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On equivariant self-homotopy equivalences of G-CW complexes

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Abstract. Let G be a finte group. We give a short exact sequence for calculating the group $\mathscr{E}_G(X)$ of based G-homotopy classes of based G-self-homotopy equivalences of a G-CW complex X under certain conditions.

0. Introduction

For a based G-space X, the set $\mathscr{E}_G(X)$ of based G-equivariant homotopy classes of based G-equivariant self-homotopy equivalences of X forms a group under composition of maps. In this paper, we study $\mathscr{E}_G(X)$ for a G-CW complex X under certain conditions. Throughout the paper, G is a finite group and H a subgroup of G, all G-CW complexes are G-connected and have G-fixed base points, and all G-maps and G-homotopies (denoted by $\simeq$) preserve the base points $*$. For a G-map $f : A \to B$ between G-CW complexes, we consider the reduced cone $CA = A \times I/(A \times \{1\}) \cup (\{*\} \times I)$, the reduced suspension $SA = CA/A \times \{0\}$ and the reduced mapping cone $C_f = B \cup_f CA$ obtained from the topological sum of B and CA by identifying each $(a, 0) \in CA$ with $f(a) \in B$, where G acts trivially on $I = [0, 1]$. Then a G-coaction of SA on C_f defines a map λ in §1, whose restriction to $\operatorname{Im} i_*$ yields the homomorphism $\lambda : i_*([SA, B]_G) \to \mathscr{E}_G(C_f)$, where $i : B \to C_f$ is the inclusion (Lemma 1.3). This homomorphism will be used in §3. In §2 $\mathscr{E}_G(C_f)$ for $A = G/H^+ \wedge S^n$, the n-fold reduced suspension of G/H^+, is studied. Here G/H denotes the left coset space of G by H with action given by $g \cdot (g'H) = (gg')H$ for $g \in G$ and $g'H \in G/H$, and G/H^+ the topological sum of G/H and a single point $*$, the base point of G/H^+. A homomorphism $\varphi \times \psi : \mathscr{E}_G(C_f) \to \mathscr{E}_G(A) \times \mathscr{E}_G(B)$ is obtained when $\dim B \leqq n - 1$ and $n \geqq 2$. The image and the kernel of this homomorphism are studied in §2 and §3, respectively. Then, a short exact sequence for calculating $\mathscr{E}_G(C_f)$ is obtained in Theorem 3.5. The non-equivariant case is due to Barcus and Barratt [1, Theorem (6.1)]. In §4 we show that if $n \geqq 2$ then $\mathscr{E}_G(G/H^+ \wedge S^n)$ is anti-

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isomorphic to the group $U(\boldsymbol{Z}(N(H)/H))$ of units of the integral group ring $\boldsymbol{Z}(N(H)/H)$ of $N(H)/H$, where $N(H)$ denotes the normalizer of H in G (Theorem 4.1). In §5 using the above anti-isomorphism and short exact sequence, we study $\mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ for each $\boldsymbol{Z}_2$-map $f : \boldsymbol{Z}_2^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$ with $n \geqq k+3 \geqq 4$ (Theorem 5.11) and further calculate $\mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ in the case of $k=1$ (Proposition 5.16). In §6 we also study $\mathscr{E}_{\boldsymbol{Z}_6}(C_f)$ for each $\boldsymbol{Z}_6$-map $f : \boldsymbol{Z}_6^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$ with $n \geqq k+3 \geqq 4$ (Theorem 6.6) and calculate $\mathscr{E}_{\boldsymbol{Z}_6}(C_f)$ in the case of $k=1$ (Proposition 6.10). We use the following notation: $[X, Y]_G$ denotes the set of based G-homotopy classes of based G-maps of X into Y. X^H denotes the H-stationary subspace $\{x \in X \mid gx = x$ for every $g \in H\}$. $(\boldsymbol{Z}_q)^k$ denotes the direct product of k-copies of $\boldsymbol{Z}_q$. The same symbol will be used for a G-map and its G-homotopy class. A G-CW complex X is called G-connected (resp. G-1-connected) if the fixed point set X^H is connected (resp. simply connected) for every subgroup H of G.

1. Preliminalies

For a G-map $f : A \to B$ between G-CW complexes we consider the sequence of the induced cofibering

$$A \xrightarrow{f} B \xrightarrow{i} C_f \xrightarrow{p} SA,$$

where i and p are G-maps with respect to the natural G-actions. The coaction

$$l : C_f \to C_f \vee SA, \tag{1.1}$$

defined by collapsing the subspace $A \times \{1/2\}$ of $C_f = B \cup_f CA$ to the base point $*$, is a G-map and defines a map

$$\lambda : [SA, C_f]_G \to [C_f, C_f]_G \tag{1.2}$$

by $\lambda(\alpha) = \nabla(1 \vee \alpha)l$ for $\alpha \in [SA, C_f]_G$, where ∇ denotes the folding map. Then we have the following, which will be used in §3.

LEMMA 1.3. *$\lambda(\alpha + \beta) = \lambda(\alpha)\lambda(\beta)$ for $\alpha \in [SA, C_f]_G$ if β belongs to the image of $i_* : [SA, B]_G \to [SA, C_f]_G$.*

PROOF. If $\beta = i\beta'$ for some $\beta' \in [SA, B]_G$, then $\lambda(\alpha)\beta = \beta$ by the definition of λ. For the natural G-comultiplication l' on SA, $(l \vee 1)l = (1 \vee l')l$. These equalities, $\lambda(\alpha)\beta = \beta$ and $(l \vee 1)l = (1 \vee l')l$, yield

$$\begin{aligned}\lambda(\alpha)\lambda(\beta) &= \nabla(\lambda(\alpha) \vee \lambda(\alpha)\beta)l = \nabla(\lambda(\alpha) \vee \beta)l \\ &= \nabla(1 \vee \nabla)(1 \vee \alpha \vee \beta)(1 \vee l')l = \lambda(\alpha + \beta).\end{aligned}$$

q.e.d.

2. Homomorphism $\varphi \times \psi$ and its image

In this section we assume that $A = G/H^+ \wedge S^n$ with $n \geqq 2$ and B is a G-CW complex; we consider the mapping cone

$$C_f = B \cup_f (G/H^+ \wedge e^{n+1})$$

of a G-map $f : A \to B$. Note that $G/H^+ \wedge S^n = \bigvee_i (g_i H/H^+ \wedge S^n)$, the one point union of n-spheres with action given by $g \cdot (g_i H/H^+) = (gg_i)H/H^+$.

Lemma 2.1. *If* $\dim B \leqq n-1$, *then* $i_* : [B, B]_G \to [B, C_f]_G$ *and* $p^* : [SA, SA]_G \to [C_f, SA]_G$ *are bijective.*

Proof. Let L be a subgroup of G. Since the fixed point set $C_f^L = B^L \cup_f (((G/H)^L)^+ \wedge e^{n+1})$, (C_f^L, B^L) is n-connected (cf. [8, II, (3.9) Theorem]). Hence $i_* : [B, B]_G \to [B, C_f]_G$ is bijective by [2, II, (5.3) Corollary]. Also $SA = G/H^+ \wedge S^{n+1}$ implies that $[SB, SA]_G = [B, SA]_G = 0$ by [2, II, (5.2) Lemma]. Therefore, the Puppe sequence (cf. [2, III, (2.2)])

$$\longrightarrow [SB, SA]_G \xrightarrow{(Sf)^*} [SA, SA]_G \xrightarrow{p^*} [C_f, SA]_G \xrightarrow{i^*} [B, SA]_G \longrightarrow$$

shows that p^* is bijective. q.e.d.

Since the suspension $S : [A, A]_G \to [SA, SA]_G$ is bijective (see §4), the above lemma allows us to define a map

$$\varphi \times \psi : [C_f, C_f]_G \to [A, A]_G \times [B, B]_G \tag{2.2}$$

by $\varphi = S^{-1} p^{*-1} p_*$ and $\psi = i_*^{-1} i^*$ under the assumption of Lemma 2.1. Namely, $S\varphi(h)$ and $\psi(h)$ are the elements uniquely determined by the G-homotopy commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA \\ \Big\downarrow{\scriptstyle \psi(h)} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle S\varphi(h)} \\ B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA. \end{array} \tag{2.3}$$

Therefore $\varphi \times \psi$ is a homomorphism of monoids, and hence a homomorphism

$$\varphi \times \psi : \mathscr{E}_G(C_f) \to \mathscr{E}_G(A) \times \mathscr{E}_G(B) \tag{2.4}$$

of groups can be defined as the restriction of the map $\varphi \times \psi$ in (2.2) to $\mathscr{E}_G(C_f)$ when $\dim B \leqq n-1$. From now on, we study the image of this homomorphism $\varphi \times \psi$. Let $ESA = (SA)^I$, the space of free paths (not necessary equivariant) in SA, and $PSA = \{\sigma \in ESA \mid \sigma(1) = *\}$, the space of paths in SA, where G acts on ESA and PSA by $(g \cdot \sigma)(t) = g \cdot \sigma(t)$ for $g \in G$ and $\sigma \in ESA$

(or PSA), and let

$$\Omega SA \xrightarrow{j} F_p \xrightarrow{q} C_f \qquad (q(x,\sigma)=x)$$

be the path fibering induced from the fibering $\Omega SA \to PSA \to SA$ by $p: C_f \to SA$, where G acts diagonally on $F_p = \{(x,\sigma) \in C_f \times PSA \mid p(x) = \sigma(0)\}$. Then a G-lifting $\imath : B \to F_p$ of $i: B \to C_f$ can be defined by $\imath(b) = (b, 0_*) \in F_p$, where 0_* denotes the constant path, $0_*(t) = *, t \in I$.

LEMMA 2.5. (i) *If* $\dim B \leqq n-1$, *then* $q_* : [B, F_p]_G \to [B, C_f]_G$ *is bijective.*
(ii) *If* B *is* G-1-*connected, then* $\imath_* : [A, B]_G \to [A, F_p]_G$ *is bijective.*

PROOF. (i) Let L be a subgroup of G. Since $SA^L = ((G/H)^L)^+ \wedge S^{n+1}$, $\pi_i(\Omega SA^L) = 0$ for all $i \leqq n-1$. Therefore, the homotopy sequence

$$\longrightarrow \pi_i(\Omega SA^L) \xrightarrow{j_*} \pi_i(F_p^L) \xrightarrow{q_*} \pi_i(C_f^L) \xrightarrow{\delta} \pi_{i-1}(\Omega SA^L) \longrightarrow$$

of the fibering $\Omega SA^L \to F_p^L \to C_f^L$ shows that $q_* : \pi_i(F_P^L) \to \pi_i(C_f^L)$ is isomorphic for all $i \leqq n-1$ and epimorphic for $i = n$. Hence, if $\dim B \leqq n-1$, then $q_* : [B, F_p]_G \to [B, C_f]_G$ is bijective in the same way as in [2, II, (5.4) Theorem].

(ii) Since $A = G/H^+ \wedge S^n$, it suffices to show that $\imath_* : \pi_n(B^H) \to \pi_n(F_p^H)$ is isomorphic by [4, Lemma 2.1′]. Let $E_p = \{(x,\sigma) \in C_f \times ESA \mid p(x) = \sigma(0)\}$, where G acts diagonally on E_p. Then the fibering

$$F_p \xrightarrow{\mathrm{u}} E_p \xrightarrow{\mathrm{r}} SA \qquad (r(x,\sigma) = \sigma(1))$$

induces the isomorphism $r_* : \pi_i(E_p^H, F_p^H) \to \pi_i(SA^H)$ for all i. Also, since $C_f^H = B^H \cup_f ((G/H)^H)^+ \wedge e^{n+1})$, Blakers-Massey Theorem implies that $p_* : \pi_i(C_f^H, B^H) \to \pi_i(SA^H)$ is isomorphic for all $i \leqq n+1$ (cf. [8, VII, (7.12) Theorem]). The inclusion $e : C_f \to E_p$ defined by $e(x) = (x, 0_{p(x)})$ is a G-homotopy equivalence satisfying $p = re$. Therefore, in particular, $(e,\imath)_* = r_*^{-1}p_* : \pi_{n+1}(C_f^H, B^H) \to \pi_{n+1}(E_p^H, F_p^H)$ and $e_* : \pi_i(C_f^H) \to \pi_i(E_p^H)$ for $i = n$ and $n+1$ are isomorphic. Thus, the equality $ei = u\imath$ gives rise to the commutative diagram

$$\begin{tikzcd}
\longrightarrow \pi_{n+1}(C_f^H) \arrow[r] \arrow[d, "e_*"', "\cong"] & \pi_{n+1}(C_f^H, B^H) \arrow[r, "\delta"] \arrow[d, "(e,\imath)_*"', "\cong"] & \pi_n(B^H) \arrow[r, "i_*"] \arrow[d, "\imath_*"'] & \pi_n(C_f^H) \arrow[r] \arrow[d, "e_*"', "\cong"] & 0 \\
\longrightarrow \pi_{n+1}(E_p^H) \arrow[r] & \pi_{n+1}(E_p^H, F_p^H) \arrow[r, "\delta"] & \pi_n(F_p^H) \arrow[r, "u_*"] & \pi_n(E_p^H) \arrow[r] & 0
\end{tikzcd}$$

whose top and bottom rows are the homotopy sequences of the pairs (C_f, B^H) and (E_p^H, F_p^H), respectively. This diagram shows that $\imath_* : \pi_n(B^H) \to \pi_n(F_p^H)$ is isomorphic by the five lemma. q.e.d.

Let $\varphi \times \psi$ be the homomorphism in (2.4). Then we show the following in the same way as in the non-equivariant case due to Rutter [6, Theorem 4.6].

Lemma 2.6. *If B is G-1-connected and $\dim B \leqq n-1$, then the image of $\varphi \times \psi$ is equal to*

$$M = \{(h_1, h_2) \in \mathscr{E}_G(A) \times \mathscr{E}_G(B) \mid h_2 f = f h_1 \text{ in } [A, B]_G\}.$$

Proof. Let (h_1, h_2) be any element of M. Then, each G-homotopy $h_2 f \simeq f h_1$ allows us to construct a G-map $h : C_f \to C_f$ such that $hi \simeq ih_2$ and $Sh_1 p \simeq ph$, that is, $\psi(h) = h_2$ and $S\varphi(h) = Sh_1$ in (2.3). Therefore, to prove $M \subset \mathrm{Im}(\varphi \times \psi)$, it suffices to show that the above element h is a G-homotopy equivalence. For each subgroup L of G, h_1 and h_2 induce the isomorphisms $h_{1*} : H_i(A^L; \boldsymbol{Z}) \to H_i(A^L; \boldsymbol{Z})$ and $h_{2*} : H_i(B^L; \boldsymbol{Z}) \to H_i(B^L; \boldsymbol{Z})$ for all i, respectively. Therefore, h induces the isomorphism $h_* : H_i(C_f^L; \boldsymbol{Z}) \to H_i(C_f^L; \boldsymbol{Z})$ for all i by the five lemma, and hence it induces the isomorphism $h_* : \pi_i(C_f^L) \to \pi_i(C_f^L)$ for all i by Whitehead Theorem. By [2, II, (5.5) Corollary], this shows that h is a G-homotopy equivalence. Thus, $M \subset \mathrm{Im}(\varphi \times \psi)$. Next, let h be any element of $\mathscr{E}_G(C_f)$. Then, $p_* h = p^* S\varphi(h)$ by the definition of φ, and each G-homotopy $ph \simeq S\varphi(h)p$ allows us to construct a G-map $\bar{h} : F_p \to F_p$ such that the diagram

$$\begin{array}{ccccccc}
\Omega SA & \xrightarrow{j} & F_p & \xrightarrow{q} & C_f & \xrightarrow{p} & SA \\
\downarrow{\scriptstyle \Omega S\varphi(h)} & & \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle S\varphi(h)} \\
\Omega SA & \xrightarrow{j} & F_p & \xrightarrow{q} & C_f & \xrightarrow{p} & SA
\end{array} \tag{2.7}$$

is G-homotopy commutative. Let $\imath : B \to F_p$ be the G-lifting of $i : B \to C_f$ in Lemma 2.5. Then, the equality $q\imath = i$ and the commutativity of the diagrams (2.3) and (2.7) yield

$$q\imath\psi(h) = i\psi(h) \simeq hi = hq\imath \simeq q\bar{h}\imath,$$

and hence $\imath\psi(h) \simeq \bar{h}\imath$ by Lemma 2.5 (i). Furthermore, let $\tau : A \to \Omega SA$ be a G-map defined by $\tau(a)(t) = (a, 1-t)$ for $a \in A$ and $t \in I$. Then, $\Omega S\varphi(h)\tau = \tau\varphi(h)$. Let $\tau_s : A \to PSA$ be a G-homotopy defined by $\tau_s(a)(t) = p(a, s(1-t))$ for $a \in A$ and $s, t \in I$, and let $h_s : A \to F_p$ be a G-homotopy defined by $h_s(a) = ((a, s), \tau_s(a))$. Then this G-homotopy h_s shows that $\imath f \simeq j\tau$. Now, these G-homotopies and the equality, $\imath\psi(h) \simeq \bar{h}\imath$, $\imath f \simeq j\tau$ and $\Omega S\varphi(h)\tau = \tau\varphi(h)$, and the commutativity of the diagram (2.7) yield

$$\imath\psi(h)f \simeq \bar{h}\imath f \simeq \bar{h}j\tau \simeq j\Omega S\varphi(h)\tau = j\tau\varphi(h) \simeq \imath f\varphi(h).$$

Hence, $\psi(h)f \simeq f\varphi(h)$ by Lemma 2.5 (ii). Thus, $\mathrm{Im}(\varphi \times \psi) \subset M$. q.e.d.

3. Kernel of $\varphi \times \psi$ and a short exact sequence

In this section we assume that $A' = G/H^+ \wedge S^{n-1}$ with $n \geqq 2$ and B' is a G-CW complex; we also assume that $f' : A' \to B'$ is any G-map and that $f = Sf' : A = SA' \to B = SB'$. Then we have

LEMMA 3.1. *If B is G-1-connected, then there is an exact sequence of groups*

$$[SA, B]_G \xrightarrow{i_*} [SA, C_f]_G \xrightarrow{p_*} [SA, SA]_G.$$

PROOF. An isomorphism $\pi_{n+1}(C_f^H, B^H) \cong \pi_{n+1}(((G/H)^H)^+ \wedge S^{n+1})$ obtained by Blakers-Massey Theorem yields an exact sequence

$$\pi_{n+1}(B^H) \xrightarrow{i_*} \pi_{n+1}(C_f^H) \xrightarrow{p_*} \pi_{n+1}(((G/H)^H)^+ \wedge S^{n+1}),$$

which implies this lemma by [4, Lemma 2.1′]. q.e.d.

Let λ be the map in (1.2) and $\varphi \times \psi$ the homomorphism in (2.4). Then we have

LEMMA 3.2. (i) *$\lambda(\alpha) = 1 + \alpha p$ for $\alpha \in [SA, C_f]_G$.*

(ii) *If B is G-1-connected and* $\dim B \leqq n-1$, *then the kernel of $\varphi \times \psi$ is isomorphic to*

$$K = i_*[SA, B]_G/(Sf)^*[SB, C_f]_G.$$

PROOF. (i) Since $C_f \simeq SC_{f'}$ by the assumption $f = Sf'$, C_f has the natural G-comultiplication $l' : C_f \to C_f \vee C_f$, and $l \simeq (1 \vee p)l'$ for the G-coaction l in (1.1). Therefore, by the definition of λ in (1.2),

$$\lambda(\alpha) = \nabla(1 \vee \alpha)(1 \vee p)l' = 1 + \alpha p.$$

(ii) The equality of (i) and the definitions of φ and ψ in (2.2) give rise to the commutative diagram

$$\begin{tikzcd}
{[SB, C_f]_G} \arrow[r, "(Sf)^*"] & {[SA, C_f]_G} \arrow[r, "\lambda"] \arrow[d, "1+p_*"'] & {[C_f, C_f]_G} \arrow[r, "i^*"] \arrow[dl, "S\varphi"'] \arrow[d, "p_*"] \arrow[drr, "\psi"] & {[B, C_f]_G} & \\
{[SA, B]_G} \arrow[ur, "i_*"] & {[SA, SA]_G} \arrow[r, "p^*", "\cong"'] & {[C_f, SA]_G} & & {[B, B]_G.} \arrow[ul, "i_*", "\cong"']
\end{tikzcd} \tag{3.3}$$

Since the row sequence in (3.3) is an exact sequence of groups if we replace λ by p^*, we have

$$\psi^{-1}(1) = 1 + \psi^{-1}(0) = 1 + p^*[SA, C_f]_G = \lambda([SA, C_f]_G). \tag{3.4}$$

Also, (3.4), (3.3) and Lemma 3.1 yield

$$\operatorname{Ker}(\varphi \times \psi) \cong (S\varphi)^{-1}(1) \cap \lambda([SA, C_f]_G)$$
$$= \lambda(i_*[SA, B]_G).$$

Moreover, by (3.3) and Lemma 3.1 we have $(Sf)^*[SB, C_f]_G \subset i_*[SA, B]_G$ and by Lemma 1.3 and (i) of this lemma we have a group isomorphism

$$\lambda(i_*[SA, B]_G) \cong i_*[SA, B]_G/(Sf)^*[SB, C_f]_G. \qquad \text{q.e.d.}$$

Now Lemmas 2.6 and 3.2 give the following theorem, which is due to Barcus and Barratt in the non-equivariant case [1, Theorem (6.1)] (cf. [5, Theorem 2.12]).

Theorem 3.5. *Let $A' = G/H^+ \wedge S^{n-1}$ with $n \geqq 2$ and B' a G-CW complex, and let $f' : A' \to B'$ be a G-map. If $B = SB'$ is G-1-connected and $\dim B \leqq n-1$, then for the mapping cone $C_f = B \cup_f (G/H^+ \wedge e^{n+1})$ of the G-map $f = Sf' : A = SA' \to B = SB'$ with the natural G-action, there is an exact sequence of groups*

$$0 \longrightarrow K \xrightarrow{\bar{\lambda}} \mathscr{E}_G(C_f) \xrightarrow{\varphi \times \psi} M \longrightarrow 1$$

with

$$K = i_*[SA, B]_G/(Sf)^*[SB, C_f]_G \qquad \text{and}$$
$$M = \{(h_1, h_2) \in \mathscr{E}_G(A) \times \mathscr{E}_G(B) \mid h_2 f = f h_1 \text{ in } [A, B]_G\}.$$

4. Anti-isomorphism: $\mathscr{E}_G(G/H^+ \wedge S^n) \cong U(\boldsymbol{Z}(N(H)/H))$ $(n \geqq 2)$

Let G be a finite group and H a subgroup of G. Note that $(G/H)^H = N(H)/H$, where $N(H)$ denotes the normalizer of H in G. Then we have

Theorem 4.1. *If $n \geqq 2$, then the group $\mathscr{E}_G(G/H^+ \wedge S^n)$ is anti-isomorphic to the group $U(\boldsymbol{Z}(N(H)/H))$ of units of the integral group ring $\boldsymbol{Z}(N(H)/H)$ of $N(H)/H$.*

Proof. To prove this theorem, it suffices to show that there is a ring anti-isomorphism $[G/H^+ \wedge S^n, G/H^+ \wedge S^n]_G \cong \boldsymbol{Z}(N(H)/H)$. Let $\{g_i H\}$ be the left decomposition of $N(H)$ with respect to H, and let the homotopy class of the composite of a map $m : S^n = H/H^+ \wedge S^n \to S^n = g_i H/H^+ \wedge S^n$ of degree m and the inclusion of $g_i H/H^+ \wedge S^n$ into $N(H)/H^+ \wedge S^n$ be identified with $m g_i H \in \boldsymbol{Z}(N(H)/H)$. Then by [4, Corollary 2.2], the restriction to $S^n = H/H^+ \wedge S^n$ and this identification yield the following isomorphism Φ of additive groups.

$$\Phi : [G/H^+ \wedge S^n, G/H^+ \wedge S^n]_G \cong \pi_n(N(H)/H^+ \wedge S^n) \cong \boldsymbol{Z}(N(H)/H).$$

Let u and v be any two elements of the set $[G/H^+ \wedge S^n, G/H^+ \wedge S^n]_G$ and $j : N(H)/H^+ \wedge S^n \to G/H^+ \wedge S^n$ the inclusion. Since v is equivariant,

$$v|(g_i H/H^+ \wedge S^n) = g_i H \cdot v|(H/H^+ \wedge S^n).$$

If $u|(H/H^+ \wedge S^n) = m_0 H + m_1 g_1 H + \cdots + m_k g_k H \in \pi_n(N(H)/H^+ \wedge S^n)$, then

$$\begin{aligned} \Phi(vu) &= vj(m_0 H + m_1 g_1 H + \cdots + m_k g_k H) \\ &= (v|(H/H^+ \wedge S^n))m_0 + \cdots + (v|(g_k H/H^+ \wedge S^n))m_k \\ &= m_0(H \cdot v|(H/H^+ \wedge S^n)) + \cdots + m_k(g_k H \cdot v|(H/H^+ \wedge S^n)) \\ &= m_0 H \cdot \Phi(v) + \cdots + m_k g_k H \cdot \Phi(v) \\ &= \Phi(u) \cdot \Phi(v). \end{aligned}$$

Thus Φ is an anti-isomorphism of rings. q.e.d.

For a finite abelian group G, let n_2 denote the number of its elements of order 2 and c the number of its cyclic subgroups (including $\{e\}$). Then we have the following theorem due to Higman (cf. [3, Theorem 4.1]).

THEOREM 4.2 (Higman). *Let G be a finite abelian group. Then*

$$U(\boldsymbol{Z}G) = \pm G \times F,$$

where F is a free abelian group of rank $(|G| + n_2 + 1)/2 - c$.

Now Theorems 4.1 and 4.2 immediately give the following.

THEOREM 4.3. *Let G be a finite abelian group and H a subgroup of G. If $n \geqq 2$, then*

$$\mathscr{E}_G(G/H^+ \wedge S^n) \cong \boldsymbol{Z}_2 \times G/H \times (\boldsymbol{Z})^k, \qquad k = (|G/H| + n_2 + 1)/2 - c,$$

where $\boldsymbol{Z}_2 = \{1, -1\}$, n_2 denotes the number of elements of order 2 and c denotes the number of cyclic subgroups of G/H.

Let E_q be the $q \times q$ identity matrix and F_q the $q \times q$ matrix defined by

$$F_q = \begin{pmatrix} \mathbf{0} & 1 \\ E_{q-1} & \mathbf{0} \end{pmatrix}. \tag{4.4}$$

If G/H is isomorphic to the cyclic group $\boldsymbol{Z}_q$ of order q, then $\mathscr{E}_G(G/H^+ \wedge S^n)$ has the torsion subgroup $\boldsymbol{Z}_2 \times \boldsymbol{Z}_q$ generated by $-E_q$ and F_q.

COROLLARY 4.5. *In the above theorem, if G/H is isomorphic to the cyclic group $\boldsymbol{Z}_q$, then*

$$\mathscr{E}_G(G/H^+ \wedge S^n) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_q \times (\boldsymbol{Z})^k, \qquad k = [q/2] + 1 - d(q),$$

where $d(q)$ *is the number of divisors of* q *and the torsion subgroup* $\boldsymbol{Z}_2 \times \boldsymbol{Z}_q$ *is generated by* $-E_q$ *and* F_q*, and, in particular,*

$$\mathscr{E}_G(G/H^+ \wedge S^n) \cong \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_q, & \text{if } q = 2, 3, 4, 6 \\ \boldsymbol{Z}_2 \times \boldsymbol{Z}_q \times (\boldsymbol{Z})^k, & \text{if } q \text{ is a prime} \geqq 5, \end{cases}$$

where $k = (q-3)/2$.

5. $\mathscr{E}_{Z_2}(C_f)$ for $f : \boldsymbol{Z}_2^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$ $(n \geqq k+3 \geqq 4)$

In this section $A = \boldsymbol{Z}_2^+ \wedge S^{n+k}$ and $B = \boldsymbol{Z}_2^+ \wedge S^n$ with $n \geqq k+3 \geqq 4$; for each Z_2-map $f : A \to B$ we consider its mapping cone

$$C_f = (\boldsymbol{Z}_2^+ \wedge S^n) \cup_f (\boldsymbol{Z}_2^+ \wedge e^{n+k+1}). \tag{5.1}$$

Since $[A, B]_{\boldsymbol{Z}_2} \cong \pi_{n+k}(\boldsymbol{Z}_2^+ \wedge S^n) \cong \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^n)$ by [4, Lemma 2.1′], the $\boldsymbol{Z}_2$-homotopy class $f \in [A, B]_{Z_2}$ can be written as $f = Sf'$ for some $f' \in [\boldsymbol{Z}_2^+ \wedge S^{n+k-1}, \boldsymbol{Z}_2^+ \wedge S^{n-1}]_{\boldsymbol{Z}_2}$ and

$$f = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \qquad f_i \in \pi_{n+k}(S^n), \ i = 1, 2. \tag{5.2}$$

We first calculate the group K in Theorem 3.5. By an argument similar to the proof of Lemma 2.1 we have

$$i_* : [SB, B]_{\boldsymbol{Z}_2} \to [SB, C_f]_{\boldsymbol{Z}_2} \text{ is epimorphic.} \tag{5.3}$$

Let η_n denote the generator of $\pi_{n+1}(S^n) = \boldsymbol{Z}_2$. Then by [7, Proposition 3.1]

$$\eta_n S f_i = f_i \eta_{n+k} \qquad \text{for any } f_i \in \pi_{n+k}(S^n) \ (n \geqq k+3 \geqq 4). \tag{5.4}$$

Since $[SB, B]_{\boldsymbol{Z}_2} \cong \pi_{n+1}(S^n) \oplus \pi_{n+1}(S^n) = \boldsymbol{Z}_2\{\eta_n\} \oplus \boldsymbol{Z}_2\{\eta_n\}$ and similarly $[SA, A]_{\boldsymbol{Z}_2} \cong \boldsymbol{Z}_2\{\eta_{n+k}\} \oplus \boldsymbol{Z}_2\{\eta_{n+k}\}$, (5.4) yields

$$(Sf)^*[SB, B]_{\boldsymbol{Z}_2} = f_*[SA, A]_{\boldsymbol{Z}_2}. \tag{5.5}$$

Now, (5.3) and (5.5) yield

$$(Sf)^*[SB, C_f]_{\boldsymbol{Z}_2} = (Sf)^* i_*[SB, B]_{\boldsymbol{Z}_2} = i_* f_*[SA, A]_{\boldsymbol{Z}_2} = 0. \tag{5.6}$$

As in the proof of Lemma 3.1 we have an exact sequence of groups

$$[SA, A]_{\boldsymbol{Z}_2} \xrightarrow{f_*} [SA, B]_{\boldsymbol{Z}_2} \xrightarrow{i_*} [SA, C_f]_{\boldsymbol{Z}_2}.$$

Therefore, (5.6) yields

$$\begin{aligned} K = i_*[SA, B]_{\boldsymbol{Z}_2} &\cong [SA, B]_{\boldsymbol{Z}_2} / f_*[SA, A]_{\boldsymbol{Z}_2} \\ &\cong \pi_{n+k+1}(S^n) \oplus \pi_{n+k+1}(S^n) / \{(f_1\eta, f_2\eta), (f_2\eta, f_1\eta)\}, \end{aligned} \tag{5.7}$$

where $\eta = \eta_{n+k}$ and $\{x, y\}$ denotes the subgroup generated by x and y. We next calculate the subgroup M of $\mathscr{E}_{\boldsymbol{Z}_2}(A) \times \mathscr{E}_{\boldsymbol{Z}_2}(B)$ in Theorem 3.5. Let $E = E_2$ be the 2×2 identity matrix and $F = F_2$ the 2×2 matrix of order 2 defined in (4.4), and let

$$a = (-E, -E), \qquad b = (F, F), \qquad c = (E, -E), \qquad \text{and} \qquad d = (E, F).$$

Then, by Corollary 4.5

$$\mathscr{E}_{\boldsymbol{Z}_2}(A) \times \mathscr{E}_{\boldsymbol{Z}_2}(B) \cong (\boldsymbol{Z}_2)^4 \text{ generated by } a, b, c \text{ and } d, \tag{5.8}$$

and for the presentation of $\boldsymbol{Z}_2$-homotopy class f in (5.2) we have

$$\begin{array}{lll} f(-E) = (-E)f & \text{and } fF = Ff & \text{always hold,} \\ f = (-E)f & \text{if and only if} & 2f_i = 0 \text{ for } i = 1 \text{ and } 2, \\ f = Ff & \text{if and only if} & f_1 = f_2, \\ f = (-F)f & \text{if and only if} & f_1 = -f_2. \end{array} \tag{5.9}$$

Now by Theorem 3.5, (5.8) and (5.9) we have

$$M \cong \begin{cases} (\boldsymbol{Z}_2)^2 & \text{if } f_1 \neq f_2,\ f_1 \neq -f_2 \text{ and } 2f_i \neq 0 \text{ for } i = 1 \text{ or } 2, \\ (\boldsymbol{Z}_2)^3 & \text{if } f_1 \neq f_2 \text{ and } 2f_i = 0 \text{ for } i = 1 \text{ and } 2, \\ (\boldsymbol{Z}_2)^3 & \text{if } f_1 = f_2 \text{ and } f_1 \neq -f_2, \\ (\boldsymbol{Z}_2)^3 & \text{if } f_1 \neq f_2 \text{ and } f_1 = -f_2, \\ (\boldsymbol{Z}_2)^4 & \text{otherwise.} \end{cases} \tag{5.10}$$

Consequently by Theorem 3.5 we have

THEOREM 5.11. *If* $n \geqq k + 3 \geqq 4$, *then for each* $\boldsymbol{Z}_2$*-map* $f : \boldsymbol{Z}_2^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$, *its* $\boldsymbol{Z}_2$*-homotopy class* $f \in [\boldsymbol{Z}_2^+ \wedge S^{n+k}, \boldsymbol{Z}_2^+ \wedge S^n]_{\boldsymbol{Z}_2}$ *can be written as* (5.2), *and for its mapping cone* C_f *there is an exact sequence of groups*

$$0 \to K \to \mathscr{E}_{\boldsymbol{Z}_2}(C_f) \to M \to 1$$

where K *and* M *are the groups in* (5.7) *and* (5.10) *respectively.*

Using this theorem, we further calculate the group $\mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ for $k = 1$. Since the group $\pi_{n+1}(S^n)$ in (5.2) is isomorphic to $\boldsymbol{Z}_2$ generated by η_n, for each $\boldsymbol{Z}_2$-map $f : A \to B$ its $\boldsymbol{Z}_2$-homotopy class $f \in [A, B]_{\boldsymbol{Z}_2}$ can be written as

$$f = \begin{pmatrix} s\eta & t\eta \\ t\eta & s\eta \end{pmatrix}, \qquad \eta = \eta_n, \qquad s, t = 0, 1.$$

Also, since the group $\pi_{n+2}(S^n)$ in (5.7) is isomorphic to $\boldsymbol{Z}_2$ generated by $\eta_n \eta_{n+1}$, the group K in (5.7) is trivial when $s \neq t$, and hence by Theorem 5.11 and

(5.10)

$$\mathscr{E}_{Z_2}(C_f) \cong (\boldsymbol{Z}_2)^3 \qquad \text{if } s \neq t. \tag{5.12}$$

We now assume that $s = t = 0$. Then the group K is isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, and hence Theorem 5.11 and (5.10) yield the exact sequence of groups

$$0 \longrightarrow \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \xrightarrow{\lambda} \mathscr{E}_{\boldsymbol{Z}_2}(C_f) \xrightarrow{\varphi \times \psi} (\boldsymbol{Z}_2)^4 \longrightarrow 1, \tag{5.13}$$

where (5.8) shows that the right-hand group $(\boldsymbol{Z}_2)^4$ is generated by a, b, c and d. Furthermore, since $C_f \simeq (\boldsymbol{Z}_2^+ \wedge S^n) \vee (\boldsymbol{Z}_2^+ \wedge S^{n+2})$ by (5.1), the right inverse $\sigma : (\boldsymbol{Z}_2)^4 \to \mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ of the homomorphism $\varphi \times \psi$ can be given by

$$\sigma(a) = -E_4, \qquad \sigma(b) = \begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix}, \qquad \sigma(c) = \begin{pmatrix} -E & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}, \qquad \sigma(d) = \begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}.$$

Therefore, (5.13) is a split extension, and hence $\mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ is isomorphic to the semi-direct product $(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2) \rtimes (\boldsymbol{Z}_2)^4$. Furthermore, for $\eta^2 = \eta_n \eta_{n+1}$ we define

$$\begin{aligned} P &= \begin{pmatrix} \eta^2 & 0 \\ 0 & \eta^2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \eta^2 \\ \eta^2 & 0 \end{pmatrix}, \\ P_4 &= \begin{pmatrix} E & P \\ \mathbf{0} & E \end{pmatrix}, \qquad Q_4 = \begin{pmatrix} E & Q \\ \mathbf{0} & E \end{pmatrix}. \end{aligned} \tag{5.14}$$

Then, P_4 and Q_4 generate $\lambda(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)$ by the definition of λ, and hence $\mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ is generated by $\sigma(a), \sigma(b), \sigma(c), \sigma(d), P_4$ and Q_4. Thus, we have

$$\mathscr{E}_{\boldsymbol{Z}_2}(C_f) \cong D_4 \times (\boldsymbol{Z}_2)^3 \qquad \text{if } s = t = 0, \tag{5.15}$$

where the direct factor D_4 is the dihedral group of order 8, and $(\boldsymbol{Z}_2)^3$ is generated by $\sigma(a), \sigma(b)$ and $\sigma(c)$. If $s = t = 1$, then the group K is isomorphic to Z_2 by (5.7) and the group M is isomorphic to $(Z_2)^4$ by (5.10). Therefore, by (5.12), (5.15) and Theorem 5.11 we have

Proposition 5.16. *If $n \geqq 4$, then for each $\boldsymbol{Z}_2$-map $f : \boldsymbol{Z}_2^+ \wedge S^{n+1} \to \boldsymbol{Z}_2^+ \wedge S^n$, its $\boldsymbol{Z}_2$-homotopy class $f \in [\boldsymbol{Z}_2^+ \wedge S^{n+1}, \boldsymbol{Z}_2^+ \wedge S^n]_{\boldsymbol{Z}_2}$ can be written as*

$$f = \begin{pmatrix} s\eta & t\eta \\ t\eta & s\eta \end{pmatrix}, \qquad \eta = \eta_n, \qquad s, t = 0, 1,$$

and for its mapping cone C_f, we have

$$\mathscr{E}_{\boldsymbol{Z}_2}(C_f) = \begin{cases} (\boldsymbol{Z}_2)^3 & \text{if } s \neq t \\ D_4 \times (\boldsymbol{Z}_2)^3 & \text{if } s = t = 0. \end{cases}$$

If $s = t = 1$, *then there is an exact sequence of groups*

$$0 \to \boldsymbol{Z}_2 \to \mathscr{E}_{\boldsymbol{Z}_2}(C_f) \to (\boldsymbol{Z}_2)^4 \to 1.$$

6. $\mathscr{E}_{\boldsymbol{Z}_6}(C_f)$ for $f : \boldsymbol{Z}_6^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$ $(n \geqq k+3 \geqq 4)$

We take $A = \boldsymbol{Z}_6^+ \wedge S^{n+k}$ and $B = \boldsymbol{Z}_2^+ \wedge S^n$ with $n \geqq k+3 \geqq 4$, where $\boldsymbol{Z}_2 = \boldsymbol{Z}_6/\boldsymbol{Z}_3$. Since $[A,B]_{\boldsymbol{Z}_6} \cong \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^n)$, each $\boldsymbol{Z}_6$-homotopy class $f \in [A,B]_{\boldsymbol{Z}_6}$ can be written as $f = Sf'$ for some $f' \in [\boldsymbol{Z}_6^+ \wedge S^{n+k-1}, \boldsymbol{Z}_2^+ \wedge S^{n-1}]_{\boldsymbol{Z}_6}$ and

$$f = \begin{pmatrix} f_1 & f_2 & f_1 & f_2 & f_1 & f_2 \\ f_2 & f_1 & f_2 & f_1 & f_2 & f_1 \end{pmatrix}, \qquad f_i \in \pi_{n+k}(S^n), \qquad i = 1, 2. \tag{6.1}$$

Let K be the group in Theorem 3.5. Then, as in §5 we have

$$K \cong \pi_{n+k+1}(S^n) \oplus \pi_{n+k+1}(S^n)/\{(f_1\eta_{n+k}, f_2\eta_{n+k}), (f_2\eta_{n+k}, f_1\eta_{n+k})\}. \tag{6.2}$$

We calculate the subgroup M of $\mathscr{E}_{\boldsymbol{Z}_6}(A) \times \mathscr{E}_{\boldsymbol{Z}_6}(B)$ in Theorem 3.5. Let E_q be the $q \times q$ identity matrix and F_q the $q \times q$ matrix of order q defined in (4.4), and let

$$a = (F_6, F_2), \qquad b = (-E_6, -E_2), \qquad c = (E_6, -E_2) \qquad \text{and} \qquad d = (E_6, F_2).$$

Then by Corollary 4.5

$$\mathscr{E}_{\boldsymbol{Z}_6}(A) \times \mathscr{E}_{\boldsymbol{Z}_6}(B) \cong \boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^3 \text{ generated by } a, b, c \text{ and } d, \tag{6.3}$$

and

$$\begin{aligned} & f(-E_6) = (-E_2)f \text{ and } fF_6 = F_2 f \text{ always hold,} \\ & f = (-E_2)f \qquad \text{if and only if} \quad 2f_i = 0 \text{ for } i = 1 \text{ and } 2, \\ & f = F_2 f \qquad \text{if and only if} \quad f_1 = f_2, \\ & f = (-F_2)f \qquad \text{if and only if} \quad f_1 = -f_2 \end{aligned} \tag{6.4}$$

for f in (6.1). Now by Theorem 3.5, (6.3) and (6.4) we have

(6.5)

$$M \cong \begin{cases} \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^2 & \text{if } f_1 \neq f_2,\ f_1 \neq -f_2 \text{ and } 2f_i \neq 0 \text{ for } i = 1 \text{ or } 2, \\ \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 & \text{if } f_1 \neq f_2 \text{ and } 2f_i = 0 \quad \text{for } i = 1 \text{ and } 2, \\ \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 & \text{if } f_1 = f_2 \text{ and } f_1 \neq -f_2, \\ \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 & \text{if } f_1 \neq f_2 \text{ and } f_1 = -f_2, \\ \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^4 & \text{otherwise.} \end{cases}$$

Consequently by Theorem 3.5 we have

Theorem 6.6. *If $n \geqq k+3 \geqq 4$, then for each $\boldsymbol{Z}_6$-map $f : \boldsymbol{Z}_6^+ \wedge S^{n+k} \to \boldsymbol{Z}_2^+ \wedge S^n$, its $\boldsymbol{Z}_6$-homotopy class $f \in [\boldsymbol{Z}_6^+ \wedge S^{n+k}, \boldsymbol{Z}_2^+ \wedge S^n]_{\boldsymbol{Z}_6}$ can be written as* (6.1), *and for its mapping cone C_f there is an exact sequence of groups*

$$0 \to K \to \mathscr{E}_{\boldsymbol{Z}_6}(C_f) \to M \to 1$$

where K and M are the groups in (6.2) *and* (6.5) *respectively.*

We further calculate the group $\mathscr{E}_{\boldsymbol{Z}_6}(C_f)$ for $k=1$. Since the group $\pi_{n+1}(S^n)$ in (6.1) is isomorphic to $\boldsymbol{Z}_2$ generated by η_n, we have $f_1 = s\eta$, $f_2 = t\eta$, $\eta = \eta_n$ with $s, t = 0, 1$ in (6.1). Also, since the group $\pi_{n+2}(S^n)$ in (6.2) is isomorphic to $\boldsymbol{Z}_2$ generated by $\eta_n\eta_{n+1}$, the group K in (6.2) is trivial when $s \neq t$, and hence by Theorem 6.6 and (6.5)

$$\mathscr{E}_{\boldsymbol{Z}_6}(C_f) \cong \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 \qquad \text{if } s \neq t. \tag{6.7}$$

We now assume that $s = t = 0$. Then the group K is isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, and hence Theorem 6.6 and (6.5) yield the exact sequence of groups

$$0 \longrightarrow \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \xrightarrow{\lambda} \mathscr{E}_{\boldsymbol{Z}_6}(C_f) \xrightarrow{\varphi \times \psi} \boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^3 \longrightarrow 1, \tag{6.8}$$

where (6.3) shows that the right-hand group $\boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^3$ is generated by a, b, c and d. Furthermore, since $C_f \simeq (\boldsymbol{Z}_2^+ \wedge S^n) \vee (\boldsymbol{Z}_6^+ \wedge S^{n+2})$, the right inverse $\sigma : \boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^3 \to \mathscr{E}_{\boldsymbol{Z}_2}(C_f)$ of the homomorphism $\varphi \times \psi$ can be given by

$$\sigma(a) = \begin{pmatrix} F_2 & \mathbf{0} \\ \mathbf{0} & F_6 \end{pmatrix}, \qquad \sigma(b) = -E_8,$$

$$\sigma(c) = \begin{pmatrix} -E_2 & \mathbf{0} \\ \mathbf{0} & E_6 \end{pmatrix}, \qquad \sigma(d) = \begin{pmatrix} F_2 & \mathbf{0} \\ \mathbf{0} & E_6 \end{pmatrix},$$

where F_q is the matrix in (4.4). Therefore, the sequence (6.8) is a split extension, and hence $\mathscr{E}_{\boldsymbol{Z}_6}(C_f) \cong (\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2) \rtimes (\boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^3)$. Let P_8 and Q_8 be 8×8 matrices defined by

$$P_{12} = (P \quad P \quad P), \qquad Q_{12} = (Q \quad Q \quad Q),$$

$$P_8 = \begin{pmatrix} E_2 & P_{12} \\ \mathbf{0} & E_6 \end{pmatrix}, \qquad Q_8 = \begin{pmatrix} E_2 & Q_{12} \\ \mathbf{0} & E_6 \end{pmatrix},$$

where P and Q are the 2×2 matrices in (5.14). Then, P_8 and Q_8 generate $\lambda(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)$ by the definition of λ, and hence $\mathscr{E}_{\boldsymbol{Z}_6}(C_f)$ is generated by $\sigma(a), \sigma(b), \sigma(c), \sigma(d), P_8$ and Q_8. Thus, we have

$$\mathscr{E}_{\boldsymbol{Z}_6}(C_f) \cong D_4 \times \boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^2 \qquad \text{if } s = t = 0, \tag{6.9}$$

where the direct factor $\boldsymbol{Z}_6 \times (\boldsymbol{Z}_2)^2$ is generated by $\sigma(a), \sigma(b)$ and $\sigma(c)$. If $s = t = 1$, then the group K is isomorphic to Z_2 by (6.2) and the group M is isomorphic to $Z_3 \times (Z_2)^4$ by (6.5). Therefore, by (6.7), (6.9) and Theorem 6.6 we have

PROPOSITION 6.10. *If $n \geqq 4$, then for each $\boldsymbol{Z}_6$-map $f : \boldsymbol{Z}_6^+ \wedge S^{n+1} \to \boldsymbol{Z}_2^+ \wedge S^n$, its $\boldsymbol{Z}_6$-homotopy class $f \in [\boldsymbol{Z}_6^+ \wedge S^{n+1}, \boldsymbol{Z}_2^+ \wedge S^n]_{\boldsymbol{Z}_6}$ can be written as*

$$f = \begin{pmatrix} s\eta & t\eta & s\eta & t\eta & s\eta & t\eta \\ t\eta & s\eta & t\eta & s\eta & t\eta & s\eta \end{pmatrix}, \qquad \eta = \eta_n, \qquad s, t = 0, 1,$$

and for its mapping cone C_f we have

$$\mathscr{E}_{\boldsymbol{Z}_6}(C_f) = \begin{cases} \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 & \text{if } s \neq t \\ D_4 \times \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^3 & \text{if } s = t = 0. \end{cases}$$

If $s = t = 1$, then there is an exact sequence of groups

$$0 \to \boldsymbol{Z}_2 \to \mathscr{E}_{\boldsymbol{Z}_6}(C_f) \to \boldsymbol{Z}_3 \times (\boldsymbol{Z}_2)^4 \to 1.$$

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