism on sections, $C^\infty(H) \to$

$C_f^\infty(T\mathbf{R}^2)$. Here, $C^\infty(H)$ denotes the set of sections of $H \subset TM$ over M and $C_f^\infty(T\mathbf{R}^2)$ denotes the set of vector fields along f. Hence we can construct a vector field $s^\circ : M \to H \subset TM$ such that $w' = (df) \circ s^\circ$. Obviously we have $s^\circ|\partial M = 0$, since $w'|\partial M = 0$, and w is expressed by $w = (df) \circ (\overline{s_\partial} + s^\circ) + t_\partial \circ f$. Note that the vector field $\overline{s_\partial} + s^\circ$ is tangent to ∂M on ∂M. This completes the proof. □

4. Stein factorization

In §3, we gave the local normal forms of a stable map $f : M \to \mathbf{R}^2$ with $S(f) = \emptyset$ around singular points of $f|\partial M$. In this section, we investigate the structure of the Stein factorization of a stable map $f : M \to \mathbf{R}^2$. Our purpose is to show that (b) implies (a) in Theorem 1. So, throughout this section we assume $S(f) = \emptyset$ and the condition (I).

Definition 1. Let M be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbf{R}^2$ a stable map with $S(f) = \emptyset$. Then $p \in S(f|\partial M)$ is a *simple point* if the connected component of $f^{-1}(f(p))$ containing p intersects $S(f|\partial M)$ only at p.

Let $\mathscr{F}_I$ (or $\mathscr{F}_{II}$) be the set of fold points of $S(f|\partial M)$ around which f is expressed by the local normal form $(Y_1, Y_2) = (X_1^2 + T, X_2)$ (resp. $(X_1^2 - T, X_2)$) as in Proposition 1. Note that a point in $\mathscr{F}_I$ is always simple and that $\mathscr{F}_{II}$ may contain non-simple points. We denote the set of non-simple points by $\mathscr{T}$. Let $\mathscr{C}$ be the set of cusp points of $f|\partial M$. Note that a cusp point is always simple, since $f|\partial M$ is a stable map. We denote the images of $\mathscr{F}_I$, $\mathscr{F}_{II}$, $\mathscr{C}$ and $\mathscr{T}$ by q_f in W_f by $W\mathscr{F}_I$, $W\mathscr{F}_{II}$, $W\mathscr{C}$ and $W\mathscr{T}$, respectively. Furthermore, we put $\Sigma = q_f(S(f|\partial M))$. Note that, $\Sigma = W\mathscr{F}_I \cup W\mathscr{F}_{II} \cup W\mathscr{C}$. For $p \in W_f$, we define as follows:

p: regular point $\Leftrightarrow p \in W_f - \Sigma$,
p: fold point of type I $\Leftrightarrow p \in W\mathscr{F}_I$,
p: fold point of type II $\Leftrightarrow p \in W\mathscr{F}_{II}$,
p: cuspidal point $\Leftrightarrow p \in W\mathscr{C}$,
p: tridental point $\Leftrightarrow p \in W\mathscr{T}$.

Definition 2. Let M be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbf{R}^2$ a stable map with $S(f) = \emptyset$. For any $y \in \mathbf{R}^2$, an embedding of a closed interval $\alpha : J \to \mathbf{R}^2$ is called a *transverse arc* at y if y is in $\alpha(\mathrm{Int}\, J)$, α is transverse to $f|\partial M$, and $\alpha(J) \cap f(S(f|\partial M)) = \{y\} \cap f(S(f|\partial M))$. For $x \in M$, if $\alpha : J \to \mathbf{R}^2$ is a transverse arc at $f(x)$, then the component of $f^{-1}(\alpha(J))$ containing x is called a *transverse manifold* at x and is denoted by $T(x)$.

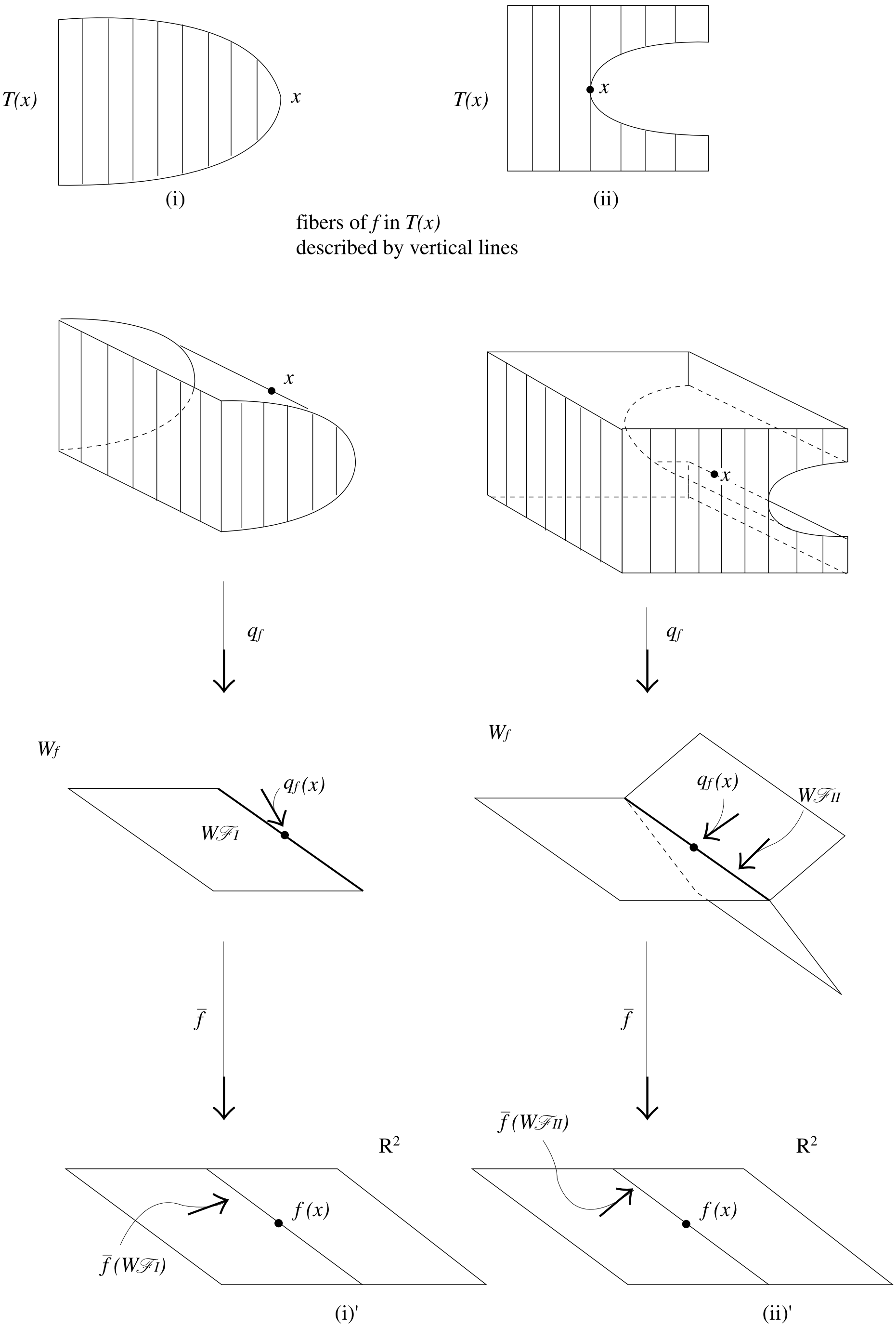
T(x)
x
(i)
(ii)
fibers of f in T(x)
described by vertical lines
x
q_f
W_f
$q_f(x)$
$W\mathscr{F}_I$
$W\mathscr{F}_{II}$
$\bar{f}$
$\bar{f}(W\mathscr{F}_{II})$
R^2
$f(x)$
$\bar{f}(W\mathscr{F}_I)$
(i)'
(ii)'

Fig. 1

Let us first consider simple singular points of $f|\partial M$. By using local normal forms obtained in § 3 and by repeating Levine's argument as described in [9, Chapter I], we obtain the following propositions, the proofs of which are easy exercises. In [9], Levine considers compact 3-dimensional manifolds without boundary, while we treat the case with boundary. Thus a main difference from the argument of [9] is the structures of the transverse manifolds. But, we can easily obtain the structures of transverse manifolds based on the local normal forms near singularities of $f|\partial M$ as described in Propositions 1, 2 and 3.

Proposition 4. *Let x be a simple point in $\mathscr{F}_I$ (or $\mathscr{F}_{II}$). Then the transverse manifold, $T(x)$, of f at x is as in Figure* 1 (i) *(resp. Figure* 1 (ii)*), and the Stein factorization W_f and the map $\bar{f}$ near $q_f(x)$ are as in Figure* 1 (i)$'$ *(resp. Figure* 1 (ii)$'$*).*

Proposition 5. *Let x be a cusp point in $\mathscr{C}$. Then the transverse manifold, $T(x)$, of f at x, the Stein factorization W_f and the map $\bar{f}$ near $q_f(x)$ are as in Figure* 2.

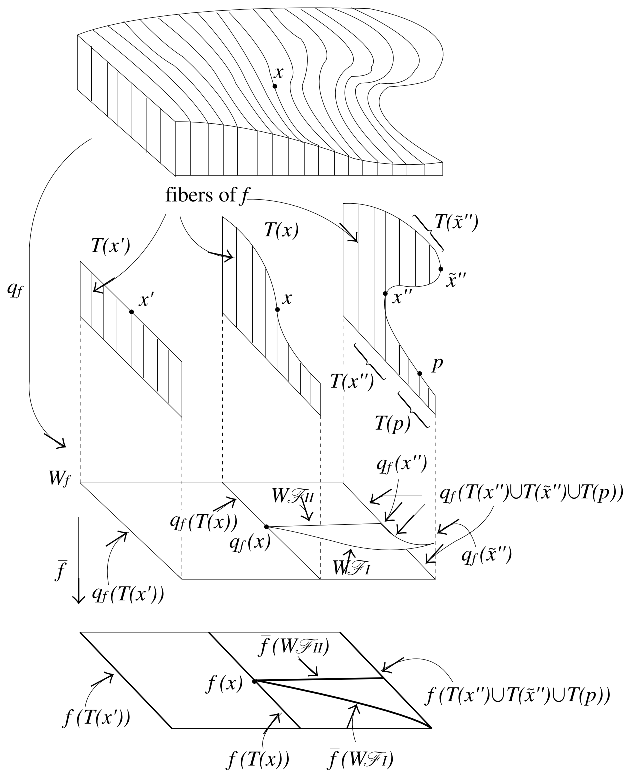

Fig. 2

Let us now consider a non-simple singular point of $f|\partial M$.

PROPOSITION 6. *Let x be a non-simple point in $S(f|\partial M)$. Then there exists a neighborhood of $q_f(x)$ in the Stein factorization W_f as in Figure* 3.

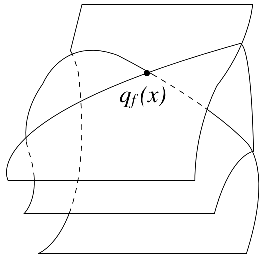

Fig. 3

PROOF. Since $f|\partial M$ is stable, $f(S(f|\partial M))$ forms a normal crossing around $f(x)$. Furthermore, non-simple points must belong to $\mathscr{F}_{II}$. By the condition (I), a component of $f^{-1}(f(x))$ containing x is homeomorphic to a closed interval, and it contains two singular points of $f|\partial M$.

As in Levine [9, p. 15, 1.4] we investigate how the fibers are situated around a non-simple point. Then we see that the connected component of $f^{-1}(U)$ containing x is as in Figure 4, where U is a certain compact neighborhood of $f(x)$ in $\mathbf{R}^2$. Thus, the corresponding Stein factorization is easily seen to be as in Figure 3. □

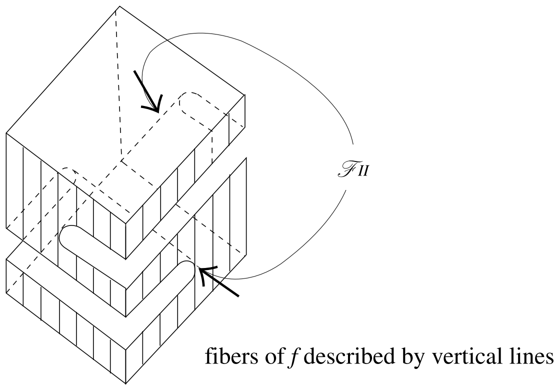

Fig. 4

Summarizing the above results, we obtain the following proposition.

PROPOSITION 7. *Let M be a compact orientable 3-manifold with non-empty boundary, and let $f: M \to \mathbf{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition* (I). *For each $x \in M$, there exists a neighborhood of $q_f(x)$ in W_f which is homeomorphic to one of the polyhedrons as in Figure 5. Moreover, W_f is a 2-dimensional polyhedron.*

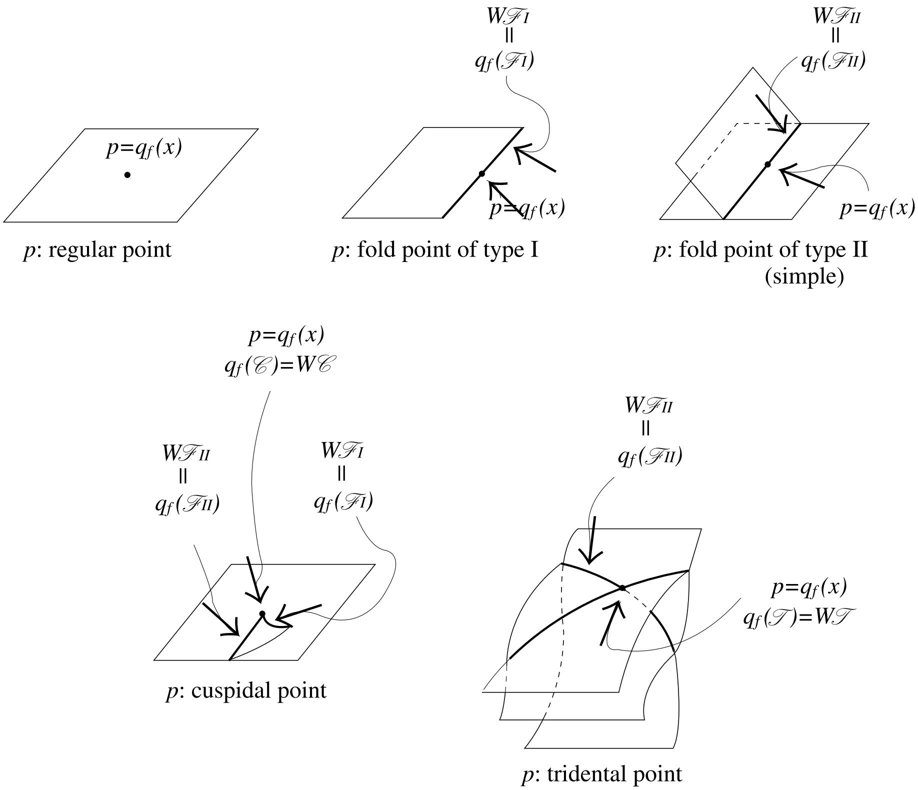

Fig. 5

REMARK 1. Note that $W_f - \Sigma$ has a natural structure of a C^∞-manifold of dimension two which is induced from $\mathbf{R}^2$ by the local homeomorphism $\bar{f}$, and that $\Sigma - (W\mathscr{C} \cup W\mathscr{T})$ also has a natural structure of a C^∞-manifold of dimension one.

5. Immersion lift from M to $\mathbf{R}^3$

In this section, we prove Theorem 1. We may suppose that M and $\mathbf{R}^2$ are oriented. Then each connected component of fibers of f which is homeomorphic to a closed interval has the induced orientation.

We first prove the implication (a) ⇒ (b) in Theorem 1. Since $f = \pi \circ F$ for an immersion F and a submersion π, we have $S(f) = \emptyset$. Let r be a point of $f(M)$. Then by Propositions 1, 2 and 3, for every $x \in f^{-1}(r)$, there exists an open neighborhood U of x in M such that U satisfies one of the following:

(1) $U \cap f^{-1}(r) \approx (-1, 1)$ $(x \in \mathrm{Int}\, M \cup \mathscr{F}_{II})$,

(2) $U \cap f^{-1}(r) \approx [0, 1)$ $(x \in (\partial M \cap (M \backslash S(f|\partial M))) \cup \mathscr{C})$,

(3) $U \cap f^{-1}(r)$ is a point $(x \in \mathscr{F}_I)$,

where "$\approx$" denotes a homeomorphism. Thus, $f^{-1}(r)$ is a disjoint union of 1-dimensional manifolds with or without boundary and discrete points. By the compactness of $f^{-1}(r)$, $f^{-1}(r)$ must be homeomorphic to a finite disjoint union of circles, closed intervals and points. However, since $f^{-1}(r) \subset \{r\} \times \mathbf{R}$, $f^{-1}(r)$ cannot contain circles. This implies the condition (I) and hence (b).

The remainder of this section is devoted to the proof of the implication (b) ⇒ (a) in Theorem 1 or its restatement, Proposition 9.

Set $Y = \{re^{\sqrt{-1}\theta} \in \mathbf{C} \mid 0 \leq r \leq 1, \theta = \pi/3, \pi, 5\pi/3\}$, $Y_0 = \{re^{\sqrt{-1}\theta} \in Y \mid r \neq 0, \theta = \pi\}$, $Y_1 = \{re^{\sqrt{-1}\theta} \in Y \mid r \neq 0, \theta = \pi/3\}$ and $Y_2 = \{re^{\sqrt{-1}\theta} \in Y \mid r \neq 0, \theta = 5\pi/3\}$. Define $\sigma : Y \to [-1, 1/2]$ by $\sigma(z) = \mathrm{Re}\, z$. Assume that $x \in \mathscr{F}_{II} - \mathscr{T}$. Then, there exist homeomorphisms $\Lambda : q_f(T(x)) \to Y$ and $\lambda : f(T(x)) \to [-1, 1/2]$ such that $\sigma \circ \Lambda = \lambda \circ \bar{f}|q_f(T(x))$. We say that $\Lambda^{-1}(Y_0)$ is the *stem* and $\Lambda^{-1}(Y_1)$ and $\Lambda^{-1}(Y_2)$ are the *arms* of $q_f(T(x))$. The transverse manifold $T(x)$, its image $q_f(T(x))$ in W_f and their images in $\mathbf{R}^2$ are described in Figure 6. The fibers of f in $T(x)$ are described by vertical lines with arrows consistent with their orientations. The two arms in $q_f(T(x))$ are classified into the *upper arm* α_+ and the *lower arm* α_- by the images of the upper branch $\tilde{\alpha}_+$ and the lower branch $\tilde{\alpha}_-$ respectively in $T(x)$. The upper branch $\tilde{\alpha}_+$ contains the upper part of the fiber passing through the point x as in Figure 6, and the lower branch $\tilde{\alpha}_-$ contains the lower part.

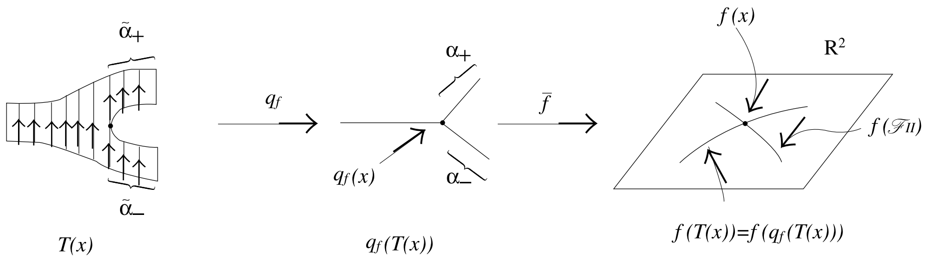

Fig. 6

Since W_f is a polyhedron by Proposition 7, we can take sufficiently small regular neighborhoods $N(p)$ of $p \in W\mathscr{C} \cup W\mathscr{T}$ so that $N(p) \cap N(p') = \emptyset$ if $p \neq p'$, and that $N(p)$ coincides with a component of $\bar{f}^{-1}(D)$ for some $D \subset \mathbf{R}^2$, where D is homeomorphic to $I \times I$, $I = [0,1]$. Moreover, if c is a connected component of $W\mathscr{F}_I - \bigcup_p \operatorname{Int} N(p)$ (or $W\mathscr{F}_{II} - \bigcup_p \operatorname{Int} N(p)$), then c has a regular neighborhood $N(c)$ relative boundary in W_f which is homeomorphic to $I \times c$ (or $Y \times c$ resp.). In fact, since $\bar{f}$ is an immersion on $W_f - \Sigma$, a regular neighborhood $N(c)$ is homeomorphic to an I-bundle (or Y-bundle resp.) over c. When $c \subset W\mathscr{F}_I - \bigcup_p \operatorname{Int} N(p)$, this I-bundle is immersed in $\mathbf{R}^2$ and hence trivial. Furthermore, suppose that $c \subset W\mathscr{F}_{II} - \bigcup_p \operatorname{Int} N(p)$ and $N(c)$ contains a non-trivial Y-bundle over a circle c_1 in c which exchanges the arms along c_1. Then for a section s of the sub I-bundle consisting of the stems along c_1, $q_f^{-1}(s)$ forms a non-orientable I-bundle, i.e., Möbius band. This contradicts the induced orientations of fibers.

We may assume that $N(c) \cap N(c') = \emptyset$ if $c \neq c'$. We may also assume $(\bigcup_p N(p)) \cup (\bigcup_c N(c)) = N(\Sigma)$, the regular neighborhood of Σ.

Definition 3. Let M be a compact orientable 3-manifold with non-empty boundary, and let $f: M \to \mathbf{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (I). Then a continuous map $g: W_f \to \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ is said to be an *immersion lift* of $\bar{f}$ to $\mathbf{R}^3$ if $\bar{f} = \pi \circ g$ and the following conditions (1), (2), (3) and (4) are satisfied.

(1) $g|(W_f - \Sigma)$ is a smooth immersion with normal crossings.

(2) $g|\Sigma$ is an injection, and $g|(\Sigma - (W\mathscr{C} \cup W\mathscr{T}))$ is a smooth embedding.

(3) $g|N(\Sigma)$ is an injection, and $g|(N(\Sigma) - \Sigma)$ is a smooth embedding.

(4) For each $x \in \mathscr{F}_{II} - \mathscr{T}$, we have $\pi' \circ g(a) > \pi' \circ g(b)$ for any point a of the upper arm and any point b of the lower arm of $q_f(T(x))$, where $\pi': \mathbf{R}^3 \to \mathbf{R}$ is the projection to the last coordinate.

Proposition 8. *Let M be a compact orientable* 3*-manifold with non-empty boundary, and let $f: M \to \mathbf{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition* (I). *Then there exists an immersion lift $g: W_f \to \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ of the form $g(x) = (\bar{f}(x), h_0(x))$.*

Proof. Let p be a point of $W\mathscr{C} \cup W\mathscr{T}$. Then we define $g|(N(p) \cap \Sigma): N(p) \cap \Sigma \to \mathbf{R}^2 = \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$ by $g|(N(p) \cap \Sigma) = \bar{f}|(N(p) \cap \Sigma)$. Then $g|(N(p) \cap \Sigma)$ is injective. Moreover, g can be extended all over Σ by separating normal crossing points of $\bar{f}|(\Sigma - (W\mathscr{C} \cup W\mathscr{T}))$ into extra dimension. Thus we can define $g|\Sigma$ so that $g|\Sigma$ satisfies the above condition (2).

Let us extend g over $N(\Sigma)$. First, we lift the neighborhoods $N(p)$, $p \in W\mathscr{C} \cup W\mathscr{T}$, to $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ so that $g|N(p)$ satisfies the condition (4), and so that the angle between the images of two arms contained in $N(p) - \operatorname{Int} N(p)$

is δ $(0 < \delta < \pi)$ and that the image of each stem contained in $N(p) - \operatorname{Int} N(p)$ is horizontal. To extend g all over $N(\Sigma)$, let $\mathscr{S}$ be the set of the connected components of $\Sigma - \bigcup_p \operatorname{Int} N(p)$, $p \in W\mathscr{C} \cup W\mathscr{T}$. We consider lifts on each $N(c)$, $c \in \mathscr{S}$. Let $\Pi : N(c) \to c$ be the natural bundle projection whose fibers are homeomorphic to $I = [0, 1]$ if $c \subset W\mathscr{F}_I$ or to Y if $c \subset W\mathscr{F}_{II}$.

First, for $c \subset W\mathscr{F}_I$, define $g : N(c) \to \mathbf{R}^3$ by $x \mapsto (\bar{f}(x), h_0(\Pi(x)))$, where $h_0 : c \to \mathbf{R}$ is the smooth function which gives the third coordinate. Second, for $c \subset W\mathscr{F}_{II}$, $N(c)$ is homeomorphic to $Y \times c$. Then define $g : N(c) \to \mathbf{R}^3$ by $x \mapsto (\bar{f}(x), h_0(\Pi(x)) + Z(x))$, where $h_0 : c \to \mathbf{R}$ is the smooth function which gives the third coordinate and $Z : N(c) \to \mathbf{R}$ is defined as follows: if x belongs to a stem, then we define $Z(x) = 0$, and if x belongs to an upper (resp. lower) arm, then we define $Z(x) = \|\bar{f}(x) - \bar{f}(\Pi(x))\| \tan \delta/2$ (resp. $-\|\bar{f}(x) - \bar{f}(\Pi(x))\| \tan \delta/2$). Here note that our construction of the lifts on $N(p)$ and on $N(c)$ are consistent, and then we may assume that $g|N(\Sigma)$ is an injection and that $g|(N(\Sigma) - \Sigma)$ is a smooth embedding by choosing a sufficiently small δ. Thus a lift on $N(\Sigma)$ which satisfies the conditions (3) and (4) has been constructed.

Finally, we can extend the lift to whole W_f by using an argument similar to that of [7, pp. 26–27] and complete the proof. □

Proposition 9. *Let M be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbf{R}^2$ a stable map with $S(f) = \emptyset$ and the condition* (I). *Then there exists an immersion $F : M \to \mathbf{R}^3$ which makes the following diagram commutative.*

$$\begin{array}{ccc} & & \mathbf{R}^3 \\ & \overset{F}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{\ f\ } & \mathbf{R}^2 \end{array}$$

Proof. We use the same notations as in the proof of Proposition 8, and construct an immersion lift $F : M \to \mathbf{R}^3$ based on $g : W_f \to \mathbf{R}^3$.

First, let us construct a lift on $q_f^{-1}(N(\Sigma))$ to $\mathbf{R}^3$. We lift $q_f^{-1}(N(p)) \times$ $(p \in W\mathscr{C} \cup W\mathscr{T})$ as the top figure in Figure 2 and Figure 4, and then we lift the other part of $q_f^{-1}(N(\Sigma))$ as the top figures in (i)$'$, (ii)$'$ of Figure 1 so that $F|q_f^{-1}(N(\Sigma))$ is expressed by $x \mapsto g(q_f(x)) + (0, 0, h_0(x))$, where $h_0 : q_f^{-1}(N(\Sigma)) \to \mathbf{R}$ is an orientation preserving embedding on each q_f-fiber. In the construction, we can arrange so that the orientation of the F-image of each oriented fiber of q_f contained in $\{r\} \times \mathbf{R}$ $(r \in \mathbf{R}^2)$ coincides with that of the last coordinate of $\mathbf{R}^3$. By (3) of Definition 3, we can construct the lift $F|q_f^{-1}(N(\Sigma))$ as an embedding.

Similarly, for $q_f^{-1}(\overline{W_f - N(\Sigma)})$, we can construct a smooth function

$h_1 : q_f^{-1}(\overline{W_f - N(\Sigma)}) \to \mathbf{R}$, where $h_1 = h_0$ on $q_f^{-1}(\overline{W_f - N(\Sigma)}) \cap q_f^{-1}(N(\Sigma))$, and define $F|q_f^{-1}(\overline{W_f - N(\Sigma)})$ by $x \mapsto g(q_f(x)) + (0, 0, h_1(x))$ so that the restriction of h_1 to each q_f-fiber (which is homeomorphic to a closed interval by the condition (I)) is an orientation preserving embedding, and that $F|q_f^{-1}(\overline{W_f - N(\Sigma)})$ is an immersion. This completes the proof of Proposition 9. □

Now we have completed the proof of Theorem 1 by proving (b) ⇒ (a) by Proposition 9 and (a) ⇒ (b) at the beginning of this section. We give some remarks before closing the section.

Remark 2. The condition $S(f) = \emptyset$ does not imply the condition (I) in Theorem 1 as follows. Let N be an annulus, and consider $M = N \times S^1$. Let $\rho : N \to \mathbf{R}$ be a height function as in Figure 7 such that ρ is non-singular, while $\rho|\partial M$ is a Morse function with exactly four critical points, and that ρ contains a fiber homeomorphic to S^1. Then define $\rho \times \mathrm{id} : N \times S^1 \to \mathbf{R} \times S^1$ by $(x, t) \mapsto (\rho(x), t)$. Finally, consider an embedding $\eta : \mathbf{R} \times S^1 \to \mathbf{R}^2$ and we define $f = \eta \circ (\rho \times S^1) : M \to \mathbf{R}^2$. This f is stable, $S(f) = \emptyset$, and we can find a point $r \in \mathbf{R}^2$ such that $f^{-1}(r)$ is homeomorphic to S^1.

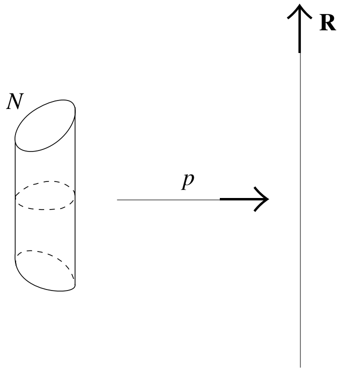

Fig. 7

However, the condition (I) does imply $S(f) = \emptyset$ under the condition that $S(f) \cap \partial M = \emptyset$. To show this, suppose $S(f) \neq \emptyset$. Then there exists a definite fold or an indefinite fold point as a singularity of f. If M contains a definite fold point $p \in \mathrm{Int}\, M$, then there must exist a fiber near p which contains a connected component homeomorphic to S^1. If M contains an indefinite fold point $p' \in \mathrm{Int}\, M$, then the connected component of the fiber containing p' cannot be diffeomorphic to a closed interval or a point. Hence, if $S(f) \neq \emptyset$, then f does not satisfy the condition (I). Thus the condition (I) implies $S(f) = \emptyset$, provided that $S(f) \cap \partial M = \emptyset$.

REMARK 3. Haefliger [5, Théorème 1] showed that for a stable map from a closed 2-manifold N into $\mathbf{R}^2$, there exists an immersion lift to $\mathbf{R}^3$ with respect to the standard projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ if and only if each connected component of its singular set has an orientable (or non-orientable) neighborhood if the number of cusps on the connected component is even (resp. odd).

Let F be an immersion lift of a stable map $f : M \to \mathbf{R}^2$ as in Theorem 1. Then the stable map $f|\partial M : \partial M \to \mathbf{R}^2$ is also lifted to $\mathbf{R}^3$ by $F|\partial M$. Then, by Haefliger [5], each connected component of $S(f|\partial M)$ must have an even number of cusps, since ∂M is an orientable closed surface.

In fact, cusps of $f|\partial M$ correspond exactly to cuspidal points of W_f by q_f. From the structure of W_f obtained in Proposition 7, the connected components of $\mathscr{F}_I$ and those of $\mathscr{F}_{II}$ must connect one after the other alternately at cusp points of $f|\partial M$ as their connecting points, and all of them must form circles. Hence, the number of cusps on each circle is even. Therefore, the stable map $f|\partial M$ automatically satisfies the condition of Haefliger.

REMARK 4. Kushner-Levine-Porto [7] have given a sufficient condition for the existence of an immersion lift to $\mathbf{R}^4$ with respect to the projection $\pi : \mathbf{R}^4 \to \mathbf{R}^2$, $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$, for a stable map from a closed orientable 3-manifold to $\mathbf{R}^2$. Of course, there is no immersion lift to $\mathbf{R}^3$ for a closed 3-manifold.

6. Embedding lift from M to $\mathbf{R}^n$

In §5, we considered the existence problem of an immersion lift F to $\mathbf{R}^3$ for a stable map from M into $\mathbf{R}^2$. We will consider the embedding lift to $\mathbf{R}^n$, $n = 3, 4$ and $n \geq 5$.

REMARK 5. There is a stable map f which satisfies the condition (b) in Theorem 1 but has no embedding lifts to $\mathbf{R}^3$.

We take the compact orientable 3-manifold with boundary $S^2 \times S^1 - \operatorname{Int} D^3$ for M. No stable map from M into $\mathbf{R}^2$ can have an embedding lift F to $\mathbf{R}^3$. In fact, if M is embedded into $\mathbf{R}^3$, then $\partial M = S^2$ bounds an embedded 3-ball in $\mathbf{R}^3$ by the theorem of Schönflies. This means that M itself is homeomorphic to D^3; a contradiction. We identify $M = S^2 \times S^1 - \operatorname{Int} D^3$ with $D^2 \times I \cup_\varphi S^2 \times I$ and give an immersion $i : M \to \mathbf{R}^3$ as in Figure 8, where $\varphi : D^2 \times \partial I \to S^2 \times \partial I$ is a handle attaching map. We can see that the map $f = \pi \circ i$ is stable by Lemma 2. Moreover, $S(f) = \emptyset$ and f satisfies the condition (I).

In this example, two cusps appear around each component of $\varphi(D^2 \times \partial I)$. The upper and lower arms in $q_f(T(x)) \subset W_f$ at the fold points $x \in \partial M$ of type $\mathscr{F}_{II}$ are drawn in the figure so as to satisfy the condition (4) of Definition

3. We understand that it is difficult to modify the immersion lift of $\bar{f}$ to an embedding keeping this condition.

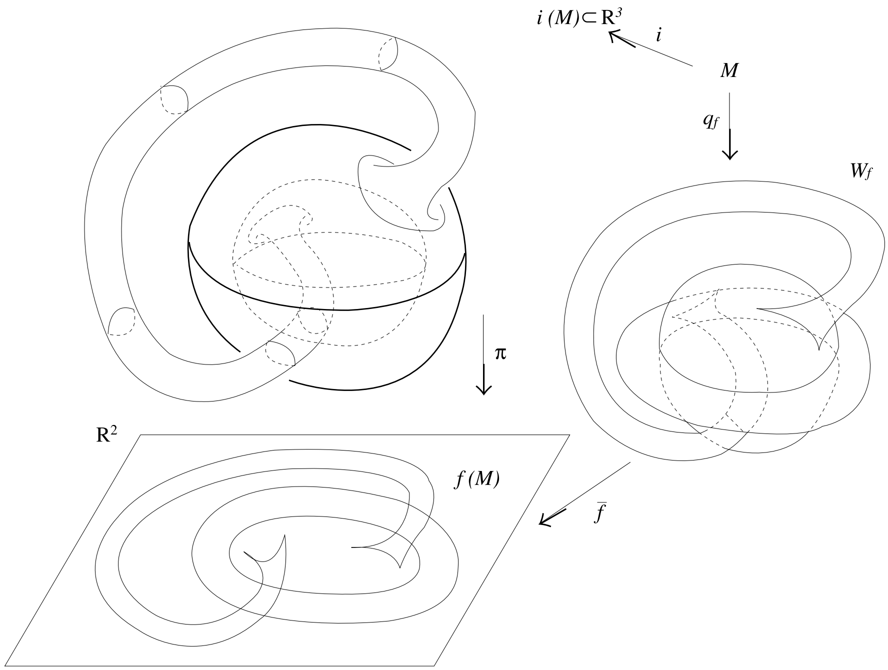

Fig. 8

Remark 6. There is a stable map f which satisfies the condition (b) in Theorem 1 but has no embedding lifts to $\mathbf{R}^4$.

Let M be a punctured lens space $L(2n,q)^\circ$. It is a compact orientable 3-manifold with boundary S^2. Then we can construct a stable map $f: M \to \mathbf{R}^2$ with $S(f) = \emptyset$ and our condition (I) by Lemma 2. However, it has been shown in [3] that a punctured lens space $L(2n,q)^\circ$ cannot be embedded in $\mathbf{R}^4$. Hence f cannot have an embedding lift to $\mathbf{R}^4$.

Definition 4. Let M be a compact orientable 3-manifold with non-empty boundary, and let $f: M \to \mathbf{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (I). Then, a continuous map $g_e: W_f \to \mathbf{R}^n$ is said to be an *embedding lift* of $\bar{f}$ to $\mathbf{R}^n$ if g_e satisfies $\bar{f} = \pi \circ g_e$ with respect to the projection $\pi: \mathbf{R}^n \to \mathbf{R}^2$, $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2)$, and the following.

(1) g_e is a topological embedding.
(2) $g_e|(W_f - \Sigma)$ is a smooth embedding.
(3) $g_e|(\Sigma - (W\mathscr{C} \cup W\mathscr{T}))$ is a smooth embedding.
(4) $g_e(N(\Sigma)) \subset \mathbf{R}^3 \times \{0\} \subset \mathbf{R}^n$, and $g_e|N(\Sigma)$ satisfies the condition (4) of

Definition 3 as a map into $\mathbf{R}^3$.

Remark 7. In the example given in Remark 5 (see Figure 8), we can see that $\bar{f}$ has a lift to $\mathbf{R}^3$ which is a topological embedding. But we have no embedding lift of $\bar{f}$ as defined in Definition 4, because it contradicts the following proposition.

Proposition 10. *Let M be a compact orientable 3-manifold with non-empty boundary, and let $f : M \to \mathbf{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (I). If there exists an embedding lift $g_e : W_f \to \mathbf{R}^n$ of $\bar{f}$ with respect to $\pi : \mathbf{R}^n \to \mathbf{R}^2$, $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2)$, then there exists an embedding lift $F_e : M \to \mathbf{R}^n$ of f. In particular, for $n \geq 5$, there always exists an embedding lift F_e of f.*

Proof. By virtue of the condition (4) of Definition 4, we can construct an embedding lift on $q_f^{-1}(N(\Sigma))$ so that $F_e(q_f^{-1}(N(\Sigma))) \subset \mathbf{R}^3 \times \{0\}$ by using an argument similar to that in the proof of Proposition 9.

Then, we construct the lift on $q_f^{-1}(\overline{W_f - N(\Sigma)})$ as follows. By the construction of $F_e|q_f^{-1}(N(\Sigma))$, we have $F_e(q_f^{-1}(p)) \subset \bar{f}(p) \times \mathbf{R} \times \{0\} \subset \mathbf{R}^3\{0\} \subset \mathbf{R}^n$ for any $p \in N(\Sigma)$. Hence, we can construct F_e on $q_f^{-1}(\overline{W_f - N(\Sigma)})$ by $x \mapsto g_e(q_f(x)) + (0, 0, h_0(x), 0, \dots, 0)$, where h_0 is an orientation preserving embedding on each q_f-fiber. Since $g_e|q_f(\overline{W_f - N(\Sigma)})$ is a smooth embedding, we can arrange so that $F_e(x) \neq F_e(x')$ if $q_f(x) \neq q_f(x')$. Thus an embedding lift F_e of f has been constructed.

The existence of an immersion lift $g : W_f \to \mathbf{R}^3$ is guaranteed by our Proposition 8. In general, the lift $g|(W_f - N(\Sigma))$ has normal crossings. However, if $n \geq 5$, then we can separate the normal crossings into extra dimensions in $\mathbf{R}^n$ by Thom's transversality theorem so that g satisfies $\pi \circ g = \bar{f}$. Therefore, for $n \geq 5$, we can always construct an embedding lift from W_f to $\mathbf{R}^n$ and hence from M to $\mathbf{R}^n$. This completes the proof. □

7. Applications

In this section, first we prove Theorem 2 as an application of the results obtained in §4. For a closed orientable 3-manifold $\hat{M}$, Burlet-de Rham [1] have proved that there exists a special generic map $f : \hat{M} \to \mathbf{R}^2$ if and only if $\hat{M}$ is diffeomorphic to S^3 or to a connected sum $\sharp^k(S^2 \times S^1)$, where a special generic map is a stable map which has only definite fold points as its singularities. Saeki [12] has obtained a characterization of graph manifolds by using simple stable maps (defined in [12]), where a graph manifold is defined to be a 3-manifold built up of S^1-bundles over surfaces attached along their torus boundaries. As an analogy, we consider the structure of source manifolds of the boundary special generic maps defined as follows.

Definition 5. Let M be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbf{R}^2$ a stable map with $S(f) = \emptyset$. Then f is called a *boundary special generic map* if $S(f|\partial M) = \mathscr{F}_I$.

Lemma 3. *Let M be a compact orientable 3-manifold with non-empty boundary. Then any boundary special generic map $f : M \to \mathbf{R}^2$ satisfies the condition* (I).

Proof. Let r be a point in $f(M)$ and r' a point such that $r' \notin f(M)$. Consider a smooth embedding $C : [0,1] \to \mathbf{R}^2$ such that $C(0) = r'$, $C(1) = r$ and C is transverse to $f|\partial M$. Then $f|f^{-1}(C([0,1])) : f^{-1}(C([0,1])) \to C([0,1])$ is a non-singular function on a surface with boundary, and each singularity of $f|\partial M$ in $f^{-1}(C([0,1]))$ belongs to $\mathscr{F}_I$ so that only arcs appear or disappear in the inverse image. Set

$$A = \{t \in [0,1] \mid f^{-1}(C(t)) \not\supset S^1\}.$$

Then we have (1) $A \ni 0$, in particular, $A \neq \emptyset$, (2) A is open, and (3) the complement of A is open. Since $[0,1]$ is connected, we see $A = [0,1]$. Hence $f^{-1}(r)$ does not contain a circle component. Then the result follows as in the proof of (a) $\Rightarrow$ (b) in Theorem 1 given at the beginning of § 5. □

Proof of Theorem 2. Suppose that M is a compact orientable 3-dimensional handlebody. Then, we can construct a boundary special generic map f from M into $\mathbf{R}^2$ as in Figure 9, where i is an embedding so that $\pi \circ i$ has only singularities of type $\mathscr{F}_I$ at ∂M.

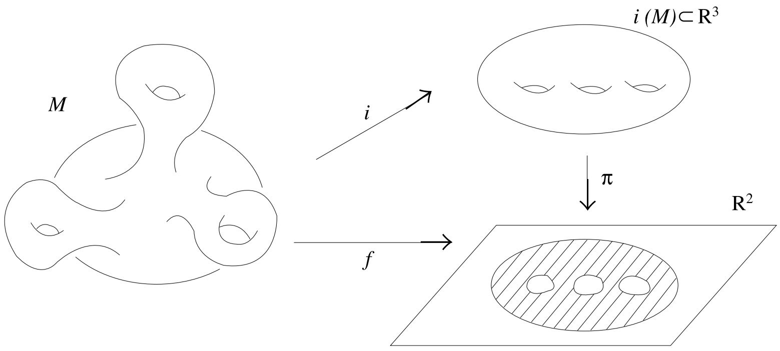

Fig. 9

Conversely, suppose that $f : M \to \mathbf{R}^2$ is a boundary special generic map. Then W_f must be a connected surface with non-empty boundary by Lemma 3 and Propositions 4 and 7. Since M is compact, so is W_f. By the smooth structure of $\overline{W_f - N(\Sigma)}$ defined in Remark 1, the continuous map $q_f|q_f^{-1}(\overline{W_f - N(\Sigma)})$ is a differentiable map, and moreover a submersion. Here, note that $\operatorname{rank} d(f|\partial M)_x = \dim \mathbf{R}^2$ for all $x \in \partial M \cap q_f^{-1}(\overline{W_f - N(\Sigma)})$. So, by applying Lemma 3 and Ehresmann's fibration theorem ([2] and [8, p. 23]), $q_f^{-1}(\overline{W_f - N(\Sigma)})$ has a structure of an I-bundle over $\overline{W_f - N(\Sigma)}$. On the other hand, by the local structure given by Proposition 4 for the fold points of type $\mathscr{F}_I$, we see that $q_f^{-1}(N(\Sigma))$ is a trivial I-bundle over $N(\Sigma)$ which is homeomorphic to $\partial W_f \times I$. Thus we see that M is an I-bundle over a compact connected surface W_f with non-empty boundary and hence that M is a 3-dimensional handlebody. □

Let us prove Theorem 3 as an application of the arguments in §5 and 6.

PROOF OF THEOREM 3. If there exists an embedding lift $g_e : W_f \to \mathbf{R}^3$, then there also exists an embedding lift $F_e : \hat{M} - \operatorname{Int} D^3 \to \mathbf{R}^3$ by Proposition 10. Since $\partial(\hat{M} - \operatorname{Int} D^3) = S^2$, S^2 is embedded in $\mathbf{R}^3$ by F_e. By the theorem of Schönflies, $S^2 = \partial(\hat{M} - \operatorname{Int} D^3)$ bounds a 3-ball in $\mathbf{R}^3$; i.e., $\hat{M} - \operatorname{Int} D^3$ must be homeomorphic to D^3. Hence $\hat{M} = (\hat{M} - \operatorname{Int} D^3) \cup D^3 \approx D^3 \cup D^3 \approx S^3$, where each "$\approx$" denotes a homeomorphism. This completes the proof. □

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