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A multivariate growth curve model with differing numbers of random effects

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Abstract. In this paper we consider a multivariate growth curve model with covariates and random effects. The model is a mixed MANOVA-GMANOVA model which has multivariate random-effects covariance structures. Test statistics for a general hypothesis concerning the adequacy of a family of the covariance structures are proposed. A modified LR statistic for the hypothesis and its asymptotic expansion are obtained. The MLE's of unknown mean parameters are obtained under the covariance structures. The efficiency of the MLE is discussed. A numerical example is also given.

1. Introduction

Suppose that we obtain repeated measurements of m response variables on each of p occasions (or treatments) for each of N individuals and that we can use observations of r covariates for each individual. Let $\boldsymbol{x}_j^{(g)}$ be an mp-vector of measurements on the j-th individual in the g-th group arranged as

$$\boldsymbol{x}_j^{(g)} = (x_{11j}^{(g)}, \dots, x_{1mj}^{(g)}, \dots\dots, x_{p1j}^{(g)}, \dots, x_{pmj}^{(g)})',$$

and assume that $\boldsymbol{x}_j^{(g)}$'s are independently distributed as $N_{mp}(\boldsymbol{\mu}_j^{(g)}, \Omega)$, where Ω is an unknown $mp \times mp$ positive definite matrix, $j = 1, \dots, N_g$, $g = 1, \dots, k$. Further, we assume that mean profiles of $\boldsymbol{x}_j^{(g)}$ are m-variate growth curves with r covariates, i.e.,

$$\boldsymbol{\mu}_j^{(g)} = (B' \otimes I_m)\xi^{(g)} + \Theta' \boldsymbol{c}_j^{(g)},$$

where B is a $q \times p$ within-individual design matrix of rank q $(\leq p)$, $B' \otimes I_m$ is the Kronecker product of B' and the $m \times m$ identity matrix, $\boldsymbol{c}_j^{(g)}$'s are r-vectors of observations of covariates, $\xi^{(g)}$'s are mq-vectors of unknown parameters, Θ is an unknown $r \times mp$ parameter matrix. Let

$$X = [\boldsymbol{x}_1^{(1)}, \dots, \boldsymbol{x}_{N_1}^{(1)}, \dots\dots, \boldsymbol{x}_1^{(k)}, \dots, \boldsymbol{x}_{N_k}^{(k)}]', \qquad N = N_1 + \cdots + N_k.$$

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Then the model of X can be written as

$$X \sim N_{N\times mp}(A\Xi(B\otimes I_m)+C\Theta, \Omega\otimes I_N), \tag{1.1}$$

where

$$A=\begin{pmatrix} \mathbf{1}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{N_k} \end{pmatrix}$$

is an $N\times k$ between-individual design matrix, $\mathbf{1}_n$ is an n-vector of ones, $C=[\boldsymbol{c}_1^{(1)},\dots,\boldsymbol{c}_{N_1}^{(1)},\dots\dots,\boldsymbol{c}_1^{(k)},\dots,\boldsymbol{c}_{N_k}^{(k)}]'$ is a fixed $N\times r$ matrix of covariates, rank $[A,C]=k+r$ $(\le N-p)$, $\Xi=[\xi^{(1)},\dots,\xi^{(k)}]'$ is an unknown $k\times mq$ parameter matrix. Without loss of generality, we may assume that $BB'=I_q$. The mean structure of (1.1) is a mixed MANOVA-GMANOVA model, and the GMANOVA portion is an extension of Potthoff and Roy [5] to the multiple-response case. For the model (1.1) in the single-response case $(m=1)$, see Yokoyama and Fujikoshi [10] and Yokoyama [12]. This type of models has been discussed by Chinchilli and Elswick [2], Verbyla and Venables [9], etc. For a comprehensive review of the literature on such models, see, e.g., von Rosen [7] and Kshirsagar and Smith [3, p. 85].

In this paper we consider a family of covariance structures

$$\Omega_s=(B_s'\otimes I_m)\Delta_s(B_s\otimes I_m)+I_p\otimes\Sigma_s, \qquad 0\le s\le q, \tag{1.2}$$

which is based on random-coefficients models with differing numbers of random effects (see Lange and Laird [4]), where Δ_s and Σ_s are arbitrary $ms\times ms$ positive semi-definite and $m\times m$ positive definite matrices respectively, B_s is the matrix which is composed of the first s rows of B. This family is a generalization of a multivariate random-effects covariance structure proposed by Reinsel [6]. In fact, the covariance structures (1.2) can be introduced by assuming the following model:

$$\boldsymbol{x}_j^{(g)}=\boldsymbol{\mu}_j^{(g)}+(B_s'\otimes I_m)\boldsymbol{\eta}_j^{(g)}+\boldsymbol{\varepsilon}_j^{(g)},$$

where $\boldsymbol{\eta}_j^{(g)}$ is an ms-vector of random effects distributed as $N_{ms}(\mathbf{0},\Delta_s)$, $\boldsymbol{\varepsilon}_j^{(g)}$ is an mp-vector of random errors distributed as $N_{mp}(\mathbf{0},I_p\otimes\Sigma_s)$, $\boldsymbol{\eta}_j^{(g)}$'s and $\boldsymbol{\varepsilon}_j^{(g)}$'s are mutually independent. This implies that $V(\boldsymbol{x}_j^{(g)})=\Omega_s$. In §2 we derive a canonical form of the model (1.1). A test statistic for testing $H_{0s}:\Omega=\Omega_s$ vs. H_{1s}: not H_{0s} in the model (1.1) has been proposed by Yokoyama [13]. In §3 we propose test statistics for the hypothesis

$$H_{0s}:\Omega=\Omega_s \qquad \text{vs.} \qquad H_{1t}:\Omega=\Omega_t \tag{1.3}$$

in the model (1.1), where $1\le s<t\le q$. Since the exact likelihood ratio

($=$ LR) statistic for the hypothesis is complicated, it is suggested to use a modified LR statistic, which is the LR statistic for a modified hypothesis. An asymptotic expansion of the null distribution of the statistic is obtained. By making this strong assumption that $\Omega = \Omega_s$, we can expect to have efficient estimators. In §4 we obtain the maximum likelihood estimators ($=$ MLE's) of unknown mean parameters under the covariance structures (1.2). In comparison with the MLE of Ξ when no special assumptions about Ω are made, we show how much gains can be obtained for the maximum likelihood estimation of Ξ by assuming that Ω has the structures (1.2). In §5 we give a numerical example of the results of §4.

2. Canonical form of the model

In order to transform (1.1) to a model which is easier to analyze, we use a canonical reduction. We define the submatrices B_t^- and $B_{s\cap t}^-$ of B by $B = [B_t', B_t^{-\prime}]'$, $B_t = [B_s', B_{s\cap t}^{-\prime}]'$. Let $\bar{B}$ be a $(p-q)\times p$ matrix such that $\bar{B}\bar{B}' = I_{p-q}$ and $B\bar{B}' = 0$. Then $G = [B_s', B_{s\cap t}^{-\prime}, B_t^{-\prime}, \bar{B}']' = [G_1', G_2', G_3', G_4']'$ is an orthogonal matrix of order p, where $G_1' = [\boldsymbol{g}_1^{(1)}, \dots, \boldsymbol{g}_1^{(s)}]' : p \times s$, $G_2' = [\boldsymbol{g}_2^{(1)}, \dots, \boldsymbol{g}_2^{(t-s)}]' : p \times (t-s)$, $G_3' = [\boldsymbol{g}_3^{(1)}, \dots, \boldsymbol{g}_3^{(q-t)}]' : p \times (q-t)$, $G_4' = [\boldsymbol{g}_4^{(1)}, \dots, \boldsymbol{g}_4^{(p-q)}]' : p \times (p-q)$. Therefore, $Q = G \otimes I_m = [Q_1', Q_2', Q_3', Q_4']' = [Q_1^{(1)\prime}, \dots, Q_1^{(s)\prime}, Q_2^{(1)\prime}, \dots, Q_2^{(t-s)\prime}, Q_3^{(1)\prime}, \dots, Q_3^{(q-t)\prime}, Q_4^{(1)\prime}, \dots, Q_4^{(p-q)\prime}]'$ is an orthogonal matrix of order mp. Further, let $H = [H_1, H_2]$ be an orthogonal matrix of order N such that H_1 is an orthonormal basis matrix on the space spanned by the column vectors of C. Then, letting $Y = H_2'XQ' = [Y_1, Y_2, Y_3, Y_4] = [Y_1^{(1)}, \dots, Y_1^{(s)}, Y_2^{(1)}, \dots, Y_2^{(t-s)}, Y_3^{(1)}, \dots, Y_3^{(q-t)}, Y_4^{(1)}, \dots, Y_4^{(p-q)}]$, $[Y_1, Y_2, Y_3] = Y_{(123)}$ and $[Y_2, Y_3] = Y_{(23)} = [Y_{(23)}^{(1)}, \dots, Y_{(23)}^{(q-s)}]$, the model (1.1) can be reduced to a canonical form

$$H'XQ' = \begin{bmatrix} & Z & \\ Y_{(123)} & & Y_4 \end{bmatrix} \sim N_{N\times mp}\left(\begin{bmatrix} & \mu & \\ \tilde{A}\Xi & & 0 \end{bmatrix}, \Psi \otimes I_N\right), \tag{2.1}$$

where

$$\mu = H_1'A[\Xi, 0] + H_1'C\Theta Q', \qquad \tilde{A} = H_2'A,$$

$$\Psi = Q\Omega Q' = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{pmatrix}.$$

Here we note that (Θ, Ξ) is an invertible function of (μ, Ξ). In fact, Θ can be expressed in terms of μ and Ξ as

$$\Theta = (H_1'C)^{-1}\mu Q - (H_1'C)^{-1}H_1'A\Xi(B \otimes I_m). \tag{2.2}$$

3. Tests for a family of covariance structures

We consider the LR test for the hypothesis (1.3) in the multivariate growth curve model (1.1). This is equivalent to considering the LR test for the hypothesis

$$(3.1) \qquad H_{0s} : \Psi = \begin{pmatrix} \Delta_s + I_s \otimes \Sigma_s & 0 \\ 0 & I_{p-s} \otimes \Sigma_s \end{pmatrix} \text{ vs.}$$

$$H_{1t} : \Psi = \begin{pmatrix} \Delta_t + I_t \otimes \Sigma_t & 0 \\ 0 & I_{p-t} \otimes \Sigma_t \end{pmatrix}$$

in the model (2.1). Since the elements of μ in (2.1) are free parameters, for testing the hypothesis (3.1) we may consider the LR test formed by only the density of Y. The model for Y is

$$(3.2) \qquad Y \sim N_{n \times mp}([\tilde{A}\Xi, 0], \Psi \otimes I_n),$$

where $n = N - r$. Let $L(\Xi, \Psi)$ be the likelihood function of Y. It is easy to see that the MLE of Ξ under H_{0s} or H_{1t} is given by $\hat{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)}$. Then we have

$$\begin{aligned} g(\Psi) &= -2 \log L(\hat{\Xi}, \Psi) \\ &= n \log|\Psi| + \operatorname{tr} \Psi^{-1}[Y_{(123)} - \tilde{A}\hat{\Xi} \;\, Y_4]'[Y_{(123)} - \tilde{A}\hat{\Xi} \;\, Y_4]. \end{aligned}$$

As is seen later on, the minimum of $g(\Psi)$ under H_{0s} or H_{1t} is complicated. For simplicity, we consider the LR test for a modified hypothesis

$$(3.3) \qquad \tilde{H}_{0s} : \Psi = \begin{pmatrix} \Psi_{11} & 0 \\ 0 & I_{p-s} \otimes \Sigma_s \end{pmatrix} \text{ vs. } \tilde{H}_{1t} : \Psi = \begin{pmatrix} \Psi_{(12)(12)} & 0 \\ 0 & I_{p-t} \otimes \Sigma_t \end{pmatrix},$$

where Ψ_{11} and $\Psi_{(12)(12)}$ are assumed to be arbitrary $ms \times ms$ and $mt \times mt$ positive definite matrices respectively, and

$$\Psi_{(12)(12)} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}.$$

We note that the difference between H_{0s} and $\tilde{H}_{0s}$ is whether or not Ψ_{11} satisfies a restriction that $\Psi_{11} \geq I_s \otimes \Sigma_s$, and so is the difference between H_{1t} and $\tilde{H}_{1t}$. It is easily seen that

$$\begin{aligned} (3.4) \qquad \min_{\tilde{H}_{0s}} g(\Psi_{11}, \Sigma_s) &= n \log\left|\frac{1}{n} S_{11}\right| \\ &\quad + n(p-s) \log\left|\frac{1}{n(p-s)} \left(\sum_{i=1}^{q-s} S_{(23)(23)}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right| \\ &\quad + nmp \end{aligned}$$

and

$$\text{(3.5)} \qquad \min_{\tilde{H}_{1t}} g(\Psi_{(12)(12)}, \Sigma_t) = n \log\left|\frac{1}{n} S_{(12)(12)}\right| + n(p-t)\log\left|\frac{1}{n(p-t)}\left(\sum_{i=1}^{q-t} S_{33}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right| + nmp,$$

where

$$S = Y'[I_n - \tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}']Y = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix},$$

$$S_{(12)(12)} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

and $S_{\alpha\alpha}^{(ii)} = Y_\alpha^{(i)\prime}[I_n - \tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}']Y_\alpha^{(i)}$. The minimum (3.4) is achieved at

$$\Psi_{11} = \frac{1}{n} S_{11}, \qquad \Sigma_s = \frac{1}{n(p-s)}\left(\sum_{i=1}^{q-s} S_{(23)(23)}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right).$$

Therefore, we can obtain the LR test statistic

$$\text{(3.6)} \qquad \tilde{\Lambda}_{s,t} = \frac{|S_{(12)(12)}|\left|\frac{1}{p-t}\left(\sum_{i=1}^{q-t} S_{33}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right|^{p-t}}{|S_{11}|\left|\frac{1}{p-s}\left(\sum_{i=1}^{q-s} S_{(23)(23)}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right|^{p-s}}$$

for testing $\tilde{H}_{0s}$ vs. $\tilde{H}_{1t}$, which may be also used for testing H_{0s} vs. H_{1t}. The statistic $\tilde{\Lambda}_{s,t}$ can be expressed in terms of the original observations, using

$$\text{(3.7)} \qquad Y_4^{(j)\prime} Y_4^{(j)} = Q_4^{(j)} V_{xx\cdot c} Q_4^{(j)\prime}, \qquad S_{\alpha\alpha} = Q_\alpha V_{xx\cdot ca} Q_\alpha', \qquad S_{\alpha\alpha}^{(ii)} = Q_\alpha^{(i)} V_{xx\cdot ca} Q_\alpha^{(i)\prime},$$

where $V_{xx\cdot c} = V_{xx} - V_{xc}V_{cc}^{-1}V_{cx}$, $V_{xx\cdot ca} = V_{xx\cdot c} - V_{xa\cdot c}V_{aa\cdot c}^{-1}V_{ax\cdot c}$ and

$$\text{(3.8)} \qquad V = [X, C, A]'[X, C, A] = \begin{pmatrix} V_{xx} & V_{xc} & V_{xa} \\ V_{cx} & V_{cc} & V_{ca} \\ V_{ax} & V_{ac} & V_{aa} \end{pmatrix}.$$

We can decompose the statistic $\tilde{\Lambda}_{s,t}$ as

$$\tilde{\Lambda}_{s,t} = \tilde{\Lambda}_1 \tilde{\Lambda}_2, \tag{3.9}$$

where

$$\tilde{\Lambda}_1 = \frac{|S_{11\cdot 2}|}{|S_{11}|} = \frac{|S_{11\cdot 2}|}{|S_{11\cdot 2} + S_{12}S_{22}^{-1}S_{21}|} \tag{3.10}$$

and

$$\tilde{\Lambda}_2 = \frac{|S_{22}|\left|\frac{1}{p-t}\left(\sum_{i=1}^{q-t} S_{33}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right|^{p-t}}{\left|\frac{1}{p-s}\left(\sum_{i=1}^{t-s} S_{22}^{(ii)} + \sum_{i=1}^{q-t} S_{33}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right)\right|^{p-s}}. \tag{3.11}$$

The statistics $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are the LR statistics for $\Psi_{12} = 0$ and $\Psi_{22} = I_{t-s} \otimes \Sigma_s$, respectively.

LEMMA 3.1. *When the hypothesis H_{0s} is true, it holds that*
(i) *$\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are independent,*

(ii) $$E(\tilde{\Lambda}_1^h) = \frac{\Gamma_{ms}(\frac{1}{2}\{n-k-m(t-s)\}+h)\Gamma_{ms}(\frac{1}{2}(n-k))}{\Gamma_{ms}(\frac{1}{2}\{n-k-m(t-s)\})\Gamma_{ms}(\frac{1}{2}(n-k)+h)},$$

(iii) $$E(\tilde{\Lambda}_2^h) = \frac{(p-s)^{m(p-s)h}}{(p-t)^{m(p-t)h}} \frac{\Gamma_{m(t-s)}(\frac{1}{2}(n-k)+h)}{\Gamma_{m(t-s)}(\frac{1}{2}(n-k))} \times \frac{\Gamma_m(\frac{1}{2}\{n(p-t)-k(q-t)\}+(p-t)h)\Gamma_m(\frac{1}{2}\{n(p-s)-k(q-s)\})}{\Gamma_m(\frac{1}{2}\{n(p-t)-k(q-t)\})\Gamma_m(\frac{1}{2}\{n(p-s)-k(q-s)\}+(p-s)h)},$$

where $\Gamma_m(n/2) = \pi^{m(m-1)/4}\prod_{j=1}^m \Gamma((n-j+1)/2)$.

PROOF. Under H_{0s}, it is easy to verify that $S_{11\cdot 2} \sim W_{ms}(n-k-m(t-s), \Psi_{11})$, $S_{12}S_{22}^{-1}S_{21} \sim W_{ms}(m(t-s), \Psi_{11})$, $S_{22} \sim W_{m(t-s)}(n-k, I_{t-s} \otimes \Sigma_s)$, $\sum_{i=1}^{q-t} S_{33}^{(ii)} \sim W_m((n-k)(q-t), \Sigma_s)$ and $\sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)} \sim W_m(n(p-q), \Sigma_s)$. Further, these statistics are independent. Therefore, $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are independent. The h-th moment of $\tilde{\Lambda}_1$ follows from that $\tilde{\Lambda}_1$ is distributed as a lambda distribution $\Lambda_{ms}(m(t-s), n-k-m(t-s))$. The h-th moment of $\tilde{\Lambda}_2$ can be written as

$$E(\tilde{\Lambda}_2^h) = 2^{m(t-s)h}\frac{(p-s)^{m(p-s)h}}{(p-t)^{m(p-t)h}} \frac{\Gamma_{m(t-s)}(\frac{1}{2}(2h+n-k))}{\Gamma_{m(t-s)}(\frac{1}{2}(n-k))} E\left[\frac{|W_2|^{(p-t)h}}{|W_1+W_2|^{(p-s)h}}\right],$$

where W_1 and W_2 are independently distributed, $W_1 \sim W_m(n_1, I_m)$, $W_2 \sim W_m(n_2, I_m)$, $n_1 = (2h+n-k)(t-s)$, $n_2 = n(p-t) - k(q-t)$. Here, letting $U = |W_1 + W_2|$ and $V = |W_2||W_1+W_2|^{-1}$, it is easy to verify that U and V

are independent, $V \sim \Lambda_m(n_1, n_2)$, U has the same distribution as $\prod_{i=1}^m U_i$ where the U_i are independent and $U_i \sim \chi^2_{n_1+n_2-i+1}$. The h-th moment of $\tilde{\Lambda}_2$ can be obtained from the above fact.

Using Lemma 3.1, we can obtain an asymptotic expansion of the null distribution of statistic $-n\rho \log \tilde{\Lambda}_{s,t}$ by expanding its characteristic function.

Theorem 3.1. *When the hypothesis H_{0s} is true, an asymptotic expansion of the distribution function of statistic $-n\rho \log \tilde{\Lambda}_{s,t}$ is*

$$P(-n\rho \log \tilde{\Lambda}_{s,t} \le x) = P(\chi^2_f \le x) + O(M^{-2}) \tag{3.12}$$

for large $M = n\rho$, where $f = \frac{1}{2}m(t-s)(mt+ms+1)$ and ρ is defined by

$$\begin{aligned} fn(1-\rho) = \frac{1}{12}m(t-s)\{&6(mt+ms+1)k + 6(mt+1)ms \\ &+ 2m^2(t-s)^2 + 3m(t-s) - 1 + \frac{1}{(p-t)(p-s)} \\ &\times \{6(p-q)^2k^2 - 6(m+1)(p-q)k + 2m^2 + 3m - 1\}\}. \end{aligned}$$

In the single-response case ($m = 1$), the asymptotic expansion (3.12) agrees with the results in Yokoyama [12].

We now consider the exact LR criterion $\Lambda_{s,t}^{n/2}$ for H_{0s} vs. H_{1t}. Let

$$\hat{\Psi}_{11} = \frac{1}{n}S_{11}, \qquad \hat{\Sigma}_s = \frac{1}{n(p-s)}\left(\sum_{i=1}^{q-s} S^{(ii)}_{(23)(23)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right),$$

$$\hat{\Psi}_{(12)(12)} = \frac{1}{n}S_{(12)(12)}, \qquad \hat{\Sigma}_t = \frac{1}{n(p-t)}\left(\sum_{i=1}^{q-t} S^{(ii)}_{33} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right).$$

If it holds that

$$\hat{\Psi}_{11} - I_s \otimes \hat{\Sigma}_s \ge 0 \qquad \text{and} \qquad \hat{\Psi}_{(12)(12)} - I_t \otimes \hat{\Sigma}_t \ge 0, \tag{3.13}$$

it is easy to show that $\Lambda_{s,t} = \tilde{\Lambda}_{s,t}$. Unless (3.13) holds, we need to solve the problem of minimizing

$$\begin{aligned} \frac{1}{n}g(\Delta_s, \Sigma_s) &= \log|\Delta_s + I_s \otimes \Sigma_s| + \operatorname{tr}(\Delta_s + I_s \otimes \Sigma_s)^{-1}\hat{\Psi}_{11} \\ &\quad + (p-s)(\log|\Sigma_s| + \operatorname{tr}\Sigma_s^{-1}\hat{\Sigma}_s) \end{aligned}$$

or

$$\begin{aligned} \frac{1}{n}g(\Delta_t, \Sigma_t) &= \log|\Delta_t + I_t \otimes \Sigma_t| + \operatorname{tr}(\Delta_t + I_t \otimes \Sigma_t)^{-1}\hat{\Psi}_{(12)(12)} \\ &\quad + (p-t)(\log|\Sigma_t| + \operatorname{tr}\Sigma_t^{-1}\hat{\Sigma}_t) \end{aligned}$$

under H_{0s} or H_{1t}, respectively. However, since the problem is not easy, we consider a lower bound denoted in terms of characteristic roots (Anderson [1]) for the minimum of $g(\Delta_s, \Sigma_s)/n$ or $g(\Delta_t, \Sigma_t)/n$. Let $l_1 \geq \cdots \geq l_{ms}$ and $l_1^* \geq \cdots \geq l_m^*$ be the characteristic roots of $\hat{\Psi}_{11}$ and $\hat{\Sigma}_s$ respectively, and let $d_1 > \cdots > d_{mt}$ and $d_1^* > \cdots > d_m^*$ be ones of $\hat{\Psi}_{(12)(12)}$ and $\hat{\Sigma}_t$, respectively. Then, as test statistics for H_{0s} vs. H_{1t}, we obtain

$$\bar{\Lambda}_{s,t} = \begin{cases} \tilde{\Lambda}_{s,t}, & \text{if } \hat{\Psi}_{11} - I_s \otimes \hat{\Sigma}_s \geq 0, \\ R_s, & \text{elsewhere,} \end{cases} \tag{3.14}$$

and

$$\Lambda_{s,t}^* = \begin{cases} \tilde{\Lambda}_{s,t}, & \text{if } \hat{\Psi}_{(12)(12)} - I_t \otimes \hat{\Sigma}_t \geq 0, \\ R_t, & \text{elsewhere,} \end{cases} \tag{3.15}$$

where

$$R_s = \frac{|\hat{\Psi}_{(12)(12)}|\,|\hat{\Sigma}_t|^{p-t}}{|\hat{\Psi}_{11}|\left\{\prod_{j=1}^{m} l_{js}\exp\left(\frac{l_j^*}{l_{js}} - 1\right)\right\}^{p-s}},$$

$$R_t = \frac{|\hat{\Psi}_{(12)(12)}|\left\{\prod_{j=1}^{m} d_{jt}\exp\left(\frac{d_j^*}{d_{jt}} - 1\right)\right\}^{p-t}}{|\hat{\Psi}_{11}|\,|\hat{\Sigma}_s|^{p-s}}.$$

The statistics $\bar{\Lambda}_{s,t}$ and $\Lambda_{s,t}^*$ are approximate LR statistics for H_{0s} vs. $\tilde{H}_{1t}$ and $\tilde{H}_{0s}$ vs. H_{1t}, respectively. In the single-response case ($m = 1$), we have $\bar{\Lambda}_{s,t} \leq \Lambda_{s,t} \leq \Lambda_{s,t}^*$.

4. The MLE's of unknown mean parameters

In this section we obtain the MLE's of unknown mean parameters in the multivariate growth curve model (1.1) with $\Omega = (B_s' \otimes I_m)\Delta_s(B_s \otimes I_m) + I_p \otimes \Sigma_s$ $(= \Omega_s)$ and consider the efficiency of the MLE of Ξ. This model is reduced to the same canonical form as in (2.1), but the covariance matrix Ψ is given by

$$\Psi = \begin{pmatrix} \Delta_s + I_s \otimes \Sigma_s & 0 \\ 0 & I_{p-s} \otimes \Sigma_s \end{pmatrix}.$$

It is easily seen that the MLE's of μ and Ξ are given by

$$\hat{\mu} = Z \qquad \text{and} \qquad \hat{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)}, \tag{4.1}$$

respectively. Therefore, from (2.2) the MLE of Θ is given by

(4.2) $$\hat{\Theta} = (H_1'C)^{-1}ZQ - (H_1'C)^{-1}H_1'A(\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)}(B \otimes I_m).$$

We now express the MLE's $\hat{\Xi}$ and $\hat{\Theta}$ in terms of the original observations or the matrix V in (3.8). Noting that

$$(\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)} = (A'H_2H_2'A)^{-1}A'H_2H_2'X(B' \otimes I_m),$$
$$(H_1'C)^{-1}H_1' = (C'C)^{-1}C',$$

we have the following theorem.

THEOREM 4.1. *The MLE's of Ξ and Θ in the multivariate growth curve model* (1.1) *with $\Omega = \Omega_s$ are given as follows:*

$$\begin{aligned}\hat{\Xi} &= [A'(I_N - P_C)A]^{-1}A'(I_N - P_C)X(B' \otimes I_m)\\ &= V_{aa\cdot c}^{-1}V_{ax\cdot c}(B' \otimes I_m),\end{aligned}$$

$$\begin{aligned}\hat{\Theta} &= (C'C)^{-1}C'X - (C'C)^{-1}C'A[A'(I_N - P_C)A]^{-1}A'(I_N - P_C)X(B'B \otimes I_m)\\ &= V_{cc}^{-1}[V_{cx} - V_{ca}V_{aa\cdot c}^{-1}V_{ax\cdot c}(B'B \otimes I_m)],\end{aligned}$$

where $P_C = C(C'C)^{-1}C'$.

On the other hand, the MLE of Ξ when Ω has no structures, i.e., is arbitrary positive definite is given by

(4.3) $$\tilde{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'H_2'XS^{*-1}(B' \otimes I_m)[(B \otimes I_m)S^{*-1}(B' \otimes I_m)]^{-1},$$

where $S^* = X'H_2[I_n - \tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}']H_2'X$. The result (4.3) is an extension of Chinchilli and Elswick [2] to a multivariate case. It is easily seen that

$$\tilde{A}'\tilde{A} = V_{aa\cdot c}, \qquad \tilde{A}'H_2'X = V_{ax\cdot c}, \qquad S^* = V_{xx\cdot c} - V_{xa\cdot c}V_{aa\cdot c}^{-1}V_{ax\cdot c} = V_{xx\cdot ac}.$$

These imply that

(4.4) $$\tilde{\Xi} = V_{aa\cdot c}^{-1}V_{ax\cdot c}V_{xx\cdot ac}^{-1}(B' \otimes I_m)[(B \otimes I_m)V_{xx\cdot ac}^{-1}(B' \otimes I_m)]^{-1}.$$

The estimators $\hat{\Xi}$ and $\tilde{\Xi}$ have the following properties.

THEOREM 4.2. *In the multivariate growth curve model* (1.1) *with $\Omega = \Omega_s$ it holds that both the estimators $\hat{\Xi}$ and $\tilde{\Xi}$ are unbiased, and*

$$V(\mathrm{vec}(\hat{\Xi})) = \Psi_s \otimes M,$$

$$V(\mathrm{vec}(\tilde{\Xi})) = \left\{1 + \frac{m(p-q)}{N-(k+r)-m(p-q)-1}\right\}\Psi_s \otimes M,$$

where $M = [A'(I_N - P_C)A]^{-1}$ *and*

$$\Psi_s = \begin{pmatrix} \Delta_s + I_s \otimes \Sigma_s & 0 \\ 0 & I_{q-s} \otimes \Sigma_s \end{pmatrix}.$$

Proof. Since $\hat{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}' Y_{(123)}$, we have

$$E(\hat{\Xi}) = \Xi \qquad \text{and} \qquad V(\text{vec}(\hat{\Xi})) = \Psi_s \otimes (\tilde{A}'\tilde{A})^{-1},$$

which imply the result on $\hat{\Xi}$. By an argument similar to the one in Yokoyama [12], it can be shown that for any positive definite covariance matrix Ω, $E(\tilde{\Xi}) = \Xi$ and

$$V(\text{vec}(\tilde{\Xi})) = \left\{1 + \frac{m(p-q)}{N-(k+r)-m(p-q)-1}\right\}[(B \otimes I_m)\Omega^{-1}(B' \otimes I_m)]^{-1} \otimes M.$$

Under the assumption that $\Omega = \Omega_s$, it holds that $[(B \otimes I_m)\Omega^{-1}(B' \otimes I_m)]^{-1} = \Psi_s$, which proves the desired result.

From Theorem 4.2, we obtain

$$V(\text{vec}(\tilde{\Xi})) - V(\text{vec}(\hat{\Xi})) = \frac{m(p-q)}{N-(k+r)-m(p-q)-1}\Psi_s \otimes M > 0, \tag{4.5}$$

which implies that $\hat{\Xi}$ is more efficient than $\tilde{\Xi}$ in the model (1.1) with $\Omega = \Omega_s$. This shows that we can get a more efficient estimator for Ξ by assuming multivariate random-effects covariance structures. Especially, when p is large relative to N, we can obtain greater gains.

As mentioned in §3, the MLE's of unknown variance parameters Σ_s and Δ_s in the model (1.1) with $\Omega = \Omega_s$ become very complicated. On the other hand, the usual unbiased estimators of Σ_s and Δ_s may be defined by

$$\tilde{\Sigma}_s = \frac{1}{n(p-s)-k(q-s)}\left(\sum_{i=1}^{q-s} S_{(23)(23)}^{(ii)} + \sum_{j=1}^{p-q} Y_4^{(j)\prime} Y_4^{(j)}\right) \qquad \text{and}$$

$$\tilde{\Delta}_s = \frac{1}{n-k}S_{11} - I_s \otimes \tilde{\Sigma}_s,$$

respectively. However, there is the possibility that the use of $\tilde{\Delta}_s$ can lead to a nonpositive semi-definite estimate of Δ_s. These estimators can be expressed in terms of the original observations, again using (3.7).

5. An example

In this section we apply the results of §4 to the data (see, e.g., Srivastava and Carter [8, p. 227]) of the price indices of hand soaps packaged in four ways, estimated by twelve consumers. Each consumer belongs to one of two groups. It is known (Yokoyama [11]) that the model (1.1) in the case $m = 1$,

$p = 4$ and $N = 12$ with

$$E(X) = \begin{pmatrix} \mathbf{1}_6 \\ \mathbf{0} \end{pmatrix} \xi \mathbf{1}_4' + \mathbf{1}_{12}(\theta_1, \theta_2, \theta_3, \theta_4) \qquad \text{and}$$

$$V(\mathrm{vec}(X)) = (\delta^2 \mathbf{1}_4 \mathbf{1}_4' + \sigma^2 I_4) \otimes I_{12}$$

is adequate to the observation matrix $X : 12 \times 4$. Now we estimate how much gains can be obtained for the maximum likelihood estimation of ξ by assuming the random-effects covariance structure. Since $k = q = r = s = 1$, $[\mathbf{1}_4'(\delta^2 \mathbf{1}_4 \mathbf{1}_4' + \sigma^2 I_4)^{-1} \mathbf{1}_4]^{-1} = (4\delta^2 + \sigma^2)/4$, $M = 1/3$, $\hat{\delta}^2 = .01353$ and $\hat{\sigma}^2 = .00976$, it follows from Theorem 4.2 and (4.5) that $V(\hat{\xi})/V(\tilde{\xi}) = 2/3$ and $\hat{V}(\tilde{\xi}) - \hat{V}(\hat{\xi}) = (4\hat{\delta}^2 + \hat{\sigma}^2)/24 = .00266$.

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References

[1] T. W. Anderson, Asymptotic theory for principal component analysis, Ann. Math. Statist. **34** (1963), 122–148.
[2] V. M. Chinchilli and R. K. Elswick, A mixture of the MANOVA and GMANOVA models, Commun. Statist.-Theor. Meth. **14** (**12**), (1985), 3075–3089.
[3] A. M. Kshirsagar and W. B. Smith, Growth Curves, New York: Dekker, 1995.
[4] N. Lange and N. M. Laird, The effect of covariance structure on variance estimation in balanced growth-curve models with random parameters, J. Amer. Statist. Assoc. **84** (1989), 241–247.
[5] R. F. Potthoff and S. N. Roy, A generalized multivariate analysis of variance model useful especially for growth curve problems, Biometrika **51** (1964), 313–326.
[6] G. Reinsel, Estimation and prediction in a multivariate random effects generalized linear model, J. Amer. Statist. Assoc. **79** (1984), 406–414.
[7] D. von Rosen, The growth curve model: a review, Commun. Statist.-Theor. Meth. **20** (**9**), (1991), 2791–2822.
[8] M. S. Srivastava and E. M. Carter, An Introduction to Applied Multivariate Statistics, New York: North-Holland, 1983.
[9] A. P. Verbyla and W. N. Venables, An extension of the growth curve model, Biometrika **75** (1988), 129–138.
[10] T. Yokoyama and Y. Fujikoshi, Tests for random-effects covariance structures in the growth curve model with covariates, Hiroshima Math. J. **22** (1992), 195–202.
[11] T. Yokoyama, LR test for random-effects covariance structure in a parallel profile model, Technical Report, **93-3**, Statistical Research Group, Hiroshima University, 1993.
[12] T. Yokoyama, Statistical inference on some mixed MANOVA-GMANOVA models with random effects, Hiroshima Math. J. **25** (1995), 441–474.

[13] T. Yokoyama, Tests for a family of random-effects covariance structures in a multivariate growth curve model, J. Statist. Plann. Inference **65** (1997), 281–292.

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